

A REMARK ON MARTIN'S CONJECTURE

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ABSTRACT. We prove that the strong Martin conjecture is false. The counterexample is the first-order theory of infinite atomic Boolean algebras. We show that for this class of Boolean algebras, the classification of their $(\omega + \omega)$ -elementary theories can be reduced to the classification of the elementary theories of their quotient algebras modulo the Frechét ideals.

Martin's conjecture is a strengthening of Vaught's conjecture. Let T be a complete consistent theory in $L_{\omega, \omega}$. Define $L_1(T)$ to be the smallest fragment of $L_{\omega_1, \omega}$ containing $L_{\omega, \omega} \cup \{\bigwedge_{\varphi \in p} \varphi(\bar{x}) \mid p \in S_n(T), n < \omega\}$. For a model A of T , let $T_1(A)$ be the complete theory of A in the language $L_1(T)$. Martin's conjecture says that, if T has fewer than 2^{\aleph_0} countable models, then $T_1(A)$ is \aleph_0 -categorical for all countable models A of T . The statement implies that, if T has fewer than 2^{\aleph_0} countable models, then the Scott ranks of all countable models of T are $\leq \omega + \omega$; thus in particular, T has at most countably many models up to isomorphism.

In [W] C. M. Wagner considered a strengthening of Martin's conjecture which she called strong Martin conjecture. This is by adding to the conclusion of Martin's conjecture the statement that, if T has 2^{\aleph_0} countable models, then T has 2^{\aleph_0} distinct completions in $L_1(T)$. Since $L_1(T)$ is a countable fragment of $L_{\omega+\omega, \omega}$, this implies that T has 2^{\aleph_0} models with distinct complete theories in $L_{\omega+\omega, \omega}$.

Wagner verified the strong Martin conjecture for theories of linear orders and of one unary function. In this paper we make the remark that the strong Martin conjecture is false. The counterexample is in the theory of Boolean algebras. Throughout this paper let us fix the signature $L = \langle \cup, \cap, C, 0, 1 \rangle$ of Boolean algebras. Our main result is the following theorem.

Theorem 1. *There is a complete consistent theory T in $L_{\omega, \omega}$ such that*

- (i) *T has 2^{\aleph_0} countable models, and*
- (ii) *There are at most countably many models of T with distinct complete theories in $L_{\omega+\omega, \omega}$.*

Before further explaining the theorem let us recall Tarski's analysis of the elementary theories of Boolean algebras (c.f. [K], §18). For any Boolean algebra A , the *Ershov-Tarski ideal* $I(A)$ is the ideal of A defined by

$$I(A) = \{x \cup y \mid x \text{ is atomic and } y \text{ is atomless}\}.$$

By induction we can define for $n \in \omega$ the n -th iterated *Ershov-Tarski ideal* $I^n(A)$ of A as follows. Let $I^0(A) = \{0\}$ and $I^1(A) = I(A)$. For $n > 0$, let $\pi_n : A \rightarrow A/I^n(A)$ be the canonical homomorphism. Then let $I^{n+1}(A) = \pi_n^{-1}[I(A/I^n(A))]$.

The Tarski invariants are triples taken from the countable set

$$\{(-1, 0, 0), (\omega, 0, 0)\} \cup \{(k, l, m) \mid k \in \omega, l \in \{0, 1\}, m \in \omega \cup \{\omega\}, l + m \neq 0\}.$$

To be specific, for a Boolean algebra A , the *elementary invariant* $\text{inv}(A)$ of A is defined as follows. $\text{inv}(A) = (-1, 0, 0)$ if $A = \{0\}$ is the trivial Boolean algebra. $\text{inv}(A) = (\omega, 0, 0)$ if $I^n(A) \neq A$ for all $n \in \omega$. Otherwise, let k be the least $n \in \omega$ such that $I^n(A) \neq A$ but $I^{n+1}(A) = A$; let $l = 0$ iff there is no atomless element in $A/I^k(A)$; and let m be the number of atoms in $A/I^k(A)$. Then $\text{inv}(A) = (k, l, m)$.

Tarski's analysis culminates in his theorem that the elementary invariants are complete for elementary theories of Boolean algebras. Thus in particular, there are only countably many distinct complete elementary theories of Boolean algebras.

Now coming back to our theorem, the theory T in question is just the theory of infinite atomic Boolean algebras. By Tarski's analysis, the models of T are exactly those which have elementary invariant $(0, 0, \omega)$, and T is in fact a complete consistent theory. It is also well-known that T has 2^{\aleph_0} countable models (c.f., e.g., [I]). Thus to establish Theorem 1, it only remains to verify clause (ii) of the conclusion. For this we prove the following theorem. For any Boolean algebra A , we denote by $F(A)$ the ideal generated by the atoms of A , and call it the *Frechét ideal* of A .

Theorem 2. *Let A and B be infinite atomic Boolean algebras. Then $A \equiv_{\omega+\omega} B$ iff $A/F(A) \equiv B/F(B)$.*

Now Theorem 1 follows from Theorem 2 immediately by Tarski's analysis. Note also that to get Theorem 1 we only use one direction of the equivalence in the conclusion of Theorem 2, so we are obtaining additional information from Theorem 2.

The rest of the paper is organized as follows. Section 1 contains the proof of the inessential direction of Theorem 2. In section 2 we prove the other direction. Then we conclude with some corollaries of the proofs and a discussion of some related problems, which constitute section 3.

1. From algebras to quotients.

In this section we show that if two arbitrary Boolean algebras are $(\omega + \omega)$ -elementarily equivalent, then their quotients modulo the Frechét ideals are elementarily equivalent. This is done by observing that any first order sentence describing a property of the quotient can be translated to an infinitary sentence describing a property of the original algebra.

Lemma 1. *For any sentence $\varphi \in L_{\omega, \omega}$ there is a sentence $\varphi^* \in L_{\omega+\omega, \omega}$ such that, for any Boolean algebra A ,*

$$A/F(A) \models \varphi \iff A \models \varphi^*.$$

Proof. For $a, b \in A$, let $a \setminus b$ be an abbreviation of $a \cap C(b)$. Let $\psi(x_1, x_2)$ be an $L_{\omega+\omega, \omega}$ formula expressing that $(x_1 \setminus x_2) \cup (x_2 \setminus x_1)$ is in the Frechét ideal, e.g.,

$\bigvee_{n,m \in \omega} B_{n,m}(x_1, x_2)$ where $B_{n,m}$ states that there are distinct atoms a_1, \dots, a_n and b_1, \dots, b_m such that $x_1 \cup a_1 \cup \dots \cup a_n = x_2 \cup b_1 \cup \dots \cup b_m$. For all formulas $\varphi(\bar{x})$ we define $\varphi^*(\bar{x})$ by induction on the form of φ .

If φ is the atomic formula $x_1 = x_2$, then φ^* is just $\psi(x_1, x_2)$. If φ is $x_1 \cup x_2 = x_3$, then φ^* is $\psi(x_1 \cup x_2, x_3)$. If φ is $x_1 \cap x_2 = x_3$, then φ^* is $\psi(x_1 \cap x_2, x_3)$. If φ is $C(x_1) = x_2$, then φ^* is $\psi(C(x_1), x_2)$. For nonatomic formulas the induction is the trivial one, i.e., φ^* is built up from the atomic cases in exactly the same fashion as φ is.

By induction it is easy to see that for any formula $\varphi(\bar{x}) \in L_{\omega, \omega}$, Boolean algebra A , tuples $\bar{c} = \langle c_1, \dots, c_n \rangle \in A/F(A)$ and $\bar{d} = \langle d_1, \dots, d_n \rangle \in A$ with $d_i \in c_i$ for every $i = 1, \dots, n$, we have $A/F(A) \models \varphi(\bar{c})$ iff $A \models \varphi^*(\bar{d})$.

In particular, if φ is a sentence, then $A/F(A) \models \varphi$ iff $A \models \varphi^*$. \square

Now using Lemma 1 we can show one direction of Theorem 2 in its full generality.

Lemma 2. *Let A and B be Boolean algebras. If $A \equiv_{\omega+\omega} B$ then $A/F(A) \equiv B/F(B)$.*

Proof. Suppose $A \equiv_{\omega+\omega} B$. Then for any sentence $\varphi \in L_{\omega, \omega}$,

$$A/F(A) \models \varphi \Leftrightarrow A \models \varphi^* \Leftrightarrow B \models \varphi^* \Leftrightarrow B/F(B) \models \varphi. \quad \square$$

In fact a lot more is true in this direction. For any formula $J(x) \in L_{\omega+\omega, \omega}$ defining an ideal in an arbitrary Boolean algebra, we have that $A \equiv_{\omega+\omega} B$ implies $A/J(A) \equiv B/J(B)$.

2. From quotients to algebras.

In this section we prove the backward direction of Theorem 2, i.e., if two models of T have elementarily equivalent quotients modulo Frechét ideals, then they are $(\omega + \omega)$ -elementarily equivalent. We use games to deal with these notions of elementary equivalence. For the first-order elementary equivalence we use the standard Ehrenfeucht-Fraissé game (c.f., e.g. [H]). For the $(\omega + \omega)$ -elementary equivalence we use the game notion defined below.

Let $n \in \omega$ and M, N be models of T . The game $G_n(M, N)$ is played in the same manner as an $n + 1$ step Ehrenfeucht-Fraissé game:

$$\begin{array}{l} G_n(M, N) \\ \text{I} \quad x_0 \quad x_1 \quad \cdots \quad x_n \\ \text{II} \quad y_0 \quad y_1 \quad \cdots \quad y_n \end{array}$$

where for each $i \leq n$ the elements x_i and y_i are from different structures. Suppose $a_0, \dots, a_n \in M$, $b_0, \dots, b_n \in N$ are all the elements played by the players. Then player II wins if $(M, a_0, \dots, a_n) \equiv (N, b_0, \dots, b_n)$. By a standard argument we have that $M \equiv_{\omega+\omega} N$ iff player II has a winning strategy in $G_n(M, N)$ for any $n \in \omega$.

Let us denote by $E_n(M, N)$ the $n + 1$ step Ehrenfeucht-Fraissé game played on the structures M and N . For notational convenience we also denote by $H_n(M, N)$ the $n + 1$ step Ehrenfeucht-Fraissé game played on the structures $M/F(M)$ and $N/F(N)$. Then $M/F(M) \equiv N/F(N)$ iff player II has winning strategies in the games $H_n(M, N)$ for all $n \in \omega$.

For any Boolean algebra A and $a \in A$, we denote by \hat{a} the restricted algebra with domain $\{x \in A \mid x \leq a\}$.

Lemma 3. *Let $n \in \omega$, $a_0, \dots, a_n \in M$ and $b_0, \dots, b_n \in N$. Then $(M, a_0, \dots, a_n) \equiv (N, b_0, \dots, b_n)$ iff*

- (a) $(\widehat{a_0}, a_0 \cap a_1, \dots, a_0 \cap a_n) \equiv (\widehat{b_0}, b_0 \cap b_1, \dots, b_0 \cap b_n)$, and
- (b) $(\widehat{C(a_0)}, C(a_0) \cap a_1, \dots, C(a_0) \cap a_n) \equiv (\widehat{C(b_0)}, C(b_0) \cap b_1, \dots, C(b_0) \cap b_n)$.

Proof. Fix an arbitrary $m \in \omega$. Denote $E = E_m((M, a_0, \dots, a_n), (N, b_0, \dots, b_n))$, $F_1 = E_m((\widehat{a_0}, a_0 \cap a_1, \dots, a_0 \cap a_n), (\widehat{b_0}, b_0 \cap b_1, \dots, b_0 \cap b_n))$ and $F_2 = E_m((\widehat{C(a_0)}, C(a_0) \cap a_1, \dots, C(a_0) \cap a_n), (\widehat{C(b_0)}, C(b_0) \cap b_1, \dots, C(b_0) \cap b_n))$.

On the one hand, a winning strategy for player II in E is also a winning strategy for player II in either F_1 or F_2 . On the other hand, the game E can be split into two boards with the games F_1 and F_2 each played on a board, in the following sense. Any element $c \in M$ (or $d \in N$) played in E is split into two elements $a_0 \cap c$ (or $b_0 \cap d$) and $C(a_0) \cap c$ (or $C(b_0) \cap d$), to be played on the boards of F_1 and F_2 , respectively. And elements played on the boards of F_1 and F_2 can be joined to give elements to be played in E . In an obvious manner, the winning strategies of player II in F_1 and F_2 can thus be combined to give a winning strategy for player II in E . \square

By unraveling the definition of $G_n(M, N)$, Lemma 3 immediately implies the following lemma.

Lemma 4. *Let $n \in \omega$. Player II has a winning strategy in $G_{n+1}(M, N)$ iff*

- (a) for any $a \in M$ there is $b \in N$ such that player II has winning strategies in both $G_n(\widehat{a}, \widehat{b})$ and $G_n(\widehat{C(a)}, \widehat{C(b)})$, and
- (b) for any $b \in N$ there is $a \in M$ such that player II has winning strategies in both $G_n(\widehat{a}, \widehat{b})$ and $G_n(\widehat{C(a)}, \widehat{C(b)})$.

The following lemma is our main lemma, which, in an essential way, uses our assumption that the Boolean algebras we consider are infinite atomic Boolean algebras.

Lemma 5. *For any $n \in \omega$, if player II has a winning strategy in $H_{n+1}(M, N)$, then so does player II in $G_n(M, N)$.*

Proof. By induction on n . The base case is $n = 0$. Without loss of generality we assume that the player I has played $a \in M$ in his first (and only) move in $G_0(M, N)$. We are to show that there is $b \in N$ such that $(M, a) \equiv (N, b)$. By Lemma 3, this is equivalent to find a b such that $\widehat{a} \equiv \widehat{b}$ and $\widehat{C(a)} \equiv \widehat{C(b)}$. For this we play the game $H_1(M, N)$. Let player I play $a/F(M)$ in his first move. Suppose player II responds according to her winning strategy and plays $c \in N/F(N)$. There are three cases.

Case 1. $a \notin F(M)$ and $C(a) \notin F(M)$. In this case we let b be an arbitrary element of c . Then since c was played according to a winning strategy of player II, it is neither $0/F(N)$ nor $1/F(N)$. These conditions imply that all of the algebras \widehat{a} , $\widehat{C(a)}$, \widehat{b} and $\widehat{C(b)}$ are infinite atomic Boolean algebras. Therefore we have $\widehat{a} \equiv \widehat{b}$ and $\widehat{C(a)} \equiv \widehat{C(b)}$, as required.

Case 2. $a \in F(M)$. Let m be the number of distinct atoms below a . In this case c must be $0/F(N)$. Arbitrarily choose m distinct atoms in N and let b be their

union. Then \hat{a} is in fact isomorphic to \hat{b} . Both $\widehat{C(a)}$ and $\widehat{C(b)}$ are infinite atomic Boolean algebras, so $\widehat{C(a)} \equiv \widehat{C(b)}$.

Case 3. $C(a) \in F(M)$. This is similarly handled as for Case 2. And we are done with the base case.

In general we assume that player II has a winning strategy σ in $H_{n+2}(M, N)$. Consider $G_{n+1}(M, N)$ and use Lemma 4. By symmetry it suffices to verify clause (a) of Lemma 4. Let $a \in M$. We obtain the required $b \in N$ as follows. First play the game $H_{n+2}(M, N)$ and let player I's first move be $a/F(M)$. Let player II responds according to σ and suppose she plays $c \in N/F(N)$. Then we pick $b \in c$ in exactly the same way as in the above base case. We claim that player II has winning strategies in both $G_n(\hat{a}, \hat{b})$ and $G_n(\widehat{C(a)}, \widehat{C(b)})$.

Again by symmetry, it suffices to verify the statement for $G_n(\hat{a}, \hat{b})$. By the inductive hypothesis, it suffices to show that player II has a winning strategy in $H_{n+1}(\hat{a}, \hat{b})$. Note that $\hat{a}/F(\hat{a}) = a/\widehat{F(M)}$ and $\hat{b}/F(\hat{b}) = b/\widehat{F(N)}$. Therefore by unraveling the definition of $H_{n+1}(\hat{a}, \hat{b})$, we have that σ is a winning strategy for player II in this game. \square

Now it follows immediately from Lemma 5 that if A and B are infinite atomic Boolean algebras and $A/F(A) \equiv B/F(B)$, then $A \equiv_{\omega+\omega} B$. This completes the proof of Theorem 2.

3. Some further problems.

Martin's conjecture holds, in fact in a strong sense, for Boolean algebras. It is well-known that every complete consistent first-order theory of Boolean algebras has either one or 2^{\aleph_0} countable models (see e.g. [I]).

One could naturally ask: what is the minimal countable ordinal α such that, for any complete consistent first-order theory T of Boolean algebras with 2^{\aleph_0} countable models, there are 2^{\aleph_0} countable models of T which are pairwise *non- α -elementarily* equivalent? The minimal such limit ordinal might just be $\omega \cdot 3$.

It seems that to answer the above question we need to classify the $(\omega + \omega)$ -elementary theories for Boolean algebras with other elementary invariants. As a matter of fact, Theorem 2 can be thought as the first step along this line of research. A modification of the proof of Theorem 2 by considering maximal atomless elements in Boolean algebras yields the following.

Theorem 3. *There are only countably many Boolean algebras of elementary invariant $(0, 1, \omega)$ with distinct complete theories in $L_{\omega+\omega, \omega}$.*

We formulate some problems to consider for the next step. For this we need some definitions first. Let J and J' be operations that assign to every Boolean algebra A an ideal of A. Let π_J^A be the canonical homomorphism from A onto $A/J(A)$. Let $J \circ J'$ be the operation defined by

$$(J \circ J')(A) = (\pi_{J'}^A)^{-1}[J(A/J'(A))].$$

It is easy to check that \circ is associative. For $n \in \omega$ define the iterate J^n by induction as follows. Let $J^0 = I^0$ and $J^1 = J$. For $n > 0$, let $J^{n+1} = J \circ J^n$. The earlier definition of iterated Ershov-Tarski ideals conforms to the current definition. We

can also iterate into transfinite by defining for limit ordinals λ the operation J^λ by $J^\lambda(A) = \bigcup_{\alpha < \lambda} J^\alpha(A)$. We call an operation ω -short if it is $I^m \circ F \circ I^n$ for some $m, n \in \omega$ or I^α for some $\alpha < \omega + \omega$. Note that ω -short operations are definable by $L_{\omega+\omega, \omega}$ formulas.

For a pair of Boolean algebras A and B , let us consider the following three statements:

- (A) $A \equiv_{\omega+\omega} B$.
- (B) For any formula $J(x) \in L_{\omega+\omega, \omega}$ defining an ideal in an arbitrary Boolean algebra, $A/J(A) \equiv B/J(B)$.
- (C) For any ω -short operation J , $A/J(A) \equiv B/J(B)$.

By the method of Lemmas 1 and 2 we have that $(A) \Rightarrow (B) \Rightarrow (C)$. We do not know if any of the reverse directions holds.

Finally note that the proof of Theorem 2 also yields the following result.

Theorem 4. *Let α be an ordinal. Suppose A and B are Boolean algebras such that for any $\beta < \alpha$, $A/F^\beta(A)$ and $B/F^\beta(B)$ are atomic. Then $A \equiv_{\omega \cdot (1+\alpha)} B$ iff $A/F^\alpha(A) \equiv B/F^\alpha(B)$. In particular, if A and B are superatomic Boolean algebras with distinct superatomicity types (γ, n) and (δ, m) respectively, then $A \equiv_{\omega \cdot (1+\alpha)} B$ iff $\alpha < \min(\gamma, \delta)$.*

An immediate consequence of the theorem is the well known fact that if A is a superatomic Boolean algebra with superatomicity type (γ, n) , then the Scott rank of A is $\leq \omega \cdot (1 + \gamma)$.

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