# Distributed Space-Time Coding for Two-Way Wireless Relay Networks

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***Abstract*— We consider distributed space-time coding for two-way wireless relay networks, where communication between two terminals is assisted by relay nodes. We compare existing and new protocols that operate over 2, 3 or 4 times slots. Particularly, a new class of relaying protocols, termed as *partial decode-and-forward* (PDF), is proposed for the 2 time slots transmission. We show that the proposed *amplify and forward* (AF) protocols achieve the diversity order of $\min\{N,T\}\left(1-\frac{\log\log P}{\log P}\right)$, where $N$ is the number of relays, $P$ is the total power of the network, and $T$ is the number of symbols transmitted during each time slot. When linear dispersion (LD) codes with random unitary matrices are used, the proposed PDF protocols resemble random linear network coding, where the former operates on unitary group and the latter works on finite field.**

## I. Introduction

Several works on wireless networks consider the exploitation of spatial diversity using antennas of different users in the network [1]–[4]. In [1], spatial diversity is exploited by extending existing strategies from one-way relay channels [5], which include *amplify-and-forward* (AF) and *decode and forward* (DF) protocols. In [2], a distributed linear dispersion (LD) space-time code was proposed using the AF protocol, where the diversity gain and coding gain are analyzed. Distributed space-time block coding for DF protocols was proposed in [3] while its randomized version was given in [4]. These works [1]–[4] consider only unidirectional communication.

Two-way communication is another common communication scenario where two parties transmit information to each other. The two-way channel (TWC) was first considered by Shannon [6], who derived inner and outer bounds on the capacity region. Recently, the two-way relay channel (TWRC) has drawn renewed interest. AF and DF protocols for one-way relay channels are extended to the half-duplex Gaussian TWRC in [7]. In [8], algebraic network coding [9] is used to increase the sum-rate of two users. These works [7], [8] consider only a single relay.

In this paper, we consider distributed space-time coding (DSTC) for two-way wireless relay networks with fading. For 2 time slots transmission, a class of relaying protocols, termed as *partial decode-and-forward* (PDF) is proposed, under which each relay removes part of the noise before sending the signal to two terminals. The 2 time slots protocols cannot make use of the direct transmission between the two terminals, and we consider protocols using 3 time slots to overcome this drawback. In each protocol, the relays encode using a distributed linear dispersion (LD) code [10]. Using an approach similar to that in [2], we show that the proposed AF protocols achieve the diversity order $\min\{N,T\}\left(1-\frac{\log\log P}{\log P}\right)$. Moreover, with the aid of cyclic redundancy check (CRC), PDF can achieve a diversity order $\min\{N,T\}$.

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## II. Network Model

We consider a wireless network with $N$ relay nodes $\mathbb{R}_i$, $i=1,\ldots,N$, and two terminal nodes $\mathbb{T}_m$, $m=1,2$. The two terminals exchange information with the assistance of relays between them. Every node has only a single antenna that cannot transmit and receive simultaneously. Denote the channel between $\mathbb{T}_1$ and $\mathbb{R}_i$ as $f_i$, and the channel between $\mathbb{T}_2$ and $\mathbb{R}_i$ as $g_i$. We assume that $f_i$ and $g_i$ are independent complex Gaussian random variables with zero mean and unit variance, i.e., $f_i\sim\mathcal{CN}(0,1)$ and $g_i\sim\mathcal{CN}(0,1)$. We also assume that the channel is unknown to transmitting nodes but perfectly known at receiving nodes, which can be obtained by adapting the estimation algorithms in [11].

Assume that terminal $\mathbb{T}_m$ wishes to send the signal $\mathbf{s}_m=[s_{m,1},\ldots,s_{m,T}]^T$ to the other terminal, where $s_{m,t}\in\mathcal{A}_m$, $m=1,2$, $t=1,\ldots,T$, $\mathcal{A}_m$ is a finite constellation with average power 1, and $T$ is the length of each time slot. Thus, $E\{\mathbf{s}_m^H\mathbf{s}_m\}=T$. The average power of terminal $\mathbb{T}_m$ is $P_m$, $m=1,2$. For fair comparison, we assume that the total power of all the relays is $P_3$ and each relay has equal power $P_3/N$. For convenience, the noise variance at $\mathbb{R}_i$ or $\mathbb{T}_i$ is assumed to be 1.

## III. DSTC Protocols Using 2 Time Slots

Both $\mathbb{T}_1$ and $\mathbb{T}_2$ simultaneously send their data to the relays in the first time slot. Since each terminal transmits every two timeslots, the transmit power is $\sqrt{2P_i}$. The $i$th relay receives

$$\mathbf{y}_i^r=\sqrt{2P_1}f_i\mathbf{s}_1+\sqrt{2P_2}g_i\mathbf{s}_2+\mathbf{n}_i^r. \tag{1}$$

where $\mathbf{n}_i^r$ is the $T\times 1$ vector representing the circularly complex Gaussian noise at the $i$th relay. In the second time slot, the $i$th relay transmits $\mathbf{x}_i^r$ (obtained as a function of $\mathbf{y}_i^r$ as described in the various protocols below) scaled by $\beta_i$ to maintain average power $P_3$. The $m$th terminal then receives

$$\mathbf{y}_m=\sum_{i=1}^{N}\beta_i f_i \mathbf{x}_i^r+\mathbf{n}_m, \quad (2)$$

where $\mathbf{n}_m$ is the noise vector at the $m$th terminal.

### A. Amplify and Forward (2-AF)

The AF scheme is usually related with LD codes [2], where $\mathbf{x}_i^r$ is obtained by precoding $\mathbf{y}_i^r$ with a unitary matrix $\mathbf{A}_i^r$, and is then scaled by $\beta \triangleq \sqrt{\frac{2P_3}{N(2P_1+2P_2+1)}}$. Due to symmetry, we will only consider the received signal at $\mathbb{T}_2$, which is

$$\begin{aligned}\mathbf{y}_2&=\beta\left(\sqrt{2P_1}\mathbf{S}_1\mathbf{h}+\sqrt{2P_2}\mathbf{S}_2\mathbf{g}\right)+\mathbf{w}_2,\\ &=\beta\left(\sqrt{2P_1}\mathbf{H}\mathbf{s}_1+\sqrt{2P_2}\mathbf{G}\mathbf{s}_2\right)+\mathbf{w}_2, \quad (3)\end{aligned}$$

where

$\mathbf{S}_m=[\mathbf{A}_i^r\mathbf{s}_m,\ldots,\mathbf{A}_N^r\mathbf{s}_m]$, $\mathbf{g}=[g_1^2,\ldots,g_N^2]^T$, $\mathbf{h}=[f_1g_1,\ldots,f_Ng_N]^T$, $\mathbf{w}_2=\beta\sum_{i=1}^N g_i\mathbf{A}_i^r\mathbf{n}_i^r+\mathbf{n}_2, \mathbf{H}=\sum_{i=1}^N f_ig_i\mathbf{A}_i^r, \mathbf{G}=\sum_{i=1}^N g_i^2\mathbf{A}_i^r$.

Since the true $\mathbf{s}_m$ is known at terminal $m$, the maximum-likelihood (ML) decoding of (3) can be easily obtained as

$$\hat{\mathbf{s}}_1=\underset{\tilde{\mathbf{s}}_1\in\mathcal{A}_1^N}{\operatorname{argmin}}\left\|\mathbf{y}_2-\beta\left(\sqrt{2P_1}\mathbf{H}\tilde{\mathbf{s}}_1+\sqrt{2P_2}\mathbf{G}\mathbf{s}_2\right)\right\|^2, \quad (4)$$

which can be solved by using sphere decoder [12].

### B. Partial Decode and Forward I (PDF I)

Noting that the 2-AF protocol amplifies the relay noise, we propose a new protocol to mitigate this effect. Instead of simply amplifying the received signal, each relay $i$ first decodes $\mathbf{s}_1$ and $\mathbf{s}_2$ via the ML decoder

$$\{\hat{\mathbf{s}}_{1,i},\hat{\mathbf{s}}_{2,i}\}=\underset{\tilde{\mathbf{s}}_1\in\mathcal{A}_1^N,\tilde{\mathbf{s}}_2\in\mathcal{A}_2^N}{\operatorname{argmin}}\left\|\mathbf{y}_i^r-\sqrt{2P_1}f_i\mathbf{A}_1\tilde{\mathbf{s}}_1-\sqrt{2P_2}g_i\mathbf{A}_2\tilde{\mathbf{s}}_2\right\|_2^2 \quad (5)$$

Note that (5) represents an under-determined system, which can be efficiently solved using generalized sphere decoder [13]. Since the number of unknowns in (5) is twice the number of equations, the error probability, even with ML decoding, is still high. Therefore, it is not good to send $\hat{\mathbf{s}}_{1,i}$ and $\hat{\mathbf{s}}_{2,i}$ directly, as an error at one relay may affect the decoding at the end terminals. To mitigate error propagation, we propose that each relay sends

$$\mathbf{x}_i^r=\mathbf{A}_i^r\left(\sqrt{2P_1}f_i\hat{\mathbf{s}}_{1,i}+\sqrt{2P_2}g_i\hat{\mathbf{s}}_{2,i}\right), \quad (6)$$

scaled by $\beta=\beta_i=\sqrt{\frac{P_3}{N(P_1+P_2)}}$. Note that we still use the form $\sqrt{2P_1}f_i\hat{\mathbf{s}}_{1,i}+\sqrt{2P_2}g_i\hat{\mathbf{s}}_{2,i}$ as the useful signal component in the received signal. This scheme can be considered as removing noise from the received signal while keeping channel effects.

Let $P(\Delta\mathbf{s}_{1,i},\Delta\mathbf{s}_{2,i})$ denote the pairwise error probability at relay $i$, where $\Delta\mathbf{s}_{1,i}=\mathbf{s}_1-\hat{\mathbf{s}}_{1,i}$ and $\Delta\mathbf{s}_{2,i}=\mathbf{s}_2-\hat{\mathbf{s}}_{2,i}$. The ML decoder at $\mathbb{T}_2$ can be obtained as

$$\begin{aligned}\hat{\mathbf{s}}_1=&\underset{\tilde{\mathbf{s}}_1\in\mathcal{A}_1^N}{\operatorname{argmax}}\sum_{\Delta\mathbf{s}_{1,i},\Delta\mathbf{s}_{2,i}}\prod_{i=1}^N P(\Delta\mathbf{s}_{1,i},\Delta\mathbf{s}_{2,i})\\ &\times\exp\left[-\left\|\mathbf{y}_2+\beta\sum_{i=1}^N\mathbf{A}_i^r\left(\sqrt{2P_1}f_i\Delta\mathbf{s}_{1,i}+\sqrt{2P_2}g_i\Delta\mathbf{s}_{2,i}\right)\right.\right.\\ &\left.\left.-\beta\left(\sqrt{2P_1}\mathbf{H}\tilde{\mathbf{s}}_1+\sqrt{2P_2}\mathbf{G}\mathbf{s}_2\right)\right\|^2\right], \quad (7)\end{aligned}$$

where $\mathbf{H}$ and $\mathbf{G}$ are defined in (3). When $N$ or the constellation size is large, it is hard to implement (7) directly. In high SNR, $\prod_{i=1}^N P(\Delta\mathbf{s}_{1,i},\Delta\mathbf{s}_{2,i})$ is dominated by $\Delta\mathbf{s}_{1,i}=\mathbf{0},\Delta\mathbf{s}_{2,i}=\mathbf{0}$. Thus, we approximate the ML decoding at terminal 2 by

$$\hat{\mathbf{s}}_1=\underset{\tilde{\mathbf{s}}_1\in\mathcal{A}_1^N}{\operatorname{argmin}}\left\|\mathbf{y}_2-\beta\left(\sqrt{2P_1}\mathbf{H}\tilde{\mathbf{s}}_1+\sqrt{2P_2}\mathbf{G}\mathbf{s}_2\right)\right\|^2. \quad (8)$$

The ML decoding at $\mathbb{T}_1$ can be obtained similarly.

### C. Partial Decode and Forward II (PDF II)

Both AF and PDF I transmit a weighted sum of signals from two terminals. However, this can be wasteful in terms of power as terminal $m$ already knows $\mathbf{s}_m$, $m=1,2$. In PDF II, we propose to superimpose the signals via modular arithmetic. Let $M_m$ denote the size of constellation $\mathcal{A}_m$, and $\mathcal{A}_m(j)$ denote the $j$-th element of $\mathcal{A}_m$, $m=1,2$, $j=0,\ldots,M_m-1$. Define $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathcal{A}_1(\mathbf{v}_1)=\mathbf{s}_1$ and $\mathcal{A}_2(\mathbf{v}_2)=\mathbf{s}_2$. We further set $M=\max\{M_1,M_2\}$. Without loss of generality, we assume that $M_1\geq M_2$.

In this protocol, each relay obtains $\hat{\mathbf{s}}_{1,i},\hat{\mathbf{s}}_{2,i}$ from (5) as in PDF I. Let $\mathcal{A}_1(\hat{\mathbf{v}}_{1,i})=\hat{\mathbf{s}}_{1,i}$ and $\mathcal{A}_2(\hat{\mathbf{v}}_{2,i})=\hat{\mathbf{s}}_{2,i}$. In this case

$$\mathbf{x}_i^r=\mathbf{A}_i^r\mathcal{A}_1(\operatorname{mod}(\hat{\mathbf{v}}_{1,i}+\hat{\mathbf{v}}_{2,i},M)), \quad (9)$$

where mod denotes the componentwise modular operation and $\beta=\beta_i=\sqrt{\frac{2P_3}{N}}$. Note that as fading channels are considered, the probability that there exists a pair of vectors $\{\mathbf{v}_1,\mathbf{v}_2\}$ and $\{\hat{\mathbf{v}}_1,\hat{\mathbf{v}}_2\}$ such that $\sqrt{2P_1}f_i\mathcal{A}_1(\mathbf{v}_1)+\sqrt{2P_2}g_i\mathcal{A}_2(\mathbf{v}_2)=\sqrt{2P_1}f_i\mathcal{A}_1(\hat{\mathbf{v}}_1)+\sqrt{2P_2}g_i\mathcal{A}_2(\hat{\mathbf{v}}_2)$ is vanishingly small.

Compared with (6), power is saved by using the modular operation. This scheme is very similar to network coding but has two differences: 1) this scheme is applied at the physical layer while network coding is performed at the network layer; 2) the operation in (9) is on a modular group, whereas finite field is used in network coding.

To exploit the diversity offered by multiple relays, we assume that each relay is able to determine whether $\operatorname{mod}(\hat{\mathbf{v}}_{1,i}+\hat{\mathbf{v}}_{2,i},M)=\operatorname{mod}(\mathbf{v}_1+\mathbf{v}_2,M)$ through the use of CRCs or other error detecting codes. When $M=2$, the XOR between CRCs of $\mathbf{v}_1$ and $\mathbf{v}_2$ is also the CRC of $\operatorname{mod}(\mathbf{v}_1+\mathbf{v}_2,M)$. Note that this is different from checking the correctness of $\mathbf{v}_1$ or $\mathbf{v}_2$ individually as both of them may be wrong but their modular sum is correct. Each relay will send $\mathbf{x}_i^r$ in (9) if and only if the modular sum is correct. The terminal decoder can be obtained similar to (7) and (8). In the following, we only consider the suboptimal decoder with the same form as (8).

## IV. DSTC Protocols Using 3 Time Slots

In this section, we consider three-time slots protocols. In the first time slot, $\mathbb{T}_1$ transmits and in the second time slot, $\mathbb{T}_2$ transmits. The transmission power is $\sqrt{3P_i}$ because each terminal transmits every 3 time slots. The received signal at the $i$th relay in the first and the second time slots are

$$\mathbf{y}_i^{r,1}=\sqrt{3P_1}f_i\mathbf{s}_1+\mathbf{n}_i^{r,1}, \quad (10)$$

$$\mathbf{y}_i^{r,2}=\sqrt{3P_2}g_i\mathbf{s}_2+\mathbf{n}_i^{r,2}, \quad (11)$$

respectively. In the third time slot, each relay $i$ transmits $\mathbf{x}_i^r$ scaled by $\beta_i$ to meet its power constraint. The received signals at $\mathbb{T}_1$ and $\mathbb{T}_2$ are the same as those in (2).

For amplify and forward (3-AF), $\mathbf{x}_i^r$ is obtained from

$$\mathbf{x}_i^r = \mathbf{A}_i^r(\sqrt{\lambda_i}\mathbf{y}_i^{r,1} + \sqrt{1-\lambda_i}\mathbf{y}_i^{r,2}), \quad (12)$$

where $0 \le \lambda_i \le 1$ is a power allocation coefficient at relay $i$. For decode and forward, each relay $i$ first decodes $\mathbf{s}_1$ and $\mathbf{s}_2$ by normal ML coding during the first and second time slot. The rest of DF I and DF II is similar to PDF I and PDF II respectively. The details can be found in [14].

The 3 time slots protocols have an advantage over the 2 time slots protocols in that they can exploit the direct transmission between the two terminals. Let $d$ denote the channel gain between the two terminals and $\mathbf{y}_2^1$, $\mathbf{y}_2^3$ denote the received signal at terminal 2 in time slot 1 and 3 respectively. The approximate ML decoder at terminal 2 can be obtained as

$$\hat{\mathbf{s}}_1 = \underset{\tilde{\mathbf{s}}_1 \in \mathcal{A}_1^N}{\operatorname{argmin}} \left\| \mathbf{y}_2^3 - \sqrt{\frac{3P_3}{N}}(\mathbf{\Theta}_1\tilde{\mathbf{s}}_1 + \mathbf{\Theta}_2\mathbf{s}_2) \right\|_2^2 + \left\| \mathbf{y}_2^1 - d\sqrt{3P_1}\mathbf{A}_1\tilde{\mathbf{s}}_1 \right\|_2^2, \quad (13)$$

where $\mathbf{\Theta}_1 = \sum_{i=1}^N \sqrt{\lambda_i} f_i g_i \mathbf{A}_i^r$, and $\mathbf{\Theta}_2 = \sum_{i=1}^N \sqrt{1-\lambda_i} g_i^2 \mathbf{A}_i^r$. From Section VI, the achievable diversity is increased by one due to the presence of the direct transmission.

## V. Performance Analysis and Optimization

In this section, we analyze the performance of the proposed protocols, and compare them with 4 time slots protocols, which simply apply the normal one-way relay protocol twice. For brevity, we only analyze some representative protocols.

### *A. Amplify and Forward*

The analysis of AF protocols basically follows that in [2]. We consider the pairwise error probability (PEP) of mistaking $\mathbf{s}_m$ by $\mathbf{s}'_m$ and define $\mathbf{\Delta}_m = \mathbf{s}_m - \mathbf{s}'_m$, $\mathbf{C}_m = [\mathbf{A}_1\mathbf{\Delta}_m, \dots, \mathbf{A}_N\mathbf{\Delta}_m]$, $\mathbf{M}_m = \mathbf{C}_m^H\mathbf{C}_m$ $m=1,2$. Let $\mu_m$ denote the rank of $\mathbf{M}_m$ and $P = P_1 + P_2 + P_3$ denote the total average power of the whole network. As an example, we have the following theorem on the performance of the 2-AF protocol in Section III.

**Theorem 1**: If $\log P \gg 1$, $T \ge N$, and $N \gg 1$, the PEPs of signals from $\mathbb{T}_m, m=1,2$ are upper bounded as

$$\mathrm{PEP}_m \lesssim \det{}^{-1}\mathbf{M}_m \left(\frac{8N}{T}\right)^{\mu_m} P^{-\mu_m\left(1-\frac{\log\log P}{\log P}\right)}. \quad (14)$$

*Proof:* By using [2, Theorem 1] and considering the expression of (3), we obtain

$$\mathrm{PEP}_1 + \mathrm{PEP}_2 \lesssim \underset{g_i}{\mathrm{E}} \det{}^{-1}\left[\mathbf{I}_N + \frac{P_1P_3}{N(1+2P)}\mathbf{M}_1 \mathrm{diag}(\mathbf{g})\right] + \underset{f_i}{\mathrm{E}} \det{}^{-1}\left[\mathbf{I}_N + \frac{P_2P_3}{N(1+2P)}\mathbf{M}_2 \mathrm{diag}(\mathbf{f})\right], \quad (15)$$

where $\mathbf{f} = [f_1^2, \dots, f_N^2]^T$. First note that (15) is a convex function in $P_1$ and $P_2$ given $P_3$. Since $P$ is fixed, $P_1 + P_2$ is also fixed given $P_3$. Therefore, (15) is minimized when $P_1 = P_2$ by assuming $\mathbf{M}_1 = \mathbf{M}_2$. By minimizing (15) under the conditions $P_1 = P_2$ and $P = P_1 + P_2 + P_3$, we get the optimal power allocation as $P_1 = P_2 = P/4$ and $P_3 = P/2$. By using [2, Corollary 2], we obtain (14). □

From (14), it is clear that the optimal codes $\mathbf{A}_i$, $i=1,\dots,N$ should maximize the minimum $\det^{-1}\mathbf{M}_m$, $m=1,2$. The following theorem provides a sufficient condition on the optimal design.

**Theorem 2**: If $T \ge N$, a set of matrices $\mathbf{A}_i^r$, $i=1,\dots,N$ achieves the minimum PEP if $\det\mathbf{M}_m = \|\mathbf{\Delta}_m\|^{2N}$ and diversity order $N\left(1-\frac{\log\log P}{\log P}\right)$ can be achieved.

*Proof:* Due to symmetry, we drop the subscript on $\mathbf{M}_m$ and $\mathbf{\Delta}_m$. Given any $\mathbf{\Delta}$, by Householder transformation, there exists a unitary matrix $\mathbf{U}$ such that $\mathbf{U}\mathbf{\Delta} = [\|\mathbf{\Delta}\|_2, 0, \dots, 0]^T$. Let $\tilde{\mathbf{A}}_i = \mathbf{A}_i^r\mathbf{U}^H$ be a new unitary matrix, and $\tilde{\mathbf{a}}_i$ be its first column with $\|\tilde{\mathbf{a}}_i\|_2 = 1$. Denote the matrix $\mathbf{D} = [\tilde{\mathbf{a}}_1, \dots, \tilde{\mathbf{a}}_N]$. It is easy to show that $\mathbf{M} = \|\mathbf{\Delta}\|_2^2\mathbf{D}^H\mathbf{D}$. Note that the diagonal entries of $\mathbf{D}^H\mathbf{D}$ are all ones and $\mathbf{D}^H\mathbf{D}$ is positive semidefinite. By Hadamard inequality, we obtain $\mathbf{M} \le \|\mathbf{\Delta}\|_2^{2N}$. If $\det\mathbf{M} = \|\mathbf{\Delta}\|^{2N}$, the rank of $\mathbf{M}$ is $N$. By Theorem 1, the diversity order is $N\left(1-\frac{\log\log P}{\log P}\right)$. □

From Theorem 2, it clear that if $\mathbf{S} = [\mathbf{A}_1^r\mathbf{s}, \cdots, \mathbf{A}_N^r\mathbf{s}]$ constitutes an orthogonal space time code, then it achieves the minimum PEP. For example, when $N = T = 2$ and $\mathcal{A}$ is a real set, we can choose

$$\mathbf{A}_1^r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{A}_2^r = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (16)$$

It is easy to show that Theorem 2 is satisfied with (16). In fact, (16) is a variant of Alamouti code.

When $\mathbf{A}_i^r$, $i=1,\dots,N$ are all random unitary matrices, it is clear that diversity order $N\left(1-\frac{\log\log P}{\log P}\right)$ can be achieved if the following two conditions hold

- The matrix $\mathbf{M}_m$ is full rank with high probability;
- The expectation $\mathrm{E}\{\det^{-1}\mathbf{M}\}$ is finite.

By using the same approach as the proof of Theorem 2 and noting that $\tilde{\mathbf{A}}_i = \mathbf{A}_i^r\mathbf{U}^H$ is also a random unitary matrix, we can show that verifying the above two conditions is equivalent to showing whether $\mathbf{D}^H\mathbf{D}$ is of full rank and whether $\mathrm{E}\{\det^{-1}\mathbf{D}^H\mathbf{D}\}$ is finite, where each column is drawn uniformly and independently on complex hypersphere with unit radius. When $N = T = 2$, we can show that the eigenvalues of $\mathbf{D}^H\mathbf{D}$ are $1-\sqrt{\xi}$ and $1+\sqrt{\xi}$, where $\xi \sim F_{24}$ and $F_{nm}$ is the $F$-distribution. As $F_{24}$ is a continuous distribution, the eigenvalues are zero with probability 0. Therefore, the matrix $\mathbf{D}^H\mathbf{D}$ is full rank with high probability. Also, we can show that $\frac{1}{1-\xi/4}$ is finite. Therefore, random unitary matrices achieve diversity order $2\left(1-\frac{\log\log P}{\log P}\right)$. For $N > 2$, treated in [14], the full diversity can be achieved for all random unitary matrices with probability 1. The advantage of random unitary matrices over orthogonal matrices is that the former is easy to extend to arbitrary number of relays and can be implemented distributedly. $\mathbf{A}_i^r$, $i=1,\dots,N$ can also be designed by maximizing the mutual information as that in [10].

The following theorem characterizes the SNR gain of 2-AF over 4 time slots AF (4-AF).

**Theorem 3**: Let $d_{\min}^{(4)}$ and $d_{\min}^{(2)}$ denote the minimum distance between points in the constellations used by 4-AF and 2-AF, respectively, and $R^{(4)}$ and $R^{(2)}$ denote the rates of the two constellations. Denote $P^{(4)}$ and $P^{(2)}$ as the total power in the networks in the two cases. Assuming that random unitary matrices or optimal unitary matrices are used with $T{=}N$, to achieve the same bit error rate (BER) in high SNR, we must have

$$\frac{P^{(2)}}{P^{(4)}}\approx 2^{R^{(2)}-R^{(4)}+1}\sqrt[N]{\frac{R^{(4)}}{R^{(2)}}}\left(\frac{d_{\min}^{(4)}}{d_{\min}^{(2)}}\right)^2. \quad (17)$$

The proof can be found in [14]. This theorem suggests that under an average power constraint, when the number of relays is large or the constellation size is large, 2 time slots protocols performs better if the same rate is assumed.

### *B. Partial Decode and Forward*

For the PDF I protocol, in [14], we show that diversity order of 2 can be achieved using the orthogonal codes specified in Theorem 2 in a network with 2 relays. For the PDF II protocol, we have the following theorem.

**Theorem 4**: When $T{\geq}N$, the PDF II protocol attains a diversity order $N$.

*Proof:* We first derive the PEP at each relay, i.e. the probability of mistaking $\{\mathbf{s}_1,\mathbf{s}_2\}$ by $\{\mathbf{s}_1',\mathbf{s}_2'\}$. From [15], we upper bound the PEP at any relay as:

$$\mathrm{PEP}\leq\mathbf{det}^{-1}\left(\mathbf{I}_2+\frac{P}{16}\mathbf{M}\right), \quad (18)$$

where $\mathbf{M}{=}\mathbf{C}^H\mathbf{C}$, $\mathbf{C}{=}[\mathbf{\Delta}_1,\mathbf{\Delta}_2]$, and $\mathbf{\Delta}_1{=}\mathbf{s}_1{-}\mathbf{s}_1'$, $\mathbf{\Delta}_2{=}\mathbf{s}_2{-}\mathbf{s}_2'$. By computing the righthand side of (18), we obtain

$$\mathbf{det}^{-1}\left(\mathbf{I}_2+\frac{P}{16}\mathbf{M}\right) = \frac{1}{\left(1+\frac{P}{16}\|\mathbf{\Delta}_1\|^2\right)\left(1+\frac{P}{16}\|\mathbf{\Delta}_2\|^2\right)-\left\|\frac{P}{16}\mathbf{\Delta}_1^H\mathbf{\Delta}_2\right\|^2}. \quad (19)$$

By using the inequality $\|\mathbf{x}_1\|_2^2\|\mathbf{x}_2\|_2^2\geq\|\mathbf{x}_1^H\mathbf{x}_2\|_2^2$ where equality is attained when $\mathbf{x}_1{=}c\mathbf{x}_2$, we get

$$\mathbf{det}^{-1}\left(\mathbf{I}_2+\frac{P}{16}\mathbf{M}\right)\leq\frac{1}{1+\frac{P}{8}\|\mathbf{\Delta}\|_2^2}, \quad (20)$$

where $\mathbf{\Delta}_1{=}\mathbf{\Delta}_1{=}\mathbf{\Delta}$. To achieve a diversity order 1, we require that $\mathbf{s}_m{\neq}\mathbf{s}_m'$, $m{=}1{,}2$. Defining $d_{\min}$ as the minimum distance of the constellation, we obtain $\mathrm{PEP}{\leq}\frac{8}{Pd_{\min}^2}$, which incurs a 3-dB loss at each relay compared with a single input single output (SISO) system. Applying union bound, we obtain the average error probability at relay $i$ as

$$P_e^r\leq M^{2T}\frac{8}{Pd_{\min}^2}, \quad (21)$$

where $M_1{=}M_2{=}M$ is the constellation size. Note that (21) is also a loose upperbound on the probability that $\mathrm{mod}(\hat{\mathbf{v}}_{1,i}{+}\hat{\mathbf{v}}_{2,i},M){\neq}\mathrm{mod}(\mathbf{v}_1{+}\mathbf{v}_2,M)$ at relay $i$. If $k$ relays satisfies $\mathrm{mod}(\hat{\mathbf{v}}_{1,i}{+}\hat{\mathbf{v}}_{2,i},M){=}\mathrm{mod}(\mathbf{v}_1{+}\mathbf{v}_2,M)$, by following the approach in [15], we can bound the PEP of $\mathbf{s}_1$ as

$$\mathrm{PEP}_k\leq\det{}^{-1}\left(\mathbf{I}_k+\frac{P_3}{2N}\mathbf{M}\right), \quad (22)$$

where $\mathbf{M}{=}\mathbf{C}^H\mathbf{C}$, $\mathbf{C}{=}[\mathbf{A}_1^r\mathbf{\Delta},\ldots,\mathbf{A}_k^r\mathbf{\Delta}]$, and $\mathbf{\Delta}{=}\mathcal{A}(\mathrm{mod}(\mathbf{v}_1{+}\mathbf{v}_2,M)){-}\mathcal{A}(\mathrm{mod}(\hat{\mathbf{v}}_1{+}\mathbf{v}_2,M))$. If orthogonal matrices are used, by using union bound, the overall error probability can be bounded as

$$\begin{aligned}P_e&\leq\sum_{k=0}^{N}\binom{N}{k}P_{e,k}(1-P_e^r)^k(P_e^r)^{N-k}\leq\sum_{k=0}^{N}\binom{N}{k}P_{e,k}(P_e^r)^{N-k}\\&=M^T\left(4N+8M^{2T}\right)^N(Pd_{\min}^2)^{-N}.\end{aligned} \quad (23)$$

Therefore, the diversity order of PDF II is $N$. □

The analysis of SNR gain can be refined with a more careful analysis that tightens the probability bound in (21). The simulation results in Section VI show that PDF II has a significant performance gain over other protocols.

### *C. MAC Gain*

Consider a two-way relay network with random access rather than centralized time division multiplexing. Two protocols are compared. In the first one, the two terminals cannot transmit simultaneously and the relays cannot combine received packets, which is similar to the 4 time slots protocols. The second one allows simultaneous transmissions by the two terminals and allows packet combination at the relays, which are mechanisms used in the 2 time slots protocols. We compare here the uplink to the relays. If we apply slotted ALOHA and assume that each terminal transmits with probability $p$ in a given time slot, the probability that one terminal transmits signal successfully to the relay nodes is $2p(1{-}p)$ in the first protocol, which is maximized when $p{=}1/2$. Hence, on average $1/2$ packets per time slot can be transmitted, while 2 packets per time slot can be transmitted by using the second protocol. Thus, 2 time slots protocols attain a 400% saving in uplink transmissions at the MAC layer.

## VI. Simulation Results

We consider the average BER for two-relay networks averaged over the fading gains. If the constellation size used in 2 time slots protocols is $2^B$, the constellation size used in 4 time slots protocols is $2^{2B}$ and $2^{1.5B}$ in 3 time slots protocols, to equalize the data rate in all protocols. The channel coefficients $f_i{\sim}\mathcal{CN}(0{,}1)$ and $g_i{\sim}\mathcal{CN}(0{,}1)$, $\forall i$ are used. Unless otherwise mentioned, random unitary matrices are used for LD code. We choose $P_1{=}P_2{=}P_3{=}0.5$.

Fig. 1 compares the 2 time slots protocols using BPSK with 4 time slots protocols using 4-QAM. The protocol P using the orthogonal LD code (16) is denoted as P_o. PDF I and PDF II with correct signals at the relays are used as benchmarks and denoted as PDF ideal. 4-AF has only a 0.3-dB loss over 2-AF at BER$=2\times10^{-4}$. The loss predicted by Theorem 3 is 3-dB, which may be due to the use of the union bound and other approximations in Theorem 3. We show here the performance of conventional 4-DF, where each relay decodes and transmits the decoded signals. This outperforms 4-AF, 2-AF, and PDF I in low SNR but cannot achieve the diversity order 2. To achieve the diversity order 2, 4-DF needs to be modified such that the relays transmit the decoded signal times the channel coefficient between the terminal and the relay as in PDF I, where the error made at the relay seen by the destination is bounded in contrast to the conventional 4-DF.

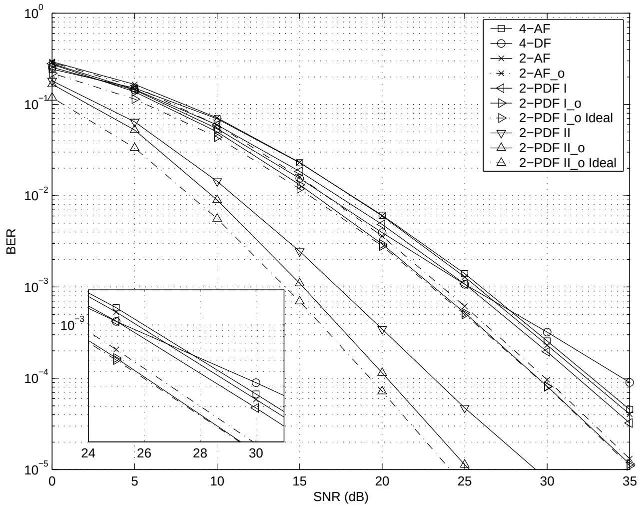


Fig. 1. Performance comparison of 2 time slots protocols and 4 time slots protocols in a network with two relays.

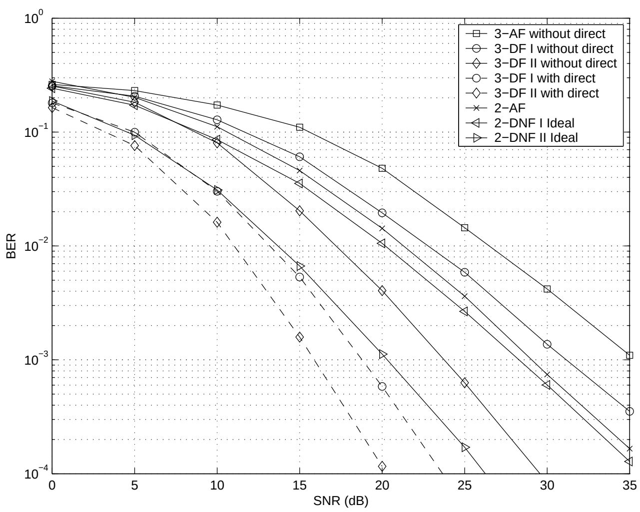


Fig. 2. Performance comparison of 2 time slots protocols and 3 time slots protocols in a network with two relays.

When the orthogonal code (16) is used, both 2-AF_o and PDF I_o can attain an additional 2.5-dB gain over those using random unitary matrices at BER=$2\times10^{-4}$. In high SNR, PDF I_o achieves almost the same performance as PDF I_o Ideal, i.e. effective denoising occurs. PDF II has a 8.3-dB gain over PDF II at BER=$2\times10^{-4}$. When orthogonal code is used, another 3-dB gain can be realized. This shows that PDF II is a promising candidate for two-way relay networks. The gain attained by PDF II is from information embedding by using modular operation, together with the use of CRC. We also find that the 2 time slots protocols perform better than their 4 time slots counterparts in high SNR.

Fig. 2 compares the performance of 2 time slots protocols using 4-QAM, and 3 time slots protocols using 8-QAM. Without direct transmission, due to noise amplification, 3-AF performs worse than the other protocols. DF I has a 2.5-dB loss over PDF I and DF II has a 2.6-dB loss over PDF II at BER=$2\times10^{-3}$. With the presence of the direct transmission with direct fading coefficient $d=1$, we see that the diversity order of 3 time slots protocols increases by one. DF I and DF II outperform both PDF I and PDF II in high SNR. Compared with Fig. 1, PDF I attains a larger gain over 2-AF as the constellation size increases, which shows that when the constellation size increases while keeping the rate constant, 2 time slots protocols perform better.

Simulation results for asymmetric channels and peak power constraints are available in [14]. 2 time slots protocols perform better than other protocols under the peak power constraints, at the expense of increasing the average transmit power. We also find that as the number of relays increases PDF I achieves a higher gain over 2-AF.

## VII. Conclusion

We have studied two-way wireless relaying protocols using LD space-time coding, that operate over 2, 3 or 4 time slots. The 2 time slots protocols allow simultaneous transmissions by the two terminals, giving MAC gain. The 2 and 3 time slots protocols allow the relays to combine packets. We proposed two new 2 time slots protocols, PDF I and PDF II. We compared the protocols analytically and by simulation. Among our results we showed that the proposed AF protocols achieve the diversity order $\min\{N,T\}\left(1-\frac{\log\log P}{\log P}\right)$, while PDF II achieves diversity order $N$.

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