The exercises are taken from the text, *Abstract Algebra* (third edition) by Dummit and Foote.

Page 356.3. Any finite abelian group $M$ is a $\mathbb{Z}$–module. Suppose $|M| = n$, then $nm = 0$ for all $m \in M$. Therefore $M$ is a torsion $\mathbb{Z}$–module. Take the infinite abelian group which is a direct product of $\mathbb{Z}/2\mathbb{Z}$. Then every elements is annihilated by 2.

Page 356.11. Let $\phi : M_1 \to M_2$ be a nonzero $R$-module homomorphism. Since $M_1$ is irreducible, $\ker \phi = 0$. Since $M_2$ is irreducible, $\text{im} \phi = M_2$. Therefore $\phi$ is an isomorphism. If $M = M_1 = M_2$, then every element in $\text{End}_R(M)$ is an isomorphism and therefore has an inverse. Hence $\text{End}_R(M)$ is a division ring.

Page 356.13. It can be verified that $\text{Hom}_R(R, R) \to R$ via $f \mapsto f(1)$ is an $R$–module isomorphism. Now If $F$ is a free $R$–module of rank $n$, then $F \cong R^n$. We will show that $\text{Hom}_R(A \times B, M) \cong \text{Hom}_R(A, M) \times \text{Hom}_R(B, M)$, where $A, B, M$ are $R$–modules.

Let $i_1 : A \to A \times B$, $i_2 : B \to A \times B$ be the inclusions, Define $\phi : \text{Hom}_R(A \times B, M) \to \text{Hom}_R(A, M) \times \text{Hom}_R(B, M)$ via $f \mapsto (f \circ i_1, f \circ i_2)$. This is an $R$-module homomorphism, since $\phi((rf + g)i_1, (rf + g)i_2) = (rfi_1 + gi_1, rfi_2 + gi_2) = r(fi_1, fi_2) + (gi_1, gi_2) = r\phi(f) + \phi(g)$. Define $\psi : \text{Hom}_R(A, M) \times \text{Hom}_R(B, M) \to \text{Hom}_R(A \times B, M)$ via $(\alpha, \beta) \mapsto ((a, b) \mapsto \alpha(a) + \beta(b))$. Then $\psi(r(\alpha_1, \beta_1) + (\alpha_2, \beta_2))(a, b) = \psi(r\alpha_1 + \alpha_2, r\beta_1 + \beta_2)(a, b) = r(\alpha_1 + \alpha_2)(a) + (r\beta_1 + \beta_2)(b) = r(\alpha_1(a) + \beta_1(b)) + \alpha_2(a) + \beta_2(b) = r\psi(\alpha_1, \beta_1)(a, b) + \psi(\alpha_2, \beta_2)(a, b)$, so $\psi$ is an $R$-module homomorphism. Clearly $\phi$ and $\psi$ are inverses.

Now $\text{Hom}_R(F, R) \cong \text{Hom}_R(R^n, R) \cong (\text{Hom}_R(R, R))^n \cong R^n \cong F$.

Page 376.9. Take $n \in \ker(\iota)$, then $0 = \iota(n) = 1 \otimes n$. By Problem 8(c), $rn = 0$ for some $r \in R$, and $\ker(\iota) \subseteq \text{Tor}(N)$. Conversely, take $n \in \text{Tor}(N)$ with $rn = 0$, then $\iota(n) = 1 \otimes n = \frac{1}{r} \otimes n = \frac{1}{r} \otimes 0 = 0$, and $n \in \ker(\iota)$.
Page 376.  

(a) Take any element \( m \in R/I \otimes R/j \), then
\[
m = \sum_{i=1}^{n} (r_i + I \otimes s_i + J) = \sum_{i=1}^{n} (1 + I \otimes r_is_i + J) = (1 + I) \otimes (\sum_{i=1}^{n} r_is_i) + J.
\]

(b) Denote \( \varphi : R/I \times R/J \to R/(I+J) \) via \((r+I, s+J) \mapsto rs+(I+J)\). This map is easily checked to be well-defined and \( R \)-balanced. By the universal property of tensor products, we get an induced \( R \)-module homomorphism \( \varphi : R/I \otimes R/J \to R/(I+J) \). Now for any \( r+(I+J) \in R/(I+J) \), it has preimage \( 1+I \otimes r+J \) under \( \varphi \). Take any \( m \in \ker(\varphi) \), by a) we can take
\[
m = 1 + I \otimes i + J = 1 + I \otimes i + J = i + I \otimes J = 0.
\]
Therefore \( \varphi \) is an isomorphism.