The exercises are taken from the text, *Abstract Algebra* (third edition) by Dummit and Foote.

Page 249. (a) $I \cap J$ is nonempty since it contains 0. Take any $a, b \in I \cap J$, then clearly $a - b \in I \cap J$. For any $r \in R$, $ra, ar \in I \cap J$. Therefore $I \cap J$ is an ideal.

(b) Same proof from (a) would also work for the arbitrary collection case.

Page 249. (a) Take $a, b \in \varphi^{-1}(J)$, then $\varphi(a), \varphi(b) \in J$, and $\varphi(a+b) = \varphi(a) + \varphi(b) \in J$, so $a + b \in \varphi^{-1}(J)$. Since $\varphi(-a) = -\varphi(a) \in J$, $-a \in \varphi^{-1}(J)$. Now $\varphi(ra) = \varphi(r)\varphi(a) \in \varphi(r)J \subseteq J$, so $r\varphi^{-1}(J) \subseteq \varphi^{-1}(J)$ for $r \in R$. Similarly, $\varphi^{-1}(J)r \subseteq \varphi^{-1}(J)$. Hence $\varphi^{-1}(J)$ is an ideal in $R$. In the case that $R$ is a subring and $\varphi$ is the inclusion, $\varphi^{-1}(J) = R \cap J$ which is an ideal of $R$.

(b) Similar to (a), $\varphi(I)$ is a subring of $S$. Take any $s \in S$, since $\varphi$ is surjective, there exists some $r \in R$ such that $\varphi(r) = s$, then $s\varphi(I) = \varphi(rI) \subseteq \varphi(I)$. Similarly, $\varphi(I)s \subseteq \varphi(I)$, and $\varphi(I)$ is an ideal of $S$.

Let $\varpi$ be the identity map from $\mathbb{Z}$ to $\mathbb{Q}$. Then $p\mathbb{Z}$ is an ideal in $\mathbb{Z}$ but $\varpi(p\mathbb{Z}) = p\mathbb{Z}$ is not an ideal in $\mathbb{Q}$, since $\mathbb{Q}$ is a field and has no proper ideal.

Page 256. Suppose $R$ is a field, and let $I$ be a nonzerod ideal of $R$, with $0 \neq a \in I$. Then since $R$ is a field, $1 = a * a^{-1} \in I$, and $R = I$. Therefore 0 is a maximal ideal of $R$.

Conversely, suppose 0 is a maximal ideal. Let $r \in R$ be a nonzero element. Then $rR$ is a nonzero ideal, since $R$ is commutative. Since 0 is maximal, we must have $rR = R$, and for some $s \in R$, $rs = 1$. Hence $R$ is a field.

Page 256. (a) We will show that the remainder $r(x)$ of a polynomial $p(x)$ dividing by $f(x)$ has degree less than $n$. We proceed by induction on the degree $d$ of $p(x)$. The base case is trivial. Suppose the statement holds for all polynomials of degree less than $k-1$. Let $p(x) = \sum_{j=1}^{k} a_j x^j$
be of degree $k$. WLOG, we assume $k \geq n$. Then if $f(x) = x^n + b_{n-1}x^{n-1} + \cdots$, $q(x) = p(x) - a_kf(x)^{k-n}$ has degree less than $k$, and we can write $q(x) = m(x)f(x) + s(x)$ for some $s(x)$ with degree less than $n$. Now $p(x) \equiv s(x)$ in the quotient ring, and we’re done.

(b) Suppose $p(x) = q(x)$, then $p(x) - q(x) = f(x)g(x)$ for some $g(x) \in R[x]$. Since the left side has degree strictly less than the right side unless $g(x) = 0$. We must have $p(x) - q(x) = 0$, which contradicts the fact that $p(x)$ and $q(x)$ are distinct polynomials.

(c) $a(x)b(x) = 0$, and since $a(x)$ and $b(x)$ have degrees less than $n$, $a(x)$ and $b(x)$ are nonzero. Therefore $a(x)$ is a zero divisor.

5. Take $\frac{a}{s} \in S^{-1}A$, if $a \notin p$, then $a \in S$, and $\frac{a}{s} \in S^{-1}A$. Furthermore, $\frac{a \cdot a}{s \cdot s} = 1_{S^{-1}A}$. Therefore, if $M$ is a maximal ideal, $M \subseteq I = p(S^{-1}A)$. We will show that $I$ is indeed an ideal. Take $\frac{p_1}{s_1}, \frac{p_2}{s_2}$ with $p_1, p_2 \in p$, and $s_1, s_2 \in S$. Then $\frac{p_1}{s_1} + \frac{p_2}{s_2} = \frac{p_1s_2 + p_2s_1}{s_1s_2} \in I$. $-(\frac{p_1}{s_1}) = \frac{-p_1}{s_1} \in I$. Now $\frac{a \cdot p_1}{s \cdot s_1} = \frac{ap_1}{ss_1} \in I.$