SOLUTIONS

[14.3.8] Let \( \alpha \) be a root of \( f(x) = x^p - x - a \), then \( \alpha + k \) is clearly also a root of \( f(x) \) for \( k \in \mathbb{F}_p \). Since \( f \) is of degree \( p \), we get all the roots this way, and the splitting field is given by \( K = \mathbb{F}_p[\alpha] \). Also, \( f \) has no roots in \( \mathbb{F}_p \), so \([K : \mathbb{F}_p] = p\). Furthermore, the Galois group can be seen as generated by the automorphism \( \alpha \mapsto \alpha + 1 \), which also has order \( p \).

[14.4.5] (a) Let \( \bar{L} \) be the Galois closure of \( L \) over \( F \), and \( L_1, \ldots, L_n \) the Galois conjugates of \( L \) over \( F \) in the algebraic closure of \( L \), then \( \bar{L} = L_1 \cdots L_n \). Suppose \( \sigma(L) = L_i \) with \( \sigma \in \text{Gal}(\bar{L}/F) \), since \( K/F \) is Galois, \( \sigma(K) = K \), and \( L_i \) contains \( K \). Note that \( L \) and each \( L_i \) are isomorphic, so all \( L_i \) are Galois over \( K \). Therefore \([\bar{L} : K][L_1 : K] \cdots [L_n : K]\) which is a power of \( p \). Since \( K/F \) is also a \( p \)-extension, we have \([\bar{L} : F]\) is a power of \( p \).

(b) Take \( F = \mathbb{Q} \), \( L = K = \mathbb{Q}(\sqrt{2}) \), then \( \bar{L} \) is of degree 6 over \( F \).

[14.4.6] Let \( K = \mathbb{F}_p(x^p, y^p) \) and \( L = \mathbb{F}_p(x, y) \), then \([L : K] = p^2\).

For \( f \in K \) consider the subfield \( L_f = K(fx + y) \). Since \((fx + y)^p = f^p x^p + y^p \in K\) it follows that \([L_f : K] \leq p \). If \( L_f = L_g \) for some \( f \neq g \), then it is easy to see that \( x, y \in L_f \), so \( L_f = L \). However, this contradicts the degree considerations above.

Consequently, this construction yields an infinite number of intermediate subfields, so by Proposition 24 the extension \( L/K \) is not simple.

[14.5.3] Notice that \( \zeta_5^4 + \zeta_5^3 + \zeta_5^2 + \zeta_5 + 1 = 0 \) implies that \( \zeta_5^4 + \zeta_5^3 + \zeta_5^2 + \zeta_5 + 1 = 0 \), so \( (\zeta_5 + \zeta_5^{-1})^2 + (\zeta_5 + \zeta_5^{-1}) - 1 = 0 \) showing that \( \alpha = \zeta_5 + \zeta_5^{-1} \) satisfies the polynomial \( x^2 + x - 1 = 0 \). Thus \( \alpha = \frac{\sqrt{5} - 1}{4} + i\sqrt{\frac{5 + \sqrt{5}}{8}} \).

However, \( \zeta_5 \) satisfies the quadratic \( x^2 - x + 1 = 0 \) over \( \mathbb{Q}(\alpha) \), which gives us

\[ \zeta_5 = \frac{\sqrt{5} - 1}{4} + i\sqrt{\frac{5 + \sqrt{5}}{8}}. \]

[14.5.7] Let \( \tau \) be the complex conjugation, then \( \tau(\zeta_n) = \overline{\zeta_n} = \zeta_n^{-1} \). Hence \( \tau \) can be identified with \( \sigma_{-1} \in G := \text{Gal}(\mathbb{Q} / (\zeta_n)/\mathbb{Q}) \) (recall \( \sigma_i(\zeta_n) = \zeta_n^i \)). Now observe that \( \zeta_n \) satisfies the quadratic polynomial

\[ x^2 - (\zeta_n + \zeta_n^{-1}) + 1 \in K^+[x] \]

so \([K : K^+] = 2\). But \( H = \{1, \sigma_{-1}\} \subseteq G \) clearly fixes \( K^+ \). Therefore, \( K^+ \) is in fact the fixed field of \( H \), and since conjugation fixes only the real numbers we get \( K^+ = K \cap \mathbb{R} \).