Memoryless Relay Strategies for Two-Way Relay Channels: Performance Analysis and Optimization

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***Abstract*— We consider relaying strategies for two-way relay channels, where two terminals transmits simultaneously to each other with the help of relays. A memoryless system is considered, where the signal transmitted by a relay depends only on its last received signal. For binary antipodal signaling, we analyze and optimize the performance of existing amplify and forward (AF) and absolute (abs) decode and forward (ADF) for two-way AWGN relay channels. A new abs-based AF (AAF) scheme is proposed, which has better performance than AF. In low SNR, AAF performs even better than ADF. Furthermore, a novel estimate and forward (EF) strategy is proposed which performs better than ADF. More importantly, we optimize the relay strategy within the class of abs-based strategies via functional analysis, which minimizes the average probability of error over all possible relay functions. The optimized function is shown to be a Lambert's W function parameterized on the noise power and the transmission energy. The optimized function behaves like AAF in low SNR and like ADF in high SNR, resp., where EF behaves like the optimized function over the whole SNR range.**

I. Introduction

Two-way communication is another popular form of transmission in which both parties simultaneously transmit information to each other. The two-way channel was first considered by Shannon [1], who derived inner and outer bounds on the capacity region. Recently, the two-way relay channel (TWRC) has drawn renewed interest from both academic and industrial communities [2]–[6] due to its potential application to cellular networks and peer-to-peer networks. AF and DF protocols for one-way relay channels are extended to the half-duplex Gaussian TWRC in [2] and the general full-duplex discrete TWRC in [3]. In [4], network coding is used to increase the sum-rate of two users. With network coding, each node in a network is allowed to perform algebraic operations on received packets instead of only forwarding or replicating received packets. Most of these works [2]–[4] focus on capacity bounds for strategies similar to those for one-way relay channels [7].

Physical layer network coding (PNC) is considered in [5] for two-way AWGN relay channels. Two partial decode and forward (PDF) schemes are proposed in [6] for distributed space time coding to achieve diversity in two-way relay fading channels with multiple relays.

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In this paper, we focus on memoryless relay operation, analyzing the bit error probability at each receiver without considering channel coding. We first analyze the performance of existing amplify and forward (AF) and absolute-based (abs-based) decode and forward (ADF) schemes for two-way AWGN relay channels using binary antipodal signaling. Noting the performance limitations of these existing schemes, we develop a number of new schemes. We classify the existing and new schemes into two categories: abs-based schemes, where the relay first takes the absolute value of the received signal, and non-abs-based schemes. Their relative performance depends on the characteristics of the uplink and downlink channels; in general, abs-based schemes perform better when the channel from the relay is noisy. The schemes we propose include an abs AF (AAF) scheme and a novel estimate and forward (EF) strategy by extending the EF in [8] for the one-way relay channel to TWRCs, both of which can substantially outperform existing schemes. Besides characterizing the performance of different schemes, we also optimize the relay strategy within the class of abs-based strategies via functional analysis, which minimizes the average probability of error over all possible relay functions. The optimized function is show to be a Lambert's W function parameterized on the noise power and the transmission energy. Interestingly, the optimized function looks like the AAF scheme in low SNR and looks like the ADF scheme in high SNR. EF has the same shape as the optimized function in all SNRs. In our extended report [9] and in [10], we generalize these results to higher order constellations, two-way channels with multiple relays, and fading channels. The results in this paper also show insight on how to design close to optimized strategies under various communication scenarios.

Notations: $\mathcal{N}(x,\sigma^2)$ denotes the Gaussian distribution $\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)$. $Q(\cdot)$ denotes the Q-function.

II. System Model

We consider a memoryless two-way relay AWGN channel with two terminals and one relay, where the two terminals have data to be transmitted to each other. The system is memoryless, which means that the signal transmitted by a relay depends only on its last received signal and no channel coding is used. The discrete-time model for the memoryless two-way relay channel can be written as

$$Y_i = f(X_1 + X_2 + N) + Z_i, \quad i = 1, 2, \tag{1}$$

where X_i and Y_i are the transmitted symbol and received symbol at terminal i, and $Z_i \sim \mathcal{N}(0,\sigma_s^2)$ is the additive white Gaussian noise (AWGN) at terminal i, $i=1,2$, and $N \sim \mathcal{N}(0,\sigma_r^2)$ is the AWGN at the relay. For simplicity, we assume that the noises at the two terminals have the same variance. To accommodate for energy limitations, we impose on X_i an average power constraint: $E\{|X_i|^2\} \leq P_s$, $i=1,2$, as well as on the output of the relay $E\{|f(X_1+X_2+N)|^2\} \leq P_r$. For notational convenience we define the random variable $U=X_1+X_2+N$. Note that (1) both applies to a half duplex system with two time slots, where the transmission from one terminal to the other takes place in a multiple-access and a broadcast time slot, or a full duplex system. Throughout this paper, we assume that there is no direct communication between the two terminals and the system is perfectly synchronized, which may be possibly attained via pilot symbols.

In this paper, binary phase shift keying (BPSK) is assumed, i.e., the random variable X_i is taken to be

$$X_i = \begin{cases} \sqrt{P_s}, & \text{with probability 1/2,} \\ -\sqrt{P_s}, & \text{with probability 1/2.} \end{cases} \tag{2}$$

Multiple relays, fading channels, and higher order constellations are considered in [9], [10]. We focus on symbol error probability as a performance metric: each terminal is assumed to perform a hypothesis test to decide which symbol was transmitted by the other terminal; we do not consider the effect of any end to end channel coding that may be applied.

III. Amplify and Forward

In this subsection, we analyze the performance of amplify and forward [2], where, more precisely, a linear function $f(\cdot)$ is used at the relay. To satisfy the average power constraint at the relay, $f(\cdot)$ is equal to

$$f(u) = \sqrt{\frac{P_r}{2P_s+\sigma_r^2}}\, u, \tag{3}$$

which yields an output at terminal i

$$Y_i = \sqrt{\frac{P_r}{2P_s+\sigma_r^2}}(X_1+X_2) + \left(\sqrt{\frac{P_r}{2P_s+\sigma_r^2}} N + Z_i\right), i=1,2. \tag{4}$$

Therefore, given x_1 and x_2 were transmitted, the conditional probability density function of the output Y_i is

$$p_{Y_i|X_1,X_2}(y_i|x_1,x_2) \sim \mathcal{N}\left(\sqrt{\frac{P_r}{2P_s+\sigma_r^2}}(x_1+x_2), \frac{P_r\sigma_r^2}{2P_s+\sigma_r^2}+\sigma_s^2\right). \tag{5}$$

As terminal i already knows x_i, the optimal decision rule is to decide on $\sqrt{P_s}$ if $u_i = y_i - \sqrt{\frac{P_r}{2P_s+\sigma_r^2}} x_i \geq 0$ and on $-\sqrt{P_s}$ otherwise.

Therefore, the average probability of error at each terminal of this scheme is

$$\begin{aligned} P_e &= \int_{-\infty}^{0} \mathcal{N}\left(u - \sqrt{\frac{P_r P_s}{2P_s+\sigma_r^2}}, \frac{P_r\sigma_r^2}{2P_s+\sigma_r^2}+\sigma_s^2\right) du \\ &= Q\left(\sqrt{\frac{P_r P_s}{P_r\sigma_r^2+2P_s\sigma_s^2+\sigma_r^2\sigma_s^2}}\right). \end{aligned} \tag{6}$$

In practice, there may be a total power constraint on the two-way relay channel, i.e., $2P_s+P_r=P$, where P is the total power. Given P, we can optimize the power allocation between terminals and the relay to achieve the minimum average error probability. Assuming that $\sigma_r^2=\sigma_s^2$, it is easy to see that the average probability of error (6) is minimized when

$$P_s = \frac{P}{4}, \quad P_r = \frac{P}{2}. \tag{7}$$

IV. Absolute value-based relay strategies

In the absence of noise at the relay, the relay gets X_1+X_2, the possible values of which are $2\sqrt{P_s}, 0, -2\sqrt{P_s}$. If the terminals do not distinguish $2\sqrt{P_s}$ from $-2\sqrt{P_s}$, they can still decode the other terminal's signal correctly by using their own signal as side information. With this intuition, we propose a class of strategies in which the relay transmits a function of the absolute value of its received signal. At each terminal, if the received signal exceeds a threshold value v, it decides that ($X_1=\sqrt{P_s}$, $X_2=\sqrt{P_s}$) or ($X_1=-\sqrt{P_s}$, $X_2=-\sqrt{P_s}$), otherwise it decides ($X_1=\sqrt{P_s}$, $X_2=-\sqrt{P_s}$) or ($X_1=\sqrt{P_s}$, $X_2=-\sqrt{P_s}$). Each terminal uses knowledge of its own signal to decode the other terminal's signal. In the presence of relay and terminal noise the average error probability at each terminal can be written as

$$\begin{aligned} P_e =& \frac{1}{2}\int_{-\infty}^{+\infty} \mathcal{N}\left(u,\sigma_r^2\right)\left[\int_{v}^{+\infty} \mathcal{N}\left(y-f(u),\sigma_s^2\right)dy\right]du \\ &+\frac{1}{2}\int_{-\infty}^{+\infty} \mathcal{N}\left(u-2\sqrt{P_s},\sigma_r^2\right)\left[\int_{-\infty}^{v} \mathcal{N}\left(y-f(u),\sigma_s^2\right)dy\right]du. \end{aligned} \tag{8}$$

A. Abs Amplify and Forward

Within this class of strategies we propose a new scheme, abs amplify and forward, where the relay takes the absolute value of the received signal and transmits a scaled and shifted version:

$$f(u) = \beta(|u|-C), \tag{9}$$

where $|u|$ denotes the absolute value of u, C is a positive constant and β is a scaling coefficient to maintain the average power constraint at the relay. From (8), we have

$$\begin{aligned} P_e =& \frac{1}{2} + \frac{1}{2}\int_{0}^{+\infty} \left(\mathcal{N}\left(u-2\sqrt{P_s},\sigma_r^2\right) + \mathcal{N}\left(u+2\sqrt{P_s},\sigma_r^2\right)\right. \\ &\left. -2\mathcal{N}\left(u,\sigma_r^2\right)\right)\left[\int_{-\infty}^{v} \mathcal{N}\left(y-\beta(u-C),\sigma_s^2\right)dy\right]du\Big). \end{aligned} \tag{10}$$

Differentiating (10) with respect to v and setting the resulting equation to zero, we obtain

$$\begin{aligned} \int_{0}^{+\infty} &\left(\mathcal{N}\left(u-2\sqrt{P_s},\sigma_r^2\right) + \mathcal{N}\left(u+2\sqrt{P_s},\sigma_r^2\right)\right. \\ &\left. -2\mathcal{N}\left(u,\sigma_r^2\right)\right)\mathcal{N}\left(v-\beta(u-C),\sigma_s^2\right)du = 0. \end{aligned} \tag{11}$$

We can minimize (10) with respect to both v and C. However, the optimal C depends on both the signal and noise powers in a complicated way, which is hard to implement in practice. Given C, the optimal detection threshold can be obtained by solving (11). In the following, we only consider two intuitively reasonable choices of C: $C=\sqrt{P_s}$ and $C=\sqrt{P_s}+\sigma_r/\sqrt{2}$. In our simulations, we have seen the optimal threshold approaches to zero as SNR increases. In practice, if SNR is not accurately known, since the optimal threshold is very close to zero, we simply set $v=0$.

B. Abs Decode and Forward

The class of abs-based strategies includes an existing scheme for the two-way AWGN relay channel called physical network coding [5]. In ADF, the relay first performs hard decisions, based on the absolute value of the received signal, to decide whether $2\sqrt{P_s}$, 0, or $-2\sqrt{P_s}$ is received. The relay does not actually decode x_1 and x_2, but only x_1+x_2. To satisfy the relay's average power constraint, $\sqrt{P_r}$ and $-\sqrt{P_r}$ are transmitted, i.e.,

$$f(u)=\begin{cases} \sqrt{P_r}, & \text{if } |u|\geq w,\\ -\sqrt{P_r}, & \text{otherwise,}\end{cases} \tag{12}$$

where w is a threshold which will be determined below.

The average error probability at each terminal (8) can be written as

$$\begin{aligned}P_e=&\frac{1}{2}+\frac{1}{2}\int_0^w\Big(\mathcal{N}\Big(u-2\sqrt{P_s},\sigma_r^2\Big)+\mathcal{N}\Big(u+2\sqrt{P_s},\sigma_r^2\Big)\\ &-2\mathcal{N}\big(u,\sigma_r^2\big)\Big)du\int_{-\infty}^{v}\Big(\mathcal{N}\Big(y+\sqrt{P_r},\sigma_s^2\Big)-\mathcal{N}\Big(y-\sqrt{P_r},\sigma_s^2\Big)\Big)dy\end{aligned} \tag{13}$$

Expression (13) has the nice property that the optimization of w and v is separated. Minimizing (13) over w and v, we obtain the optimal w as

$$w=\sqrt{P_s}\left(1+\frac{\sigma_r^2}{2P_s}\log\left(1+\sqrt{1-e^{-4P_s/\sigma_r^2}}\right)\right), \tag{14}$$

and the optimal v as $v=0$. When $\sigma_r^2\to 0$, the optimal w converges to $\sqrt{P_s}$. In practice, we can simply choose $w=\sqrt{P_s}$.

If we are given a total power P, we can optimize the power allocation between terminals and the relay to achieve the minimum average error probability. P_e in (13) can be simplified as

$$\begin{aligned}P_e=&\frac{1}{2}+\frac{1}{2}\left(1-2Q\left(\frac{\sqrt{P_r}}{\sigma_s}\right)\right)\\ &\times\left(Q\left(\frac{2\sqrt{P_s}-w}{\sigma_r}\right)+2Q\left(\frac{w}{\sigma_r}\right)-Q\left(\frac{2\sqrt{P_s}+w}{\sigma_r}\right)-1\right).\end{aligned} \tag{15}$$

We consider high SNR case, i.e., $P\gg\sigma_s^2,\sigma_r^2$, and assume $\sigma_s^2=\sigma_r^2=\sigma^2$. In this case, the optimal w in (14) can be approximated as $w=\sqrt{P_s}$. By applying Chernoff bound-type arguments and ignoring high order terms, we can obtain

$$P_e\lesssim\frac{3}{2}e^{-\frac{P_s}{2\sigma^2}}+e^{-\frac{P_r}{2\sigma^2}}. \tag{16}$$

Minimizing the right hand side of (16) and using the high SNR assumption, we can obtain $P_s=P_r=\frac{P}{3}$. This is different from the optimal power allocation for AF in (7) because ADF saves half of power to transmit redundant information as compared with AF.

C. Abs Estimate and Forward

In this subsection, we describe a strategy that lies between the simple strategies discussed so far and the optimal strategy. Instead of minimizing the error probability directly, we consider minimizing the mean squared error (MSE) of estimating $|x_1+x_2|$ at the relay.

We first consider the function $g(u)$ such that

$$g(u)=\underset{g'(u)}{\operatorname{argmin}}E\Big\{\big||x_1+x_2|-g'(u)\big|^2\Big|u\Big\}. \tag{17}$$

The objective function in (17) can be written as

$$\begin{aligned}&E\Big\{\big||x_1+x_2|-g'(u)\big|^2\Big|u\Big\}\\ =&\sum_{x_1,x_2\in\{-\sqrt{P_s},\sqrt{P_s}\}}Pr(x_1+x_2|u)\big||x_1+x_2|-g'(u)\big|^2\\ =&\sum_{x_1,x_2\in\{-\sqrt{P_s},\sqrt{P_s}\}}\frac{Pr(u|x_1+x_2)Pr(x_1+x_2)}{Pr(u)}\big||x_1+x_2|-g'(u)\big|^2.\end{aligned} \tag{18}$$

Note that $Pr(u)$ is a common factor. Therefore, minimizing (18) is equivalent to minimizing

$$\begin{aligned}&\sum_{x_1,x_2\in\{-\sqrt{P_s},\sqrt{P_s}\}}Pr(u|x_1+x_2)Pr(x_1+x_2)\big||x_1+x_2|-g'(u)\big|^2\\ =&\frac{1}{2}\mathcal{N}(u,\sigma_r^2)\big|g'(u)\big|^2+\frac{1}{4}\mathcal{N}(u-2\sqrt{P_s},\sigma_r^2)\Big|2\sqrt{P_s}-g'(u)\Big|^2\\ &+\frac{1}{4}\mathcal{N}(u+2\sqrt{P_s},\sigma_r^2)\Big|2\sqrt{P_s}-g'(u)\Big|^2.\end{aligned} \tag{19}$$

Minimizing (19) over $g'(u)$ we obtain

$$g(u)=\frac{2\sqrt{P_s}\cosh\frac{2\sqrt{P_s}u}{\sigma_r^2}}{e^{2P_s/\sigma_r^2}+\cosh\frac{2\sqrt{P_s}u}{\sigma_r^2}}. \tag{20}$$

$f(u)$ is then a scaled version of $g(u)-C$, where C is a constant as in AAF, i.e.,

$$f(u)=\begin{cases}\beta(g(u)-C), & \text{if } u\geq 0,\\ f(-u), & \text{otherwise,}\end{cases} \tag{21}$$

where $\beta\geq 0$ is a scaling factor to keep the average power constraint $E\{f^2(u)\}=P_r$. At the two terminals, there also exists an optimal decision threshold v. We can optimize v using the same way in AAF.

V. Optimized Absolute Value-Based Strategy

In this section, we optimize the average probability of error over even functions $f(\cdot)$ at the relay. Our approach generalizes the result from [11] for the one-way case. The optimized relay function for the non-abs-based operation at the relay (i.e., assuming that $f(u)$ is an odd function) is considered in [6]. We have found that for low enough SNR or very asymmetric channels, non-abs-based strategies perform better than abs-based strategies. Here, we focus on abs based schemes.

From (8), the average probability of error can be obtained as

$$\begin{aligned}P_e(f)=&\frac{1}{2}\int_0^{+\infty}\underbrace{\Big(\mathcal{N}\Big(u+2\sqrt{P_s},\sigma_r^2\Big)+\mathcal{N}\Big(u-2\sqrt{P_s},\sigma_r^2\Big)-2\mathcal{N}\big(u,\sigma_r^2\big)\Big)}_{B(u)}\\ &\times\underbrace{\left[\int_{-\infty}^{v}\mathcal{N}\big(y-f(u),\sigma_s^2\big)dy\right]}_{A(f)}du+\frac{1}{2},\end{aligned} \tag{22}$$

where the second equality holds since $B(u)$ is an even function in u. Let

$$D(u)=\mathcal{N}\left(u+2\sqrt{P_s},\sigma_r^2\right)+\mathcal{N}\left(u-2\sqrt{P_s},\sigma_r^2\right)+2\mathcal{N}\left(u,\sigma_r^2\right). \quad (23)$$

Our optimization problem is:

$$\begin{aligned} \min_{f,v} \quad & G(f)=\int_0^{+\infty}B(u)A(f)du, \\ \text{subject to} \quad & \frac{1}{2}\int_0^{+\infty}D(u)f^2(u)du\leq P_r. \end{aligned} \quad (24)$$

We consider the Lagrangian

$$\phi(f)=G(f)+\frac{\lambda}{2}\left(\int_0^{+\infty}D(u)f^2(u)du-2P_r\right), \quad (25)$$

where $\lambda\geq 0$ is the Lagrange multiplier of the average power constraint. Differentiating $\phi(f)$ with respect to $f(u)$ for each u and setting the result to zero, we obtain

$$B(u)\mathcal{N}\left(f(u)-v,\sigma_s^2\right)=\lambda f(u)D(u), \quad (26)$$

or equivalently

$$\frac{\mathcal{N}\left(f(u)-v,\sigma_s^2\right)}{f(u)}=\lambda\frac{D(u)}{B(u)}. \quad (27)$$

Since $\lambda>0, D(u)>0$, and $B(u)\geq 0$, if $|u|\geq w$; otherwise $B(u)<0$, we have

$$\begin{cases} f(u)\geq 0, & \text{if } |u|\geq w, \\ f(u)<0, & \text{otherwise}, \end{cases} \quad (28)$$

where w is the relay hard decision threshold defined in (14).

Lemma 1: For $f(u)$ satisfying

$$\begin{cases} f(u)\geq v, & \text{if } |u|\geq w, \\ f(u)<v, & \text{otherwise}, \end{cases} \quad (29)$$

$P_e(f)$ in (22) is a strictly convex function in f (when considering functions that differ on a set of non-zero measure).

Proof: Let f and g be two functions satisfying (29), and let $\lambda\in[0,1]$ and $\gamma=1-\lambda$. Clearly, $\lambda f+\gamma g$ also satisfies (29).

Note that

$$\frac{\partial^2 A(f)}{\partial f^2}=\frac{1}{2\sigma_s^2}\left(f(u)-v\right)\mathcal{N}\left(v-f(u),\sigma_s^2\right), \quad (30)$$

which is nonnegative when $f(u)\geq v$ and is negative otherwise. Since $B(u)\frac{\partial^2 A(f)}{\partial f^2}$ is nonnegative for $u\geq w$ and positive otherwise, we have

$$\begin{aligned} P_e(\lambda f+\gamma g)&=\frac{1}{2}+\frac{1}{2}\int_0^{+\infty}B(u)A(\lambda f+\gamma g)du, \\ &<\lambda P_e(f)+\gamma P_e(g). \end{aligned} \quad (31)$$

□

If $v=0$, then Eq. (27) can be further simplified to be

$$\frac{e^{-\left(f(u)/\sqrt{2\sigma_s^2}\right)^2}}{f(u)/\sqrt{2\sigma_s^2}}=\lambda 2\sqrt{\pi}\sigma_s^2\frac{\cosh\frac{2\sqrt{P_s}u}{\sigma_r^2}+e^{2P_s/\sigma_r^2}}{\cosh\frac{2\sqrt{P_s}u}{\sigma_r^2}-e^{2P_s/\sigma_r^2}}. \quad (32)$$

which can be solved to obtain the following expression for $f(u)$:

$$f(u)=\begin{cases} \sqrt{\sigma_s^2 W\left(\frac{1}{2\pi\lambda^2\sigma_s^4}\left(\frac{\cosh\frac{2\sqrt{P_s}u}{\sigma_r^2}-e^{2P_s/\sigma_r^2}}{\cosh\frac{2\sqrt{P_s}u}{\sigma_r^2}+e^{2P_s/\sigma_r^2}}\right)^2\right)}, & \text{if } u\geq w, \\ -\sqrt{\sigma_s^2 W\left(\frac{1}{2\pi\lambda^2\sigma_s^4}\left(\frac{\cosh\frac{2\sqrt{P_s}u}{\sigma_r^2}-e^{2P_s/\sigma_r^2}}{\cosh\frac{2\sqrt{P_s}u}{\sigma_r^2}+e^{2P_s/\sigma_r^2}}\right)^2\right)}, & \text{if } w>u\geq 0, \\ f(-u), & \text{if } u<0, \end{cases} \quad (33)$$

where $W(\cdot)$ denotes the Lambert W function, defined by $W(x)e^{W(x)}=x$, and λ is such that the power constraint is satisfied with equality.

Note that $f(u)$ in (33) is derived from the Lagrange dual without any assumption on the convexity of the problem, which may not be a true optimal solution. However, (33) indeed satisfies (28), which means that it is optimal within the class of functions satisfying (28). By Lemma 1 and x^2 being convex, the set of functions satisfying (28) and the power constraint of (24) is a convex function set. The optimization under the constraint (28) is thus convex and there is no duality gap. Therefore, (33) is the optimal solution when $v=0$. In high SNR, since

$$\begin{aligned} &\lim_{\sigma_r^2\to 0}\frac{\mathcal{N}\left(u+2\sqrt{P_s},\sigma_r^2\right)+\mathcal{N}\left(u-2\sqrt{P_s},\sigma_r^2\right)-2\mathcal{N}\left(u,\sigma_r^2\right)}{\mathcal{N}\left(u+2\sqrt{P_s},\sigma_r^2\right)+\mathcal{N}\left(u-2\sqrt{P_s},\sigma_r^2\right)+2\mathcal{N}\left(u,\sigma_r^2\right)} \\ &=\begin{cases} 1, & \text{if } |u|>w, \\ -1, & \text{if } w>|u|, \end{cases} \end{aligned} \quad (34)$$

from (27) we obtain

$$f(u)=\begin{cases} C_1, & \text{if } |u|>w, \\ -C_2, & \text{if } w>|u|, \end{cases} \quad (35)$$

where $C_1,C_2>0$ are constants. Substituting (35) back into (24), we find that it becomes an optimal ADF problem as in (15), which attains its minimum at $v=0$, $C_1=C_2=\sqrt{P_r}$ when $\sigma_r^2\to 0$. Note that it is easy to show that the second term in (25) dominates $\phi(f)$ in high SNR. Therefore, there is no duality gap in high SNR and the optimal solution converges to (35) or the ADF strategy. In general, the optimal v varies with SNR.

When $\sigma_r^2\to+\infty$ and $\sigma_s^2\to+\infty$, from (27), we find that

$$\begin{aligned} &\lim_{\sigma_r^2\to+\infty}\frac{\mathcal{N}\left(u+2\sqrt{P_s},\sigma_r^2\right)+\mathcal{N}\left(u-2\sqrt{P_s},\sigma_r^2\right)-2\mathcal{N}\left(u,\sigma_r^2\right)}{\mathcal{N}\left(u+2\sqrt{P_s},\sigma_r^2\right)+\mathcal{N}\left(u-2\sqrt{P_s},\sigma_r^2\right)+2\mathcal{N}\left(u,\sigma_r^2\right)} \\ &=\frac{P_s}{\sigma_r^2}\left(\frac{u^2}{\sigma_r^2}-1\right), \text{ if } |u|<\sigma_r. \end{aligned} \quad (36)$$

Therefore, in low SNR, the dual optimal function curve is like $C\left(\frac{u^2}{\sigma_r^2}-1\right)$ when $|u|<\sigma_r$ where C is a positive constant.

Note that $f(u)$ in (33) is optimal when we apply a threshold detector with $v=0$ at the two terminals. It may be possible that a function outside the convex function set satisfying (28) or a nonlinear detector may attain even a smaller average probability of error though we conjecture that such a detector does not exist.

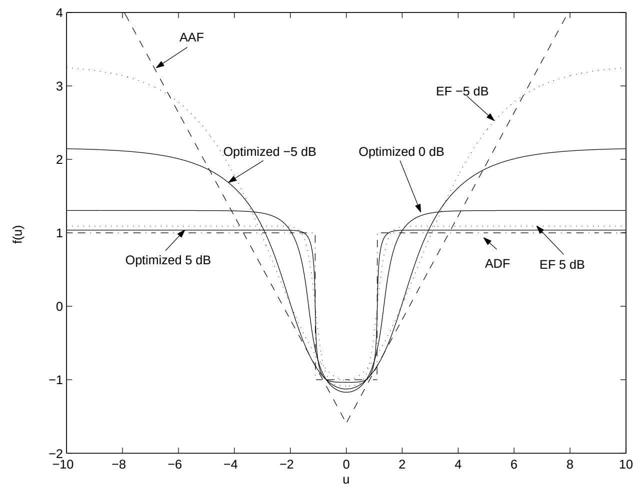

Fig. 1. Comparison of function $f(u)$ in different schemes with $\sigma_r^2 = \sigma_s^2$ and $P_r = P_s = 1$.

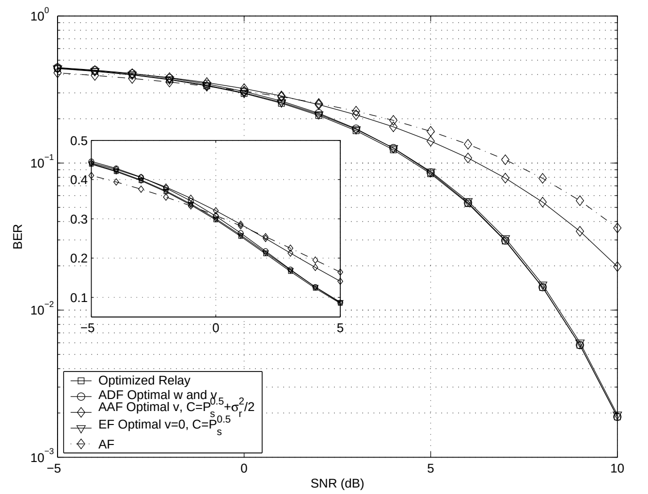

Fig. 2. Performance comparison of different schemes when $\sigma_r^2 = \sigma_s^2$ and $P_r = P_s = 1$.

VI. Simulation Results

In this section, we compare the performance of different strategies with $\sigma_r^2 = \sigma_s^2$ and $P_r = P_s = 1$ in all cases. Two-way relay AWGN channel with a single relay is considered. BPSK is used.

It is interesting to compare the relay function $f(u)$ of different relay strategies in Fig. 1. In AAF, we choose $C = \sqrt{P_s} + \sigma_r/\sqrt{2}$. Unlike ADF with a hard limiter, the optimized relay adapts its transmit power according to the signal strength it receives which is the benefit of the average power constraint. If only peak power constraint is imposed at the relay, the optimal ADF achieves the minimum average probability of error. From Fig. 1, we can also see that when SNR is small, the optimized relay function looks closer like the AAF of a "V" shape. As SNR increases, the optimized relay function looks closer like the ADF. This suggests that ADF performs well in high SNR while AAF is effective in low SNR. Interestingly, the relay function of EF has almost the same shape as the optimized relay function in all SNRs, which agrees with the simulation results in Fig. 2.

Fig. 2 compares the performance of different schemes. We can see that the performance difference between the optimized scheme and ADF is at the order of 0.01 in low SNR. When the SNR is greater than 5 dB, the two schemes perform almost identically. In low SNR, AAF also performs better than ADF. EF performs between the optimized scheme and ADF. These agree with the intuition obtained from Fig. 1. This suggests that for AWGN two-way relay channel with a single relay the suboptimal ADF with $w = 1$ and $v = 0$ seems to be a promising strategy for practical use due to its simplicity and close to optimal performance.

VII. Conclusion

We have analyzed and optimized AF and ADF relaying strategies for memoryless two-way relay channels with a binary antipodal input signal. A new AAF scheme was proposed, which attains better performance than AF. AAF performs even better than ADF in low SNR. The relay strategy was also optimized by minimizing the average probability of error over all possible relay functions. Furthermore, a novel estimate and forward strategy is proposed which performs better than ADF. We found that the optimized function looks like the AAF in low SNR, looks like the ADF in high SNR, and looks like EF in all SNRs. Interestingly, the optimized relay can be considered as waterfilling over the signal space rather than over spectral or time domain in traditional information theory. Although this work does not consider channel coding, the obtained expressions for the error probability allow a rough determination of the required rate for an end-to-end channel code. All these results can be also generalized to higher order constellations, the case with multiple relays, and channels with unequal SNRs [9], [10].

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