The exercises are taken from the text, Abstract Algebra (third edition) by Dummit and Foote.

Page 231.7. Let $Z$ denote the center of $R$. Then if $a, b \in Z$, $(ab)r = a(br) = a(rb) = r(ab)$, $(a + b)r = ar + br = ra + rb = r(a + b)$, so $ab, (a + b) \in Z$. Clearly $1 \in Z$. Also, $(-1)r = -r = r(-1)$, and $-1 \in Z$. Now $(-a)r = (-1)(ar) = (-1)(ra) = r((-1)a) = r(-a)$, and each element in $Z$ has an inverse. Since $R$ is associative and distributive, so is $Z$. Therefore $Z$ is a subring containing the identity. If $R$ is a division ring, then for $a \in Z$, there is an multiplicative inverse $a^{-1}$ of $a$. Now $ar = ra$ so $ra^{-1} = a^{-1}r$. Therefore, $a^{-1} \in Z$, and $Z$ is a field.

Page 231.13. a) $(ab)^k = a^k b^k = (a^k b) b^{k-1} = nb^{k-1} \equiv 0 \pmod{n}$

b) Suppose every prime divisor of $n$ also divides $a$. Write $n = p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}$, and $a = p_1^{m'_1} p_2^{m'_2} \cdots p_n^{m'_n} q$, where $q$ is not divisible by any $p_i$. Take $M = \max\{m_1, m_2, \cdots, m_n\}$, then clearly $a^M$ is divisible by $n$, and $\bar{a}$ is nilpotent in $Z/nZ$.

Conversely, suppose $a^k = nl$ for some integer $l$. Since $a^k$ and $a$ share the exact same prime factors, if there exists some prime factor $p$ of $n$ not dividing $a$ then $p$ does not divide $a^k$. But this is impossible since $n$ divides $a^k$. Therefore every prime divisor of $n$ is a divisor of $a$ is a necessary and sufficient condition.

$72 = 2^3 3^2$. Following the above argument, the nilpotent elements of $Z/72Z$ are 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 0.

c) Suppose $R$ has a nonzero nilpotent element $f : X \to F$, with $f^k = 0$. Since $f$ is nonzero, $f(x)$ is nonzero for some $x \in X$. But $0 = f^k(x) = f^{k-1}(x)f(x) = 0$, which means $f(x)$ is a zero-divisor. But $F$ is a field. Contradiction. Therefore such nonzero nilpotent element cannot exist.

Page 231.27. By definition, $R$ is all rational numbers $\frac{a}{b}$ such that $v_p(\frac{a}{b}) \geq 0$ together with zero. Therefore, $R$ consists of all rationals $\frac{a}{b}$ with $b$ has no factor of $p$. For a unit $u \in R$, with $u = \frac{a}{b}$, since the
inverse of $u$ is $\frac{b}{a}$, which must also be in $R$, we must have that both $a$ and $b$ have no factors of $p$.

Page 238, 4. Take any two nonzero elements $f, g \in R[[x]]$, then we can write $f = x^A \sum_{i=0}^{\infty} a_i x^i$ and $g = x^B \sum_{j=0}^{\infty} b_j x^j$, with $a_A, b_B$ nonzero. Then $fg = x^{A+B} \sum_{k=0}^{\infty} c_k x^k$, where $c_k = \sum_{l=0}^{k} a_l b_{k-l}$. In particular, since $R$ is ID, $c_0 = a_A b_B$ is nonzero, and $fg$ is nonzero. Hence $R[[x]]$ is also ID.

Page 247, 2. Suppose there is an isomorphism $f : \mathbb{Q}[x] \to \mathbb{Z}[x]$. Since only the trivial field is contained in $\mathbb{Z}[x]$, and the image of a field must also be a field, we have $f(\mathbb{Q}) = 0$, and $f$ is not an isomorphism.