The exercises are taken from the text, *Abstract Algebra* (third edition) by Dummit and Foote.

Page 376.5. Let $G_i$ be the Sylow subgroups of $A$ for $i = 1, \ldots, n$ such that $G_i$ is a direct sum of $\mathbb{Z}_{p_i r_i}$, with $p_i$ prime, and $A \cong G_1 \oplus \cdots \oplus G_n$. Now, $\mathbb{Z}_{p_k} \otimes A \cong \mathbb{Z}_{p_k} \otimes G_1 \oplus \cdots \oplus G_n \cong \oplus_{i=1}^{n} \mathbb{Z}_{p_i} \otimes G_i$. From example 3), we know that $\mathbb{Z}_{p_k} \otimes \mathbb{Z}_{p_i r_i} = 0$ for $p_i \neq p$, and $\mathbb{Z}_{p_k} \otimes \mathbb{Z}_{p_k r_i} \cong \mathbb{Z}_{p_k}$, for $p = p_k$. Therefore $\mathbb{Z}_{p_k} \otimes A \cong G_k$, where $p = p_k$.

Page 376.9. Take $n \in \ker(\iota)$, then $0 = \iota(n) = 1 \otimes n$. By Problem 8(c), $rn = 0$ for some $r \in R$, and $\ker(\iota) \subseteq \text{Tor}(N)$. Conversely, take $n \in \text{Tor}(N)$ with $rn = 0$, then $\iota(n) = 1 \otimes n = \frac{1}{r} \otimes n = \frac{1}{r} \otimes 0 = 0$, and $n \in \ker(\iota)$.

Page 376.10. a) Take any $\alpha \in M \otimes N$, then $\alpha = \sum_{i=1}^{n} m_i \otimes n_i$. Now we can write $n_i = \sum_{j=1}^{k} r_{ji} \otimes e_j$ for some $r_{ji} \in R$, so $\alpha = \sum_{i=1}^{n} m_i \otimes (\sum_{j=1}^{k} r_{ji} \otimes e_j) = \sum_{i=1}^{n} (\sum_{j=1}^{k} m_i \otimes r_{ji} \otimes e_j) = \sum_{j=1}^{k} (\sum_{i=1}^{n} m_i \otimes r_{ji} \otimes e_j) = \sum_{j=1}^{k} (\sum_{i=1}^{n} m_i \otimes r_{ji}) \otimes e_j)$. Since the representation $r_{ji}$ for $n_i$ is unique, so is the above expression, and $\sum_{i=1}^{n} m_i \otimes e_i = 0$, then $m_i = 0$ for all $i = 1, \ldots, n$.

b) In $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = 1 \otimes 2 = 2 \otimes 1 = 0 \otimes 1 = 0$.

Page 376.24. Define $\varphi : \mathbb{Z}[i] \times \mathbb{R} \rightarrow \mathbb{C}$ via $(\alpha, r) \mapsto \alpha r$. It can be checked to be $\mathbb{Z}$–balanced and we get an induced map from $\phi : \mathbb{Z}[i] \otimes \mathbb{R} \rightarrow \mathbb{C}$. Furthermore this is a ring homomorphism. Note that $\phi(\sum (\alpha_i \otimes r_i) \sum (\beta_j \otimes s_j)) = \phi(\sum \sum (\alpha_i \beta_j \otimes r_is_j)) = \sum \sum (\alpha_i \beta_j \otimes r_is_j) = \phi(\sum (\alpha_i \otimes r_i)) \phi(\sum (\beta_j \otimes s_j))$. The inverse map $\psi : \mathbb{C} \rightarrow \mathbb{Z}[i] \otimes \mathbb{R}$ is defined to be $\frac{a}{b} + \frac{c}{d} \mapsto ad + bc i \otimes \frac{1}{bd}$. After checking $\psi$ is a ring homomorphism, we can conclude $\phi$ is an isomorphism of rings.

Page 376.27. a) $(a1 + bce_2 + cde_4)(a'1 + b'e_2 + c'e_3 + d'e_4) = (aa' - bb' - cc' + dd')1 + (ab' + ba' - cd' - dc')e_2 + (ac' + ca' - bd' - db')e_3 + (ad' + da' - bc' - cb')e_4$. 

1
b) Note that $\epsilon_1 = \frac{1}{2}(e_1 + e_4)$, $\epsilon_2 = \frac{1}{2}(e_1 - e_4)$. Following the a), we get $\epsilon_1\epsilon_2 = 0$, and $\epsilon_j^2 = \epsilon_j$. $\epsilon_1 + \epsilon_2 = e_1 = 1$. The isomorphism is given by $a \mapsto (ae_1, ae_2)$.

c) $\varphi(z_1+z'_1, z_2) = ((z_1+z'_1)z_2, (z_1+z'_1)\bar{z}_2) = (z_1z_2, z_1\bar{z}_2) + (z'_1z_2, z'_1\bar{z}_2) = \varphi(z_1, z_2) + \varphi(z'_1, z_2)$. The other two conditions are similar.

d) $\Phi(\epsilon_1) = \Phi(\frac{1}{2}(e_1 + e_4)) = \frac{1}{2}(\Phi(e_1) + \Phi(e_4)) = \frac{1}{2}((1, 1) + (-1, 1)) = (0, 1)$, $\Phi(\epsilon_2) = \Phi(\frac{1}{2}(e_1 - e_4)) = \frac{1}{2}((1, 1) - (-1, 1)) = (1, 0)$. $\Phi(z_1 \otimes z_2) = \Phi(zz_1 \otimes z_2) = (zz_1z_2, zz_1\bar{z}_2) = z(z_1z_2, z_1\bar{z}_2) = z\Phi(z_1 \otimes z_2)$. Since $\Phi(\epsilon_1)$ and $\Phi(\epsilon_2)$ are basis for $\mathbb{C} \times \mathbb{C}$, and $\Phi$ is $\mathbb{C}$-linear, we have $\Phi$ is surjective. Since $ie_1 = e_3$, and $ie_2 = e_4$, $A$ is free on $\{e_1, e_2\}$, i.e. $A$ is a left-$\mathbb{C}$ module of rank at most 2. But $\mathbb{C} \times \mathbb{C}$ is of rank 2. As a surjective homomorphism between two vector spaces over $\mathbb{C}$ where image has rank $\geq$ domain, $\Phi$ must be an isomorphism of $\mathbb{C}$-algebra.