The exercises are taken from the text, Abstract Algebra (third edition) by Dummit and Foote.

Page 283.(a) Let $S$ be the set of all nonprincipal ideals of $R$, then $S$ is nonempty by assumption. Let $C$ be a chain of ideals $I_C \in S$, then it has a maximal $J = \cup I_C$. Suppose $J = (r)$ is principal, then $r \in I_C$ for some $I_C$. But then $I_C = J$ is then principal. Contradiction. Therefore $J$ is not principal, and $J \in S$. We have $J \in S$, and every chain in $S$ has a maximal element. By Zorn’s Lemma, $S$ has a maximal element.

(b) Suppose $I_a = (I, a) = R$, then $1 = ra + i$, and $b = rab + ib \in I$. Contradiction. We have $I \not\subset I_a \not\subset R$. Since $I$ is maximal in the set of nonprincipal ideals, we must have $I_a = (\alpha)$ for some $\alpha \in R$. Now $I \subset J$, and $b \in J$, so $I \not\subset I_b \subset J$. Therefore $J = (\beta)$ for some $\beta \in R$. By the definition of $J$, $I_a J = (\alpha\beta) \subset I$.

(c) If $x \in I \subset I_a = (\alpha)$, then $x = sa$ for some $s \in R$. By the definition of $J$, $s \in J$. Therefore $I \subset I_a J$, and $I = I_a J = (\alpha\beta)$. But $I$ is nonprincipal. Contradiction. $S$ must be nonempty, and $R$ is a PID.

Page 293. (a) Checking that $(\pi^{n+1})$ is an ideal in $(\pi^n)$ for non-negative $n$ is straight-forward. Let $\varphi : \mathbb{Z}[i] \rightarrow (\pi^n)/(\pi^{n+1})$ be the map of multiplication by $\pi^n$ via $\alpha \mapsto \alpha \pi^n + (\pi^{n+1})$. $\varphi$ is a group homomorphism as $\varphi(\alpha + \beta) = (\alpha + \beta) \pi^n + (\pi^{n+1}) = \alpha \pi^n + \beta \pi^n + (\pi^{n+1}) = \varphi(\alpha) + \varphi(\beta)$. Clearly this map is surjective. Furthermore, the kernel is the set of $\alpha$ such that $\pi^n \in (\pi^{n+1})$, i.e. $\alpha$ is divisible by $\pi$, and the kernel is $(\pi)$. Therefore $\mathbb{Z}[i]/(\pi) \cong (\pi^n)/(\pi^{n+1})$.

(b) From a), we know that $\mathbb{Z}[i]/(\pi) \cong (\pi^n)/(\pi^{n+1}) \cong \mathbb{Z}[i]/(\pi^{n+1})/\mathbb{Z}[i]/(\pi^n)$, and $|\mathbb{Z}[i]/(\pi^{n+1})| = |\mathbb{Z}[i]/(\pi)||\mathbb{Z}[i]/(\pi^n)|$. Since $n$ is arbitrary, $|\mathbb{Z}[i]/(\pi^n)| = |\mathbb{Z}[i]/(\pi)|^n$.

(c) We will first show that $\mathbb{Z}[i]/(\pi)$ has order $N(\pi)$ for $\pi$ irreducible. If $\pi = p$ for some prime $p \in \mathbb{Z}$, then $\mathbb{Z}[i]/(\pi)$ has order $p^2 = N(\pi)$. If not, then $\pi \bar{\pi} = p$ for some prime $p \in \mathbb{Z}$. Since $\pi$ and $\bar{\pi}$ are irreducible, the ideals generated by each of them is maximal. Furthermore, if $(\pi) = (\bar{\pi})$, then $\pi | \bar{\pi}$ and vice versa, i.e. they’re associates of each other. But this is impossible since real and imaginary parts have different signs are parity. Therefore $\pi$ and $\bar{\pi}$ are coprime. By the Chinese remainder
Now suppose similar reasoning as in a. Hence factorizations for some is a non-negative integer and N is the common norm of these elements. Therefore 3 = a^2 + 5b^2 for some a, b ∈ Z, but this is impossible.

b) If u ∈ R is a unit with uu' = 1, then 1 = N(1) = N(uu') = N(u)N(u'), where N(r) = a^2 + Db^2 for r = a + b√−D. Since N(u) is a non-negative integer and N(u)|1, we must have N(u) = 1. If u = v + w√−D, and N(u) = v^2 + Dw^2, since D ≥ 2, we must have v = ±1, and w = 0, i.e. R^× = {±1}.

c) If there’s a nonunit d dividing any of these elements, then N(d)|9, which is the common norm of these elements. Therefore 3 = N(d) = a^2 + 5b^2 for some a, b ∈ Z, but this is impossible.

2.a) Checking S^{−1}R is ID is straightforward. We’ll show that S^{−1}R is also PID. Take any ideal J ∈ S^{−1}R, and define I = {r ∈ R | s−1r ∈ J for some s ∈ S}. Note that since for any r/s ∈ S^{−1}R, r/s = r/s ∈ S^{−1}R, we can assume 1 ∈ S. Similarly, if we take a/b ∈ J, a/b ∈ S^{−1}R, it’s to see that I is indeed an ideal in R. Since R is PID, I = (a) for some a ∈ R. Now we claim that J = (a). Clearly, (a) ⊆ J. For any u/v ∈ J, u = ak for some k ∈ R, and u/v = k/a ∈ (a). Therefore the claim is proved, and S^{−1}R is PID. This is the strongest classification for a general S, since for example, if S = {1}, S^{−1}R = R, but if S = R\{0}, S^{−1}R is a field.

b) Take r/s ∈ S^{−1}R, since R is UFD, r ∈ R, r has a unique factorization s−1r. Since 1/s is a unit in S^{−1}R, r/s = 1/s s−1r is a factorization. Now suppose r/s can also be written as a product of s/ti with ti ∈ R and s/ti units. Since 1/s are units, we end up getting s−1r = t. Now these are factorizations for r ∈ R, and therefore must be unique up to units. Hence S^{−1}R is UFD. This is the strongest classification following similar reasoning as in a).
c) We’ll show that $S^{-1}R$ is ED. Since $\frac{p}{1}$ is the only prime in $S^{-1}R$, for any $\alpha = \frac{r}{s} = p^{a} \frac{a}{b}$, where $a, b$ are free of $p$, we can define a function $v(\alpha) = n$. We notice that the image of $v$ is in $\mathbb{Z}_+$, since $S$ is disjoint from $(p)$. Now for any $\beta = \frac{q}{t} = p^{m} \frac{c}{d}$, with $n \geq m$, $\alpha = \beta p^{(n-m)ad} + 0$. For $n \leq m$, we can take remainder $\gamma = \alpha$, and $v(\gamma) < v(\beta)$. Therefore $v$ is a euclidean norm.

Page 306, 12. Note that $xy - z^2 \in (x, z)$, so we have $(xy - z^2), (x, z)$ ideals of $R = \mathbb{Q}[x, y, z]$, with $(xy - z^2) \subseteq (x, z) \subseteq R$. By the third isomorphism theorem, $\mathbb{Q}[y] \cong R/(x, z) \cong R/(xy - z^2)/(x, z)/(xy - z^2)$, which is an ID. Therefore, $(xy - z^2)$ is prime. Since $\bar{xy} = \bar{z^2}$, it’s in $\bar{P}^2$. Note that $\bar{y}^k \in \bar{P}^2 \subseteq \bar{P}$ is impossible.