1. (a) If \( \{T_\alpha(X)\} \) is a collection of topologies on \( X \), show that \( \bigcap T_\alpha(X) \) is a topology on \( X \). Is \( \bigcup T_\alpha(X) \) a topology on \( X \)?

(b) Show that there is a unique smallest topology on \( X \) containing all collections \( T_\alpha(X) \), and a unique largest topology contained in all \( \{T_\alpha(X)\} \).

(c) If \( X = \{a, b, c\} \), let \( T_1 = \emptyset, X, \{a\}, \{a, b\} \) and \( T_2 = \emptyset, X, \{a\}, \{b, c\} \). What are the smallest and largest topologies, as defined above?

2. For \( T(X) \) and \( T'(X) \) two topologies on \( X \), such that \( T'(X) \) is strictly finer than \( T(X) \), what can you say about the corresponding subspace topologies induced on \( Y \subset X \)?

3. Show that for the basis \( B(X) \), the topology it generates is the intersection of all topologies on \( X \) that contain \( B(X) \). Prove the same if \( B(X) \) is a subbasis.

4. (a) Show that 
\[
B_1(\mathbb{R}) = \{(a, b) | a < b, a, b \in \mathbb{Q}\}
\]
generates the standard topology on \( \mathbb{R} \).

(b) Show that 
\[
B_2(\mathbb{R}) = \{[a, b) | a < b, a, b \in \mathbb{Q}\}
\]
and 
\[
B_3(\mathbb{R}) = \{[a, b) | a < b, a, b \in \mathbb{R}\}
\]
do not generate the same topology. Note that \( B_3 \) generates the lower limit topology.

5. A map 
\[
f : X \to Y
\]
is an open map if for every open set \( U \in T(X) \), the image \( f(U) \in T(Y) \). Let \( \pi_1 : X \times Y \to X \) and \( \pi_2 : X \times Y \to Y \) be projection maps such that \( \pi_i(x_1, x_2) = x_i \). Describe a topology on \( X \times Y \) such that \( \pi_1 \) and \( \pi_2 \) are open maps.