Erasure correction for nested receivers

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***Abstract*—We consider packet erasure or error correction coding for a nested receiver structure, where each receiver receives a subset of the packets received by the next receiver. This type of structure arises, for instance, with temporal demands, where each receiver corresponds to a deadline by which certain information must be decoded. By making a connection with our previous work on non-multicast network error correction, we find the capacity region for any given number of erasures or errors whose locations are *a priori* unknown, along with a capacity-achieving intra-session coding scheme.**

I. Introduction

We consider packet erasure/error correction coding with a nested receiver structure, where the set of packets received by each receiver i is a subset of that received by the next receiver $i+1$. A natural setting in which this type of structure arises is with temporal demands: each receiver corresponds to a particular deadline in the received packet stream by which a particular piece of information must be decoded, and has access to all earlier observations. The protocol can specify an arbitrary set of deadlines and demands.

We consider the problem of constructing codes that can correct any z packet erasures (or errors), without *a priori* knowledge of which packets will be erased (erroneous). By making a connection with our prior work on non-multicast network error correction, we characterize the capacity region of feasible demand vectors for any given nested structure (set of deadlines) and any z erasures (errors). In particular, we provide a capacity-achieving coding scheme where no coding occurs across information demanded by different receivers.

A. Background and related work

The network error correction problem, where transmissions on an unknown set of z links are arbitrarily corrupted, was introduced by Cai and Yeung [1], [2], [3] for single-source multicast. They characterized the capacity region and showed a connection between network error correction and network erasure correction by generalizing classical coding theory to the network setting. Network coding for multicast packet erasure correction was considered in [4], [5]. The problem of multicast non-coherent error correction, where the network topology and/or network code are not known *a priori*, was studied in [6], [7], [8].

For non-multicast networks, finding the capacity region of a general network even without errors is an open problem. The error-free capacity regions for some special cases, such as single-source two-sink networks [9], [10], [11] and single-source disjoint- or nested-demand networks [12] with multiple sinks, are given by the cutset bounds. On the other hand, examples of non-multicast networks whose error-free capacity regions are not given by cutset bounds appear in [13], [14].

For non-multicast error correction, in our prior work [15] we showed that unlike the error-free case cutset bounds are loose in general for single-source two-sink networks with errors, and we developed refined bounds for non-multicast networks. In this paper we build on some of the techniques developed in that work.

Martinian *et al.* [16], [17] constructed streaming codes that minimize the delay required to correct burst errors of given length.

II. Model and problem description

Consider a streaming system where at each time step one packet of unit size is transmitted, and the receiver needs to decode specific independent messages $\{M_1, M_2, \ldots, M_n\}$ at given time steps $\{m_1, m_2, \ldots, m_n\}$ respectively, under any z packet erasures. This can be viewed as a z-erasure correction problem on a *3-layer nested* network with one source and n sinks $\{t_1, t_2, \ldots, t_n\}$, constructed as follows and illustrated in Figure 1:

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- $\mathcal{I} = \{l_1, l_2, ..., l_{m_n}\}$ is the set of middle layer links.
- The source is connected to all the links in $\mathcal{I}$.
- Sink t_i is connected to links $l_1, \dots, l_{m_i}$.
- All links have unit capacity.
- At most z of the links in $\mathcal{I}$ can be erased/erroneous.

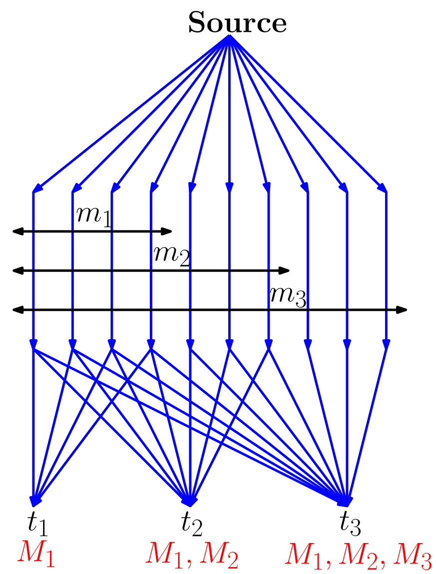

Fig. 1. 3-layer nested-network topology with three sinks.

III. Coding Scheme

A. Intra-Session Coding

An intra-session coding scheme is one in which no coding occurs across information demanded by different receivers. For a given intra-session coding scheme, let y_i^j denote the amount of information corresponding to message M_j transmitted on the link l_i. A rate vector $(u_1, u_2, \dots, u_n)$ is achievable under any z erasures by this intra-session coding scheme if the inequalities

$$\forall j : 1 \leq j \leq n \qquad u_j \leq \sum_{i \in P \cap \{1,\dots,m_i\}} y_i^j, \quad (1)$$

$$\forall i : 1 \leq i \leq m_n \qquad \sum_{j=1}^{n} y_i^j \leq 1, \quad (2)$$

are satisfied for every set $P \subseteq \mathcal{I}$ where $|P| \geq m_n - z$ (corresponding to unerased links). We assume that the packet size is large enough to accommodate an appropriate generic or random linear erasure code.

B. "As Uniform As Possible" Coding Scheme

We define an intra-session coding scheme which assigns rates for each successive receiver as uniformly as possible subject to capacity constraints imposed by assignments for previous receivers. For each receiver the process is similar to water-filling with constraints from previous receivers. For a given rate vector $(u_1, u_2, ..., u_n)$, we define a corresponding lower triangular $n \times n$ rate allocation matrix T, along with auxiliary variables $t_{i,j} \triangleq \sum_{k=j}^{i} T_{k,j}$, $d_{i,j} \triangleq \sum_{k=1}^{j}(m_k - m_{k-1})T_{i,k}$, and s_i, by the following algorithm:

Algorithm 1

$T_{1,1} = \frac{u_1}{m_1 - z}, t_{1,1} = \frac{u_1}{m_1 - z}, d_{1,1} = \frac{m_1 u_1}{m_1 - z}, s_1 = 0, m_0 = 0$
for $i = 2 \to n$ **do**
  {allocation for sink i on links $l_{m_{j-1}+1}, \dots, l_{m_j}$}
  $d_{i,0} = 0$
  $j = 1$
  while $1 - t_{i-1,j} < \frac{u_i - d_{i,j-1}}{m_i - m_{j-1} - z}$ **do**
    $T_{i,j} = 1 - t_{i-1,j}$
    $t_{i,j} = \sum_{k=j}^{i} T_{k,j}$
    $d_{i,j} = \sum_{k=1}^{j}(m_k - m_{k-1})T_{i,k}$
    $j \leftarrow j + 1$
    if $j > i$ or $u_i \leq d_{i,j}$ **then**
      return error {rate vector is unallocable}
    end if
  end while
  $s_i = j - 1$ {the uniform portion follows}
  while $j \leq i$ **do**
    $T_{i,j} = \frac{u_i - d_{i,s_i}}{m_i - m_{s_i} - z}$
    $t_{i,j} = \sum_{k=j}^{i} T_{k,j}$
    $d_{i,j} = \sum_{k=1}^{j}(m_k - m_{k-1})T_{i,k}$
    $j \leftarrow j + 1$
  end while
end for

Note that $T_{i,s_i} < T_{i,s_i+1} = T_{i,s_i+2} = ... = T_{i,i}$.

Definition 1. *A rate vector $(u_1, u_2, ..., u_n)$ is called allocable if Algorithm 1 does not return any error and the corresponding allocation matrix T is non-negative.*

Definition 2. *Given an allocable rate vector $(u_1, u_2, ..., u_n)$, the "as uniform as possible" intra-session coding scheme is defined by the allocation*

$$y_i^j = T_{j,k} \quad \forall i : m_{k-1} < i \leq m_k. \quad (3)$$

As an example, for $z = 2$ and the inputs

$$(m_1, m_2, m_3, m_4) = (10, 14, 18, 22),$$
$$(u_1, u_2, u_3, u_4) = (6, 3, 3, 4),$$

Algorithm 1 will produce:

$$T_{1,1} = t_{1,1} = \frac{u_1}{m_1 - z} = 0.75,$$
$$d_{1,1} = \frac{m_1 u_1}{m_1 - z} = 7.5, s_1 = 0,$$
$$T_{2,1} = \min(1 - t_{1,1}, \frac{u_2}{m_2 - z}) = 0.25,$$
$$t_{2,1} = T_{1,1} + T_{2,1} = 1, d_{2,1} = m_1 T_{2,1} = 2.5.$$

As $\frac{u_2}{m_2-z} \leq 1 - t_{1,1}$, we have:

$$s_2 = 1, T_{2,2} = \frac{u_2 - d_{2,s_2}}{m_2 - m_{s_2} - z} = 0.25,$$
$$t_{2,2} = 0.25, d_{2,2} = m_1 T_{2,1} + (m_2 - m_1) T_{2,2} = 3.5.$$

Since $1 - t_{2,1} = 0$, we get: $T_{3,1} = 0, t_{3,1} = 1, d_{3,1} = 0$
As $\frac{u_3 - d_{3,1}}{m_3 - m_1 - z} \leq 1 - t_{2,2}$, we have:

$$s_3 = 1, T_{3,2} = T_{3,3} = \frac{u_3 - d_{3,s_3}}{m_3 - m_{s_3} - z} = 0.5,$$
$$t_{3,2} = 0.75, t_{3,3} = 0.5, d_{3,2} = 2, d_{3,3} = 4.$$

Proceeding similarly, we obtain:

$$T = \begin{bmatrix} 0.75 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.25 & 0.5 & 0.5 \end{bmatrix}.$$

Lemma 1. *If $(u_1, u_2, ..., u_n)$ is allocable, then the corresponding allocation matrix T satisfies:*

$$T_{i,j} \leq T_{i,j+1} \quad \forall i, j : 1 \leq i \leq n, 1 \leq j < i$$

Proof: Let's prove the following statements simultaneously by induction:

$$T_{i,j} \leq T_{i,j+1} \quad \forall i, j : 1 \leq i \leq n, 1 \leq j < i,$$
$$t_{i,j} \geq t_{i,j+1} \quad \forall i, j : 1 \leq i \leq n, 1 \leq j < i.$$

The statements hold trivially for $(i, j) = (1, 1)$. Let the statements hold for all (i, j) before (k, l) in lexicographical order. For $(i, j) = (k, l)$:

Case(1): $l < s_k$:
By construction, we have:

$$T_{k,l} = 1 - t_{k-1,l}, T_{k,l+1} = 1 - t_{k-1,l+1},$$
$$t_{k,l} = t_{k,l+1} = 1.$$

Clearly, $t_{k,l} \leq t_{k,l+1}$. By induction hypothesis, we have $t_{k-1,l} \geq t_{k-1,l+1}$. Hence

$$T_{k,l} = 1 - t_{k-1,l} \leq 1 - t_{k-1,l+1} = T_{k,l+1}.$$

Case(2): $l = s_k$:
By construction, we have:

$$T_{k,l} = 1 - t_{k-1,l} < \frac{u_k - d_{k,l-1}}{m_k - m_{l-1} - z}, \quad (4)$$

$$T_{k,l+1} = \frac{u_k - d_{k,l}}{m_k - m_l - z} \leq 1 - t_{k-1,l+1}. \quad (5)$$

Hence,

$$t_{k,l} = t_{k-1,l} + T_{k,l} = 1 \geq t_{k-1,l+1} + T_{k,l+1} = t_{k,l+1}.$$

From (5) we get:

$$T_{k,l+1} = \frac{u_k - d_{k,l}}{m_k - m_l - z} = \frac{u_k - d_{k,l-1} - (m_l - m_{l-1}) T_{k,l}}{m_k - m_l - z}. \quad (6)$$

Using (4) we obtain:

$$T_{k,l}(m_k - m_{l-1} - z) < u_k - d_{k,l-1}. \quad (7)$$

Combining (6) and (7) we get:

$$T_{k,l+1} > T_{k,l}.$$

Case(3): $l > s_k$:
By construction, we have:

$$T_{k,l} = T_{k,l+1},$$
$$t_{k,l} = T_{k,l} + t_{k-1,l}, t_{k,l+1} = T_{k,l+1} + t_{k-1,l+1}.$$

By induction hypothesis, $t_{k-1,l} \geq t_{k-1,l+1}$. Hence,

$$t_{k,l} \geq t_{k,l+1}.$$

■

Lemma 2. *Given an allocable rate vector $(u_1, u_2, ..., u_n)$, the "as uniform as possible" intra-session coding scheme achieves the rate vector $(u_1, u_2, ..., u_n)$.*

Proof: Let's verify that the allocation y_i^j in (3) satisfies (1) and (2) for every set of unerased links $P \subseteq \mathcal{I}$ with $|P| \geq m_n - z$.
From Lemma 1 we have:

$$y_i^j \leq y_{i+1}^j \quad \forall i, j : 1 \leq i < m_n, 1 \leq j < n,$$

which implies:

$$\sum_{i \in P \cap \{1,...,m_i\}} y_i^j \geq \sum_{i=1}^{m_i - z} y_i^j = \sum_{i=1}^{s_i} (m_i - m_{i-1}) T_{i,j} + (m_i - m_{s_i} - z) T_{i,s_i+1} = d_{i,s_i} + (m_i - m_{s_i} - z) \frac{u_i - d_{i,s_i}}{m_i - m_{s_i} - z} = u_i,$$

which establishes (1).

By construction we have $T_{i,j} \leq 1 - t_{i-1,j}$, i.e. $t_{i,j} \leq 1$. As $\sum_{j=1}^{n} y_i^j = t_{n,k}$ for $k : m_{k-1} < i \leq m_k$, we have:

$$\sum_{j=1}^{n} y_i^j \leq 1,$$

which establishes (2). ■

IV. Main result and proof

A. Erasure Correction Capacity

Theorem 1. *The $z-$erasure correction capacity region of $\mathcal{G}_n$ is achieved by the "as uniform as possible" intra-session coding scheme.*

Theorem 1 is proved inductively using the following lemmas. Let $S = \{S_1, S_2, \ldots, S_n\}$ be an arbitrary set. We use the following notation in this section:

$$S^j = \{S_1, S_2, \ldots, S_j\},$$
$$S_i^j = \{S_i, S_{i+1}, \ldots, S_j\}.$$

Lemma 3. *Let Z be the set of random elements transmitted on $\mathcal{I} = \{l_1, l_2, ..., l_{m_n}\}$. Let $M = \{M_1, M_2, \ldots, M_n\}$ be a set of random elements, where M_i is demanded by the sink t_i. Then,*

$$I(M_{i+1}^n; Z|M_1^i) = H(Z|M_1^i).$$

Proof: Since Z is a function of $M_1, \ldots, M_n$,

$$I(M_{i+1}^n; Z|M_1^i) = H(Z|M_1^i) - H(Z|M_1^n) = H(Z|M_1^i).$$

■

Lemma 4. *Let $M = \{M_1, M_2, \ldots, M_n\}$ be a set of random elements, where M_i is demanded by the sink t_i. Let Y be the set of random elements transmitted on $\mathcal{I} = \{l_1, l_2, ..., l_{m_n}\}$, and achieving the rate vector $(u_1, u_2, \ldots, u_n)$. Then,*

$$H(Y|M_1^{i-1}) = u_i + H(Y|M_1^i).$$

Proof: By Lemma 3, $H(Y|M_1^{i-1}) = I(M_i^n; Y|M_1^{i-1})$. Therefore, by the chain rule for mutual information:

$$H(Y|M_1^{i-1}) = I(M_i; Y|M_1^{i-1}) + I(M_{i+1}^n; Y|M_1^i). \quad (8)$$

Consider the first term in expansion (8):

$$\begin{aligned} I(M_i; Y|M_1^{i-1}) &= H(M_i|M_1^{i-1}) - H(M_i|M_1^{i-1}, Y) \\ &= H(M_i|M_1^{i-1}) = H(M_i) = u_i, \end{aligned}$$

which follows from the fact that Y is a decoding set for M_i and independence of $M_1, \ldots, M_i$.

Consider the second term in expansion (8):

$$I(M_{i+1}^n; Y|M_1^i) = H(Y|M_1^i) - H(Y|M_1^n) = H(Y|M_1^i),$$

since Y is a function of $M_1, \ldots, M_n$.

Therefore,

$$H(Y|M_1^{i-1}) = u_i + H(Y|M_1^i).$$

■

Lemma 5. *Let $X = \{X_1, ..., X_t\}, Z$ and M be sets of random elements and $k \leq t$ be a positive integer. Then,*

$$\sum_{A:|A|=k, A\subset X} H(Z, A|M) \geq \binom{t-1}{k-1} H(Z, X|M).$$

Proof: Consider a subset $A = \{X_{i_1}, X_{i_2}, ..., X_{i_k}\}$ of X where $i_1 < i_2 < \ldots < i_k$. Let $A^m = \{X_{i_1}, X_{i_2}, ..., X_{i_m}\}$. By the chain rule for conditional entropy:

$$H(Z, A|M) = H(Z|M) + \sum_{j=1}^{j=k} H(X_{i_j}|M, Z, A^{j-1}). \quad (9)$$

As conditioning decreases entropy, we have

$$H(X_{i_j}|M, Z, A^{j-1}) \geq H(X_{i_j}|M, Z, X^{i_{j-1}}). \quad (10)$$

Summing (9) for all choices of A and applying (10) we get:

$$\begin{aligned} &\sum_{A:|A|=k, A\subset X} H(Z, A|M) \\ &\geq \binom{t}{k} H(Z|M) + \binom{t-1}{k-1} \sum_{j=1}^{t} H(X_j|M, Z, X^{j-1}) \\ &\geq \binom{t-1}{k-1} H(Z|M) + \binom{t-1}{k-1} \sum_{j=1}^{t} H(X_j|M, Z, X^{j-1}) \\ &= \binom{t-1}{k-1} H(Z, X|M). \end{aligned}$$

■

Let T be the allocation matrix corresponding to an allocable rate vector $(u_1, u_2, ..., u_n)$. Define integers r_i by:

$$0 = T_{i,1} = T_{i,2} = ... = T_{i,r_i} < T_{i,r_i+1}.$$

Lemma 6. *Let $(u_1, u_2, ..., u_n)$ be a rate vector attained by the "as uniform as possible" intra-session coding.*

Let T and t be the corresponding $n \times n$ lower triangular matrices. Let $X = \{X_1, X_2, ..., X_{m_n}\}$ where X_i is the random element transmitted on the link l_i. Assume that X attains the rate vector $(u_1, u_2, ..., u_n)$ and that any set of unerased packets $Y = Y_1 \cup Y_2 \cup ... \cup Y_{n-1} \subset X^{m_{n-1}}$ with $|Y| \geq m_{n-1} - z$ and satisfying

$$\left. \begin{array}{l} Y_i = \{X_{m_{i-1}+1}, ..., X_{m_i}\} \, \forall i : 0 < i \leq s_{n-1}, \\ Y_i \subset \{X_{m_{i-1}+1}, ..., X_{m_i}\} \, \forall i : s_{n-1} < i \leq n-1, \end{array} \right\} \quad (11)$$

also satisfies

$$H(Y|M_1^{n-1}) \leq \sum_{i=s_{n-1}+1}^{n-1} |Y_i|(1 - t_{n-1,i}).$$

Then, for any set of unerased packets $W = W_1 \cup W_2 \cup ... \cup W_n \subset X^{m_n}$ with $|W| \geq m_n - z$ and satisfying

$$W_i = \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : 0 < i \leq s_n,$$

$$W_i \subset \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : s_n < i \leq n,$$

we have:

$$H(W|M_1^n) \leq \sum_{i=s_n+1}^{n} |W_i|(1 - t_{n,i}).$$

Proof: Let's first prove that $m_n - m_{s_n} - z > 0$. If $s_n = 0$, since we have more than z links in $\mathcal{I}$, $m_n - m_{s_n} - z > 0$. If $s_n \geq 1$, by construction of s_n we have:

$$u_n - d_{n,s_n} > 0.$$

As T is non-negative,

$$0 \leq T_{n,s_n+1} = \frac{u_n - d_{n,s_n}}{m_n - m_{s_n} - z}.$$

Hence $m_n - m_{s_n} - z > 0$, as desired.
By construction of $T_{i,j}$ we have:

$$u_n = \sum_{i=r_n+1}^{s_n} (m_i - m_{i-1}) T_{n,i} + (m_n - m_s - z) T_{n,n}. \quad (12)$$

Let $V = V_1 \cup V_2 \cup ... \cup V_n$ with $|V| = m_n - z$ be a set of unerased packets satisfying

$$V_i = \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : 0 < i \leq s_n,$$

$$V_i \subset \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : s_n < i \leq n.$$

By Lemma 4, we have: $H(V|M_1^n) = H(V|M_1^{n-1}) - u_n$. We also have:

$$H(V|M_1^{n-1}) \leq |V_n| + H(V_1^{n-1}|M_1^{n-1}).$$

As $V' = V_1 \cup V_2 \cup ... \cup V_{n-1}$ satisfies (11) , by assumption we have:

$$H(V'|M_1^{n-1}) \leq \sum_{i=s_{n-1}+1}^{n-1} |V_i|(1 - t_{n-1,i}). \quad (13)$$

As $|V_i| = m_i - m_{i-1} \quad \forall i : 1 \leq i \leq s_n$, $s_{n-1} + 1 \leq r_n$, $t_{n-1,i} = 1 \quad \forall i : 1 \leq i \leq r_n$, and $T_{n,i} = 1 - t_{n-1,i} \quad \forall i : 1 \leq i \leq s_n$, we can rewrite (13) as:

$$H(V'|M_1^{n-1}) \leq \sum_{i=r_n+1}^{s_n} (m_i - m_{i-1}) T_{n,i} + \sum_{i=s_n+1}^{n-1} |V_i|(1 - t_{n-1,i}).$$

Hence we have:

$$H(V|M_1^n) \leq \sum_{i=r_n+1}^{s_n} (m_i - m_{i-1}) T_{n,i} + \sum_{i=s_n+1}^{n-1} |V_i|(1 - t_{n-1,i}) + |V_n| - u_n.$$

Using (12) we get:

$$\begin{aligned} H(V|M_1^n) &\leq \sum_{i=s_n+1}^{n} |V_i|(1 - t_{n-1,i} - T_{n,n}) \\ &= \sum_{i=s_n+1}^{n} |V_i|(1 - t_{n-1,i} - T_{n,i}) \\ &= \sum_{i=s_n+1}^{n} |V_i|(1 - t_{n,i}). \end{aligned} \quad (14)$$

Let $z' \leq z$ and $W = W_1 \cup W_2 \cup ... \cup W_n$ be a set of unerased packets satisfying

$$|W| = m_n - z',$$

$$W_i = \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : 0 < i \leq s_n,$$

$$W_i \subset \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : s_n < i \leq n.$$

From (14), for any $V = V_1 \cup V_2 \cup ... \cup V_n \subset W$ with $|V| = m_n - z$ and satisfying

$$V_i = \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : 0 < i \leq s_n,$$

$$V_i \subset \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : s_n < i \leq n,$$

we have:

$$H(V|M_1^n) \leq \sum_{i=s_n+1}^{n} |V_i|(1 - t_{n,i}).$$

Summing the last inequality for all possible V and applying Lemma 5 for $Z = \{W_1, W_2, ..., W_{s_n}\}$, $X =$

$\{W_{s_n+1}, W_{s_n+2}, ..., W_n\}$, $M = \{M_1, M_2, ..., M_n\}$ and $k = m_n - m_{s_n} - z$ we get:

$$\binom{m_n - m_{s_n} - z' - 1}{m_n - m_{s_n} - z - 1} H(W|M_1^n) \leq \binom{m_n - m_{s_n} - z' - 1}{m_n - m_{s_n} - z - 1} \sum_{i=s_n+1}^{n} |W_i|(1 - t_{n,i}), \quad (15)$$

where the constant on the RHS comes from the fact that any $X_j \in W_i$ with $i \in \{s_n + 1, \ldots, n\}$ appears in exactly $\binom{m_n - m_{s_n} - z' - 1}{m_n - m_{s_n} - z - 1}$ different V. The result follows by simplifying (15). ∎

Proof of Theorem 1: Let rate vector $(u_1, u_2, ..., u_n)$ be attained with $X = \{X_1, X_2, \ldots, X_{m_n}\}$. We will prove by induction that the following conditions are satisfied for each $k \in \{1, \ldots, n-1\}$:

(i) $(u_1, u_2, ..., u_{k+1})$ is attained by the "as uniform as possible" intra-session coding.

(ii) Any set of unerased packets $Y = Y_1 \cup Y_2 \cup ... \cup Y_k \subset X^{m_k}$ with $|Y| \geq m_k - z$ and satisfying

$$Y_i = \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : 0 < i \leq s_k,$$

$$Y_i \subset \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : s_k < i \leq k,$$

also satisfies

$$H(Y|M_1^k) \leq \sum_{i=s_k+1}^{k} |Y_i|(1 - t_{k,i}),$$

where s_k is defined as in Algorithm 1 using the rate vector $(u_1, u_2, \ldots, u_k)$.

Proof of (ii) when k=1: We have $r_1 = s_1 = 0$, $t_{1,1} = T_{1,1} = \frac{u_1}{m_1 - z}$. Hence it suffices to show that for any set of unerased packets $Y \subset X^{m_1}$ with $|Y| \geq m_1 - z$,

$$H(Y|M_1) \leq |Y|(1 - \frac{u_1}{m_1 - z}).$$

Let $A \subset X^{m_1}$ be a set of unerased packets with $|A| = m_1 - z$. As A is a decoding set for M_1 with rate u_1,

$$H(A|M_1) \leq H(A) - u_1 = m_1 - z - u_1. \quad (16)$$

Let $z' \leq z$ and $Y \subset X^{m_1}$ be a set of unerased packets with $|Y| = m_1 - z'$. Summing (16) for all possible $A \subset Y$ with $|A| = m_1 - z$ and applying Lemma 5 for $Z = \phi$, $X = Y$, $M = M_1$ and $k = m_1 - z$ we get:

$$\binom{m_1 - z' - 1}{m_1 - z - 1} H(Y|M_1) \leq \binom{m_1 - z'}{m_1 - z}(m_1 - z - u_1),$$

which is equivalent to

$$H(Y|M_1) \leq \frac{m_1 - z'}{m_1 - z}(m_1 - z - u_1) = |Y|(1 - \frac{u_1}{m_1 - z}).$$

Proof of (i) when k=1: **Case(1)**: $m_2 - m_1 \geq z$.
Let $A = \{X_1, X_2, \ldots, X_{m_2 - z}\}$. As A is an information set for M_2 with rate u_2, we have:

$$H(A|M_1, M_2) \leq H(A|M_1) - u_2 \leq H(X^{m_1}|M_1) + H(A - X^{m_1}|M_1) - u_2. \quad (17)$$

As $H(A - X^{m_1}|M_1) \leq H(A - X^{m_1}) \leq |A - X^{m_1}| = m_2 - m_1 - z$ and $H(X^{m_1}|M_1) \leq |X^{m_1}|(1 - \frac{u_1}{m_1 - z}) = m_1(1 - \frac{u_1}{m_1 - z})$,

$$H(A|M_1, M_2) \leq m_1(1 - \frac{u_1}{m_1 - z}) + m_2 - m_1 - z - u_2. \quad (18)$$

As $H(A|M_1, M_2) \geq 0$, (18) implies:

$$u_2 \leq m_1(1 - \frac{u_1}{m_1 - z}) + m_2 - m_1 - z.$$

The y_i^j allocation defined in (3) corresponding to the lower triangular matrix U, where $U_{1,1} = \frac{u_1}{m_1 - z}, U_{2,1} = 1 - \frac{u_1}{m_1 - z}, U_{2,2} = 1$, achieves the rate vector $(u_1, m_1(1 - \frac{u_1}{m_1 - z}) + m_2 - m_1 - z)$, hence (u_1, u_2) is attained by the "as uniform as possible" intra-session coding.
Case(2): $m_2 - m_1 < z$.
Let $A = \{X_1, X_2, \ldots, X_{m_2 - z}\}$. As A is an information set for M_2 with rate u_2, we have:

$$H(A|M_1, M_2) \leq H(A|M_1) - u_2.$$

As $H(A|M_1) \leq |A|(1 - \frac{u_1}{m_1 - z}) = (m_2 - z)(1 - \frac{u_1}{m_1 - z})$, we have:

$$H(A|M_1, M_2) \leq (m_2 - z)(1 - \frac{u_1}{m_1 - z}) - u_2. \quad (19)$$

As $H(A|M_1, M_2) \geq 0$, (19) implies:

$$u_2 \leq (m_2 - z)(1 - \frac{u_1}{m_1 - z}).$$

The y_i^j allocation defined in (3) corresponding to the lower triangular matrix U where $U_{1,1} = \frac{u_1}{m_1 - z}, U_{2,1} = 1 - \frac{u_1}{m_1 - z}, U_{2,2} = 1$ achieves the rate vector $(u_1, (m_2 - z)(1 - \frac{u_1}{m_1 - z}))$. Hence (u_1, u_2) is attained by the "as uniform as possible" intra-session coding.

Now assume that (i) and (ii) are true for $k = l < n - 1$.

Proof of (ii) when k=l+1: As (i) and (ii) are true for $k = l$, we can apply Lemma 6 for $n = l+1$, $X = X^{m_{l+1}}$, from which it follows that for any set of unerased packets

$Y = Y_1 \cup Y_2 \cup ... \cup Y_{l+1}$ with $|Y| \geq m_{l+1} - z$ and satisfying

$$Y_i = \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : 0 < i \leq s_{l+1}$$
$$Y_i \subset \{X_{m_{i-1}+1}, ..., X_{m_i}\} \quad \forall i : s_{l+1} < i \leq l+1$$

also satisfies:

$$H(Y|M_1^{l+1}) \leq \sum_{i=s_{l+1}+1}^{l+1} |Y_i|(1 - t_{l+1,i}).$$

Proof of (i) when k=l+1: **Case(1)**: $m_{l+2} - m_{l+1} \geq z$. Let $A = \{X_1, X_2, \ldots, X_{m_{l+2}-z}\}$. As A is an information set for M_{l+2} with rate u_{l+2}, we have:

$$\begin{aligned} H(A|M_1^{l+2}) &\leq H(A|M_1^{l+1}) - u_{l+2} \\ &\leq H(X^{m_{l+1}}|M_1^{l+1}) + H(A - X^{m_{l+1}}|M_1^{l+1}) \\ &\quad - u_{l+2}, \end{aligned} \tag{20}$$

$$\begin{aligned} H(A - X^{m_{l+1}}|M_1^{l+1}) &\leq H(A - X^{m_{l+1}}) \leq |A - X^{m_{l+1}}| \\ &= m_{l+2} - m_{l+1} - z. \end{aligned} \tag{21}$$

As (ii) is true for $k = l+1$ and $Y = X_1^{m_{l+1}}$:

$$H(X^{m_{l+1}}|M_1^{l+1}) \leq \sum_{i=s_{l+1}+1}^{l+1} (m_i - m_{i-1})(1 - t_{l+1,i}). \tag{22}$$

Combining (20), (21), and (22) we have:

$$\begin{aligned} H(A|M_1^{l+2}) \leq &\sum_{i=s_{l+1}+1}^{l+1} (m_i - m_{i-1})(1 - t_{l+1,i}) \\ &+ m_{l+2} - m_{l+1} - z - u_{l+2}. \end{aligned} \tag{23}$$

As $H(A|M_1^{l+2}) \geq 0$, (23) implies:

$$\begin{aligned} u_{l+2} \leq &\sum_{i=s_{l+1}+1}^{l+1} (m_i - m_{i-1})(1 - t_{l+1,i}) \\ &+ m_{l+2} - m_{l+1} - z. \end{aligned}$$

Let's define an allocation matrix U as follows: the first $l+1$ rows of U are the same as that of the matrix T corresponding to the rate vector $(u_1, u_2, ..., u_{l+1})$ and $U_{l+2,i} = 1 - t_{l+1,i}$ for $1 \leq i \leq l+2$. The y_i^j allocation defined in (3) corresponding to the lower triangular matrix U achieves the rate vector $(u_1, u_2, ..., u_{l+1}, \sum_{i=s_{l+1}+1}^{l+1}(m_i - m_{i-1})(1 - t_{l+1,i}) + m_{l+2} - m_{l+1} - z)$. Hence $(u_1, u_2, ..., u_{l+2})$ is attained by the "as uniform as possible" intra-session coding.
Case(2): $m_{l+2} - m_{l+1} < z$.
Let j be the integer satisfying $m_j < m_{l+2} - z \leq m_{j+1}$ Let $A = \{X_1, X_2, \ldots, X_{m_{l+2}-z}\}$. As A is an information set for M_{l+2} with rate u_{l+2}, we have:

$$H(A|M_1^{l+2}) \leq H(A|M_1^{l+1}) - u_{l+2}. \tag{24}$$

As (ii) is true for $k = l+1$ and $Y = A$:

$$\begin{aligned} H(A|M_1^{l+1}) &\leq \sum_{i=s_{l+1}+1}^{l+2} |A \cap X_{m_{i-1}+1}^{m_i}|(1 - t_{l+1,i}) \\ &= \sum_{i=s_{l+1}+1}^{j} (m_i - m_{i-1})(1 - t_{l+1,i}) \\ &\quad + (m_{l+2} - m_j - z)(1 - t_{l+1,j+1}). \end{aligned} \tag{25}$$

Combining (24) and (25) we get:

$$\begin{aligned} H(A|M_1^{l+2}) \leq &\sum_{i=s_{l+1}+1}^{j} (m_i - m_{i-1})(1 - t_{l+1,i}) \\ &+ (m_{l+2} - m_j - z)(1 - t_{l+1,j+1}) - u_{l+2}. \end{aligned} \tag{26}$$

As $H(A|M_1^{l+2}) \geq 0$, (26) implies:

$$\begin{aligned} u_{l+2} \leq &\sum_{i=s_{l+1}+1}^{j} (m_i - m_{i-1})(1 - t_{l+1,i}) \\ &+ (m_{l+2} - m_j - z)(1 - t_{l+1,j+1}). \end{aligned}$$

Let's define an allocation matrix U as follows: the first $l+1$ rows of U are the same as that of the matrix T corresponding to the rate vector $(u_1, u_2, ..., u_{l+1})$ and $U_{l+2,i} = 1 - t_{l+1,i}$ for $1 \leq i \leq l+2$. The y_i^j allocation defined in (3) corresponding to the lower triangular matrix U achieves the rate vector $(u_1, u_2, ..., u_{l+1}, \sum_{i=s_{l+1}+1}^{j}(m_i - m_{i-1})(1 - t_{l+1,i}) + (m_{l+2} - m_j - z)(1 - t_{l+1,j+1}))$, hence $(u_1, u_2, ..., u_{l+2})$ is attained by the "as uniform as possible" intra-session coding.

We verified that (i) and (ii) are satisfied for $k \in \{1, \ldots, n-1\}$. In particular, (i) is satisfied for $k = n-1$. Hence, the rate vector $(u_1, u_2, \ldots, u_n)$ is attained by the "as uniform as possible" intra-session coding. ■

B. Error Correction Capacity

Here we consider the z-error correction capacity region of 3-layer nested-networks. The following result is shown in [18].

Lemma 7. *Let a linear network code $\mathcal{C}$ on an acyclic network be given. For a sink node t, the following properties of $\mathcal{C}$ are equivalent:*
1) any error pattern with at most z errors can be

The proof in [18] is formally written for scalar linear codes, but easily extends to vector linear codes of vector length y by considering each link as y parallel sublinks and defining the network Hamming weight of a received vector v in terms of the number of erroneous links (not sublinks) needed to produce v.

corrected at t;
2) any erasure pattern with at most $2z$ link failures can be corrected at t.

Lemma 8. *Let a (possibly nonlinear) network code $\mathcal{C}$ on network $\mathcal{G}_n$ be given. Assume that any error pattern with at most z errors on $\mathcal{I}$ (the set of links in the middle layer of $\mathcal{G}_n$) can be corrected at sink t_i. Then, any $2z$ erasures on $\mathcal{I}$ can be corrected at sink t_i.*

Proof: By a Singleton-type argument, if two codewords agree on $m_i - 2z$ links, then there exist corresponding patterns of z errors on the remaining links such that t_i is unable to distinguish between the two codewords. Hence, any set of $m_i - 2z$ links is an information set for sink t_i, allowing correction of any $2z$ erasures. ∎

Theorem 2. *The z-error correction capacity region of $\mathcal{G}_n$ is the same as the $2z$-erasure correction capacity region of $\mathcal{G}_n$ and is achieved by the "as uniform as possible" intra-session coding for $2z$-erasures.*

Proof: By Lemma 8, the z-error correction capacity region of $\mathcal{G}_n$ is no greater than the $2z$-erasure correction capacity region of $\mathcal{G}_n$. Since the "as uniform as possible" intra-session coding for $2z$-erasures is a linear code, achievability follows from Lemma 7. ∎

C. Example

Consider an example with three receivers, parameters $m_1 = 5$, $m_2 = 8$, and $m_3 = 12$, and at most two erasures (or one error). Let (u_1, u_2, u_3) be an achievable rate vector. There are two cases:
Case(1): $T_{1,1} = \frac{u_1}{3}, T_{2,1} = T_{2,2} = \frac{u_2}{6}$. As $T_{1,1} \leq 1$ we have $u_1 \leq 3$. We also have: $T_{3,1} \leq 1 - \frac{u_1}{3} - \frac{u_2}{6}, T_{3,2} \leq 1 - \frac{u_2}{6}, T_{3,3} \leq 1$. Hence $u_3 \leq 2 + 3(1 - \frac{u_2}{6}) + 5(1 - \frac{u_1}{3} - \frac{u_2}{6})$, which is equivalent to: $5u_1 + 4u_2 + 3u_3 \leq 30$.
Case(2): $T_{1,1} = \frac{u_1}{3}, T_{2,1} = 1 - \frac{u_1}{3}, T_{2,2} = u_2 - 5(1 - \frac{u_1}{3})$. As $T_{2,2} \leq 1$ we have $u_2 \leq 1 + 5(1 - \frac{u_1}{3})$, which is equivalent to: $u_1 + 3u_2 \leq 18$. We also have: $T_{3,1} = 0, T_{3,2} \leq 6 - u_2 - \frac{5u_1}{3}, T_{3,3} \leq 1$. Hence $u_3 \leq 2 + 3(6 - u_2 - \frac{5u_1}{3})$, which is equivalent to: $5u_1 + 3u_2 + u_3 \leq 20$.

Hence the capacity region is given by

$$u_1 \leq 3, u_1 + 3u_2 \leq 18,$$
$$5u_1 + 4u_2 + 3u_3 \leq 30,$$
$$5u_1 + 3u_2 + u_3 \leq 20.$$

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References

[1] N. Cai and R. W. Yeung, "Network coding and error correction," in *ITW2002 Bangalore*, 2002.
[2] R. W. Yeung and N. Cai, "Network error correction, part I: Basic concepts and upper bounds," *Commun. Inf. Syst.*, vol. 6, no. 1, pp. 19–36, 2006.
[3] N. Cai and R. W. Yeung, "Network error correction, part II: Lower bounds," *Commun. Inf. Syst.*, vol. 6, no. 1, pp. 37–54, 2006.
[4] A. F. Dana, R. Gowaikar, R. Palanki, B. Hassibi, and M. Effros, "Capacity of wireless erasure networks," *IEEE Transactions on Information Theory*, vol. 52, pp. 789–804, 2006.
[5] R. K. D. S. Lun, M. Medard and M. Effros, "On coding for reliable communication over packet networks," *Physical Communication*, vol. 1, no. 5, Mar. 2008.
[6] S. Jaggi, M. Langberg, S. Katti, T. Ho, D. Katabi, M. Médard, and M. Effros, "Resilient network coding in the presence of byzantine adversaries," *Information Theory, IEEE Transactions on*, vol. 54, no. 6, pp. 2596 –2603, June 2008.
[7] R. Koetter and F. R. Kschischang, "Coding for errors and erasures in random network coding," *IEEE Transactions on Information Theory*, August 2008.
[8] D. Silva, F. Kschischang, and R. Kötter, "A rank-metric approach to error control in random network coding," *IEEE Transactions on Information Theory*, vol. 54, no. 9, pp. 3951–3967, Sept. 2008.
[9] E. Erez and M. Feder, "Capacity region and network codes for two receivers multicast with private and common data," in *Proc. IEEE International Symposium on Information Theory*, 2003.
[10] C. Ngai and R. Yeung, "Multisource network coding with two sinks," in *Communications, Circuits and Systems, 2004. ICCCAS 2004. 2004 International Conference on*, vol. 1, June 2004, pp. 34 – 37.
[11] A. Ramamoorthy and R. D. Wesel, "The single source two terminal network with network coding," in *in Canadian Workshop on Information Theory*, 2005.
[12] R. Koetter and M. Médard, "An algebraic approach to network coding," *IEEE/ACM Transactions on Networking*, vol. 11, no. 5, pp. 782–795, October 2003.
[13] T. Chan and A. Grant, "Mission impossible: Computing the network coding capacity region," in *Proc. IEEE International Symposium on Information Theory*, July 2008, pp. 320 –324.
[14] N. Harvey and R. Kleinberg, "Tighter cut-based bounds for k-pairs communication problems," in *Proc. 43rd Allerton Conference on Communication, Control, and Computing, Monticello, IL*, September 2005.
[15] S. Vyetrenko, T. Ho, and T. Dikaliotis, "Outer bounds on the error correction capacity region for non-multicast networks," in *Allerton conference on Communication, Control, and Computing*, September 2010.
[16] E. Martinian and C.-E.Sundberg, "Burst erasure correction codes with low decoding delay," *IEEE Trans. Inf. Theory*, vol. 50, no. 10, pp. 2494–2502, October 2005.
[17] E. Martinian and M. Trott, "Delay-optimal burst erasure code construction," in *ISIT*, July 2007.
[18] S. Vyetrenko, T. Ho, M. Effros, J. Kliewer, and E. Erez, "Rate regions for coherent and noncoherent multisource network error correction," in *Proc. IEEE International Symposium on Information Theory*, June 2009, pp. 1001 –1005.