On the classification of Polish metric spaces up to isometry

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(A) A *Polish metric space* is a complete separable metric space (X, d). Our first goal in this paper is to determine the exact complexity of the classification problem of Polish metric spaces up to isometry. Our work was motivated by a recent paper of Vershik [1998], where the author remarks (in the beginning of Section 2): "The classification of Polish spaces up to isometry is an enormous task. More precisely, this classification is not 'smooth' in the modern terminology." The first main theorem (Theorem 1 below) quantifies precisely the enormity of this task.

We will first summarize the basic ideas of a theory of complexity of classification problems, which will help to put our results in perspective. Detailed expositions can be found, e.g., in Hjorth [2000], Kechris [1999], [2000].

In mathematics one frequently deals with problems of classification of various objects up to some notion of equivalence by invariants. Quite often these objects can be viewed as forming a definable (Borel, analytic, etc.) subset X of a *standard Borel space* $\hat{X}$ (i.e., a Polish space with its associated σ-algebra of Borel sets), and the equivalence relation as a definable (Borel, analytic, etc.) equivalence relation E on X. A complete classification of X up to E consists then of finding a set of invariants I and a map $c : X \to I$ such that $xEy \Leftrightarrow c(x) = c(y)$. For this to be of interest both I and c must be as simple and concrete as possible.

For our purposes, the simplest case is when the invariants are concrete enough so that they can be represented as elements of a standard Borel space (and the map c is fairly explicitly definable). More precisely let us call E (and the classification problem it represents) *concretely classifiable* (or *smooth* or *tame*) if there is a standard Borel space Y and a Borel (measurable) map $c : X \to Y$ such that $xEy \Leftrightarrow c(x) = c(y)$.

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To apply these ideas to the problem of isometry of Polish metric spaces, we first indicate how we view any such space as an element of a standard Borel space, in other words we describe a standard Borel space of Polish metric spaces. One natural way to do that is the following. Fix a universal Polish metric space, like the *Urysohn space* $\mathbb{U}$ (which we will discuss extensively in §2 below). Then every Polish metric space is, up to isometry, a closed subspace of $\mathbb{U}$, and we can view $F(\mathbb{U})$, the standard Borel space of closed subsets of $\mathbb{U}$ with the Effros Borel structure (see §1 below), as the *space of Polish metric spaces.* Denote then by $\cong_i$ the equivalence relation of isometry between metric spaces. Our problem is to understand the complexity of $\cong_i$ on Polish metric spaces, or, equivalently, closed subsets of $\mathbb{U}$.

First let us note that if we restrict $\cong_i$ to the space $K(\mathbb{U})$ of compact subsets of $\mathbb{U}$, in other words if we consider the isometry problem for compact metric spaces, then already Gromov (see, e.g., Gromov [1999, $3.11\frac{1}{2}_+$ or 3.27]) has shown that it is concretely classifiable. However, as Vershik [1998] points out, the classification of general Polish metric spaces up to isometry, is not concretely classifiable, thus quite complicated, in some sense. But can we make this more precise and calculate how complicated it really is? This is the problem that we address in this paper.

To arrive at an answer, one first has to define in what sense a classification problem is at most as complicated as another. This is made precise by means of the concept of reducibility between equivalence relations. If E, F are equivalence relations on subsets X, Y resp., of standard Borel spaces, we say that E is *Borel reducible* to F, in symbols,

$$E \leq_B F$$

if there is a Borel map $f : X \to Y$ such that

$$xEy \Leftrightarrow f(x)Ff(y).$$

Intuitively, this means that any complete invariants for F work as well for E (after composing with f) and therefore, in some sense, the classification problem represented by E is at most as complicated as that of F. Also E is *Borel bireducible* with F, in symbols

$$E \sim_B F \Leftrightarrow E \leq_B F \;\&\; F \leq_B E,$$

means that the classification problems represented by E, F have the same complexity. Finally,

$$E <_B F \Leftrightarrow E \leq_B F \;\&\; F \not\leq_B E,$$

signifies that the classification problem of E is strictly simpler than that of F.

The (partial pre-)order $\leq_B$ imposes a hierarchy of complexity on classification problems and our goal here is to find the place of isometry of Polish metric spaces in this hierarchy. In the study of this subject several important benchmarks have been discovered, which can be used to calibrate the difficulty of specific classification problems that come up in various fields of mathematics. We will review the ones that are relevant to us here. See Becker-Kechris [1996] for more details.

For any Polish group G and Borel action $(g, x) \mapsto g \cdot x$ of G on a standard Borel space X (a *Borel G-space* for short) we denote by E_G^X the corresponding orbit equivalence relation

$$xE_G^X y \Leftrightarrow \exists g(g \cdot x = y).$$

(This is an analytic but not always Borel equivalence relation.) It turns out that among all E_G^X, with G fixed, there is a most complex, i.e., universal, one. In other words, there is a Borel G-space X such that for all Borel G-spaces Y we have $E_G^Y \leq_B E_G^X$. It is unique up to $\sim_B$ and we denote it by E_G^∞. Furthermore, letting now G vary over all Polish groups, there is a universal relation of the form E_G^∞. This is again unique up to $\sim_B$ and we call it the *universal equivalence relation induced by a Borel action of a Polish group.* One realization of it is E_G^∞, where G is either the homeomorphism group of the Hilbert cube or the isometry group of the Urysohn space (this follows from the results of Uspenskiĭ [1986], [1990] that these groups are universal Polish groups, i.e., contain every Polish group as a closed subgroup.) In many ways, that the theory of Borel reducibility makes precise, this is an enormously complex equivalence relation (it is certainly not concretely classifiable and not Borel but these are rather mild indications of its complexity). Our first result now computes the complexity of the isometry classification of Polish metric spaces as being precisely that of the universal equivalence relation induced by a Borel action of a Polish group. More precisely:

Theorem 1. *The equivalence relation of isometry of Polish metric spaces, $\cong_i$, is Borel bireducible with the universal equivalence relation induced by a Borel action of a Polish group.*

This settles the question concerning isometry for general Polish metric spaces. It is however of further interest to understand the complexity of the isometry problem for special classes of Polish metric spaces. For example, we have seen that for compact metric spaces it is concretely classifiable. The next step along these lines would be to calculate the complexity of isometry on locally compact Polish metric spaces.

Recall that $E^{\infty}_{S_\infty}$ is the universal equivalence relation induced by a Borel action of the infinite symmetric group S_∞ of all permutations of $\mathbb{N}$. This is much smaller in terms of the ordering $<_B$, than the universal equivalence relation induced by a Borel action of a Polish group. A concrete realization of $E^{\infty}_{S_\infty}$ (see Becker-Kechris [1996]) is *graph isomorphism*, i.e., the isomorphism relation between countable graphs. Then it is not hard to see that isometry of discrete Polish spaces has exactly the same complexity as graph isomorphism and therefore isometry of locally compact Polish metric spaces is at least as complex as graph isomorphism.

Our results on the isometry groups of such spaces, which we will discuss shortly, led us to the conjecture that in fact isometry of locally compact Polish metric spaces is Borel reducible to graph isomorphism, and therefore it has exactly the same complexity as graph isomorphism. Hjorth has recently shown that a weaker form of this conjecture is in fact true, namely that isometry of locally compact Polish spaces is reducible by a provably $\mathbf{\Delta}^1_2$ function to graph isomorphism. This provides strong evidence for the truth of the conjecture.

One can look further at important subclasses of locally compact spaces. Recall that a space is *0-dimensional* if it has a clopen basis. We now have:

Theorem 2. *The equivalence relation of isometry on 0-dimensional locally compact Polish metric spaces is Borel bireducible with graph isomorphism.*

At the other extreme are the connected locally compact spaces. Let us denote by E_∞ the universal equivalence relation induced by a Borel action of a countable group. Equivalently, this is the universal countable Borel equivalence relation, where a Borel equivalence relation is *countable* if all of its equivalence classes are countable (see, e.g., Dougherty-Jackson-Kechris [1994]). Again E_∞ is much smaller, in terms of $<_B$, than graph isomorphism. It is not hard to see that E_∞ is Borel reducible to the isometry of connected locally compact Polish metric spaces. Again results on their isometry groups motivated our conjecture that the isometry of connected locally compact Polish metric spaces is Borel bireducible with E_∞. In fact we conjectured this for an even wider class of such spaces which we called *pseudo-connected* (see Section 5 for the precise definition). This class contains not only the connected spaces but also the Heine-Borel spaces. (A metric space (X, d)

is *Heine-Borel* if its closed bounded subsets are compact.) Thus there are many 0-dimensional spaces (like the p-adics) which are pseudo-connected. This conjecture has now been confirmed by Hjorth.

Theorem 3 (Hjorth). *The equivalence relation of isometry of connected locally compact Polish metric spaces is Borel bireducible with the universal countable Borel equivalence relation. The same is also true for pseudo-connected and Heine-Borel locally compact Polish metric spaces.*

In another direction, we can compute exactly the complexity of isometry of another subclass, namely ultrametric Polish spaces (Recall that (X, d) is *ultrametric* if $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.) Every ultrametric space is 0-dimensional.

Theorem 4. *The equivalence relation of isometry of ultrametric Polish spaces is Borel bireducible with graph isomorphism.*

We do not know the exact complexity of isometry of 0-dimensional Polish metric spaces but John Clemens has shown that it is strictly bigger than graph isomorphism.

John Clemens has also found another proof of the result that the universal equivalence relation induced by a Borel action of a Polish group is Borel reducible to the isometry of Polish metric spaces. His method is quite different from ours and produces also very interesting lower bounds for the complexity of isometry on other classes of Polish metric spaces. These will appear in his U.C. Berkeley Ph.D. Thesis.

(B) It turns out that our work also gives some interesting applications to the study of isometries of various metric spaces. Our first result here characterizes the isometry groups of Polish metric spaces. The topology on isometry groups is always the pointwise convergence topology.

Theorem 5. *Up to (topological group) isomorphism the isometry groups of Polish metric spaces are exactly the Polish groups.*

We then consider the case of locally compact separable metric spaces (X, d), where d is not necessarily complete. For any such space it still turns out that its isometry group is Polish. We look first at the subclass of such spaces, which we called *pseudo-connected*. For such spaces, we show the following:

Theorem 6. *Let X be a pseudo-connected locally compact separable metric space. Then its isometry group is locally compact.*

This generalizes a result of van Dantzig-van der Waerden [1928] (see also Strantzalos [1974], [1989]) for the connected case.

Using this, and some further constructions, we can characterize completely the isometry groups of locally compact separable metric spaces.

Theorem 7. *Up to (topological group) isomorphism, the isometry groups of locally compact separable metric spaces are exactly the closed subgroups of products*

$$\prod_n (S_\infty \ltimes G_n^{\mathbb{N}}),$$

where (G_n) *is a sequence of locally compact Polish groups and* $S_\infty \ltimes G^{\mathbb{N}}$ *is the semi-direct product of* S_∞ *and* $G^{\mathbb{N}}$*, where* S_∞ *acts on* $G^{\mathbb{N}}$ *by* $g \cdot x(i) = x(g^{-1}(i))$*. Moreover, this class of groups is also, up to isomorphism, the same as the class of groups of isometries of locally compact Polish metric spaces, and also the same as the class of groups of isometries of* σ*-compact Polish metric spaces.*

We use this characterization to get also information about actions of such isometry groups.

Theorem 8. *Let* H *be the isometry group of a locally compact separable metric space. Let* Y *be a Borel* H*-space with associated orbit equivalence relation* E_H^Y*. Then* E_H^Y *is Borel reducible to graph isomorphism.*

This extends a result of Hjorth [2000] who proved such a theorem for countable products of locally compact Polish groups.

It is not difficult to see that, up to isomorphism, the isometry groups of 0-dimensional locally compact Polish metric spaces are exactly the closed subgroups of S_∞. We do not know an exact characterization of the isometry groups of pseudo-connected (or for that matter connected or Heine-Borel) locally compact metric spaces. Concerning ultrametric Polish spaces it is not hard to see that these isometry groups are, up to isomorphism, closed subgroups of S_∞ but we do not know if they include all of them.

(C) The paper is organized as follows: In §1 we discuss various preliminaries. In §2 we give the proof of Theorem 1 (see Theorem 2.1). A crucial tool here is the use of the Urysohn space $\mathbb{U}$. In §3 we characterize the isometry groups of Polish metric spaces and prove Theorem 5 (see Theorem 3.1). In §4 we discuss some special cases of Polish metric spaces and prove Theorems 2 and 4 (see Theorems 4.3 and 4.4). In §5 we study isometries of pseudo-connected locally compact separable metric spaces and prove Theorem 6 (see

5.6, i)), and in §6 we characterize the isometries of general locally compact separable metric spaces and prove Theorems 7 and 8 (see 6.3 and 6.9, resp.). In §7 we give the proof of Hjorth's Theorem 3 (see 7.1) and discuss its implications for the isometry problem of locally compact Polish metric spaces. In §8 we discuss certain aspects of the proof of Theorem 1, which lead to some interesting analogies with model theory. Finally, in §9 we discuss various open problems.

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1 Preliminaries

We will review here some basic background in descriptive set theory needed below.

1A. A *Polish metric space* (X, d) is a complete separable metric space. We often write it simply as X. A *Polish space* is a topological space homeomorphic to a Polish metric space. A *standard Borel space* is a measurable space $(X, \mathbf{\Sigma})$, where X is a Polish space and $\mathbf{\Sigma}$ is its σ-algebra of Borel sets. Members of $\mathbf{\Sigma}$ are called the *Borel sets* of X and a function $f : X \to Y$, where X, Y are standard Borel spaces is *Borel (measurable)* if the inverse image of a Borel set in Y is a Borel set in X. Similarly a function $g : X_0 \to Y_0$, where $X_0 \subseteq X, Y_0 \subseteq Y$, is *Borel* if the inverse image of a relatively Borel subset of Y_0 is relatively Borel in X_0. This is equivalent (see Kechris [1995, 12.2]) to saying that g admits an extension $f : X \to Y$ which is Borel. A subset $A \subseteq X$ is *analytic* if there is a standard Borel space Y and a Borel map $f : Y \to X$ with $f(Y) = A$. It is *co-analytic* if its complement is analytic.

For each Polish space X, $F(X)$ is the standard Borel space of all closed subsets of X with the *Effros Borel structure*, i.e., the σ-algebra generated by the sets of the form $\{F \in F(X) : F \cap U \neq \emptyset\}$, where U varies over the open subsets of X.

1B. A *Polish group* G is a topological group whose underlying topology is Polish. An action $(g, x) \in G \times X \mapsto g \cdot x \in X$ of G on X, where X is a standard Borel space, is Borel if the function $a(g, x) = g \cdot x$ is Borel. In this case, we also say that X is a *(standard) Borel G-space*. We denote by E_G^X

the corresponding *orbit equivalence relation* on X

$$xE_G^X y \Leftrightarrow \exists g(g \cdot x = y).$$

Clearly E_G^X is an analytic (as a subset of $X \times X$) equivalence relation.

1C. Given equivalence relations E, F on subsets X, Y, resp., of standard Borel spaces, we say that E is *Borel reducible* to Y, in symbols

$$E \leq_B F,$$

if there is a Borel map $f : X \to Y$ with

$$xEy \Leftrightarrow f(x)Ff(y).$$

We say that E, F are *Borel bireducible*, in symbols

$$E \sim_B F,$$

if $E \leq_B F$ and $F \leq_B E$. Finally, put

$$E <_B F \Leftrightarrow E \leq_B F \ \& \ F \not\leq_B E.$$

A Borel equivalence relation E on a subset X of a standard Borel space is called *concretely classifiable* (or *smooth* or *tame*) if it is Borel reducible to equality on some standard Borel space, i.e., there is a standard Borel space Y and a Borel map $f : X \to Y$ with

$$xEy \Leftrightarrow f(x) = f(y).$$

1D. In general given a class of equivalence relations $\mathcal{E}$ we say that $E \in \mathcal{E}$ is *universal* for this class if for any $F \in \mathcal{E}$, $F \leq_B E$ (see Becker-Kechris [1996, §3]). For each Polish group G there is an equivalence relation of the form E_G^X which is universal among all such equivalence relations. More precisely, given G we can find a Borel G-space X such that for every Borel G-space Y, we have

$$E_G^Y \leq_B E_G^X.$$

This is uniquely determined, up to $\sim_B$, and we denote it by E_G^∞. If $G \subseteq H$ (i.e., G is a closed subgroup of H), then $E_G^\infty \leq_B E_H^\infty$. Also if $G \times \mathbb{Z}$ is, as a topological group, isomorphic to a closed subgroup of G, then $E_G^\infty \sim_B E_G^{F(G)}$, where G acts on $F(G)$ by translation:

$$g \cdot F = gF,$$

(see Becker-Kechris [1996, pp. 42-43]).

There is a universal Polish group, i.e., a Polish group G so that every Polish group is, as a topological group, isomorphic to a closed subgroup of G (Uspenskiĭ [1986], [1990]). One realization of G, discovered by Uspenskiĭ, is the isometry group of the Urysohn space, Iso($\mathbb{U}$), which will be discussed in §2 below. (By the way, this seems to answer the question in p. 922 of Vershik [1998].) If G is a universal Polish group, then E_G^∞ is, by the above remarks, universal for all E_H^X, H a Polish group and X a Borel H-space. Thus there is a (unique up to $\sim_B$) universal equivalence relation induced by a Borel action of a Polish group.

1E. In the case of the infinite symmetric group S_∞ of all permutations of $\mathbb{N}$, with the pointwise convergence topology, there is a way to look at $E_{S_\infty}^\infty$ which ties it up with concepts of model theory (see Becker-Kechris [1996, 2.7]).

For each nonempty countable relational language $L = \{R_i\}_{i\in I}$, where I is a countable set and R_i is an n_i-ary relation symbol, denote by X_L the space

$$X_L = \prod_i 2^{(\mathbb{N}^{n_i})},$$

which is homeomorphic to the Cantor space $2^{\mathbb{N}}$. We view X_L as the space of countable infinite L-structures (normalized so that their universe is $\mathbb{N}$), identifying $x = (x_i)_{i\in I} \in X_L$ with the structure

$$\langle \mathbb{N}, R_i \rangle_{i\in I},$$

where

$$R_i(s) \Leftrightarrow x_i(s) = 1.$$

The group S_∞ acts on X_L in the obvious way: $g \cdot x = y$ means that g is an isomorphism of the structure associated to x to that associated with y. This is called the *logic action* (of S_∞ on X_L). Thus $E_{S_\infty}^{X_L}$ is simply the isomorphism relation

$$x \cong y$$

on L-structures (with universe $\mathbb{N}$).

If $\sigma \in L_{\omega_1\omega}$ is an $L_{\omega_1\omega}$-sentence, denote by $\mathrm{Mod}(\sigma)$ the set of all $x \in X_L$ which are models of σ, and by $\cong_\sigma$ the restriction of the isomorphism relation to $\mathrm{Mod}(\sigma)$.

It turns out that $E^\infty_{S_\infty}$ can be realized (up to $\sim_B$) in the form $\cong_\sigma$, for appropriate σ. For example, if $\gamma =$ the theory of (undirected) graphs, in the language $L = \{R\}$, where R is a binary relation symbol, then

$$(\cong_\gamma) \sim_B E^\infty_{S_\infty}.$$

So the relation of *graph isomorphism*, $\cong_\gamma$, is universal among equivalence relations induced by Borel actions of S_∞. For further reference, we should point out that if

$$\gamma_c = \text{the theory of connected graphs},$$

then also

$$(\cong_{\gamma_c}) \sim_B (\cong_\gamma).$$

The direction $(\cong_{\gamma_c}) \leq_B (\cong_\gamma)$ is obvious. For the other direction, Friedman-Stanley [1989] showed that if $\tau_0 =$ the theory of rooted trees (i.e., connected acyclic graphs with a distinguished vertex), then $(\cong_\gamma) \leq_B (\cong_{\tau_0})$, and one can easily modify their construction to prove the same for the theory τ_1 of rooted trees in which every vertex has at least two neighbors (see the proof of 4.3 below). Now it is not hard to see that $(\cong_{\tau_1}) \leq_B (\cong_{\gamma_c})$: Given such a rooted tree $\langle V, E, v\rangle$, let $V' = V \bigcup \{x\}$, where x is a new vertex (not in V) and let the edges E' of V' consist of the edges E of V plus the edge connecting v to x. Then $\langle V', E'\rangle$ is a tree, and $\langle V_1, E_1, v_1\rangle \cong \langle V_2, E_2, v_2\rangle \Leftrightarrow \langle V_1', E_1'\rangle \cong \langle V_2', E_2'\rangle$.

Finally it should be pointed out that graph isomorphism is $<_B$ the universal equivalence relation induced by a Borel action of a Polish group. In fact, in some sense, it is "much smaller", in the order $<_B$, than this universal equivalence relation (see Hjorth [1999]).

1F. A *countable* Borel equivalence relation is a Borel equivalence relation on a standard Borel space all of whose equivalence classes are countable. There is a universal countable Borel equivalence relation, denoted by E_∞. This is the same (up to $\sim_B$) as $E^\infty_{F_\infty}$, where F_∞ is the free group with $\aleph_0$ generators (see, e.g., Dougherty-Jackson-Kechris [1994]). Again it turns out that E_∞ is much smaller, in the order $<_B$, than graph isomorphism (see, e.g., Kechris [1999, 2000]).

2 Isometric classification of Polish metric spaces

2A. We will start by reviewing the definition and some basic properties of the *Urysohn space*, which will play a crucial role in our arguments below. We refer the reader to Gromov [1999, 3.11 $\frac{2}{3}_+$], Katětov [1988], Urysohn [1927], Uspenskiǐ [1990], and Vershik [1998], for further information and proofs of results about this space that we will use below.

A separable metric space M is called *Urysohn* if for any finite metric space X and any subspace $Y \subseteq X$ every isometric embedding $f : Y \to M$ can be extended to an isometric embedding $g : X \to M$. Urysohn [1927] showed that there is a unique, up to isometry, Polish metric space which is Urysohn. We will simply call it *the Urysohn space*, and denote it by $\mathbb{U}$. It is also characterized as the unique Polish metric space which is *universal* (i.e., every Polish metric space can be isometrically embedded into it) and *ultra-homogeneous* (i.e., any isometry between finite subsets of it can be extended to an isometry of the whole space). Finally, if M is a Urysohn space, so is its completion $\overline{M}$, and so $\overline{M}$ is isometric to $\mathbb{U}$.

So, up to isometry, any Polish metric space can be viewed as a closed subset of $\mathbb{U}$ with the induced metric. It is then natural to view the space $F(\mathbb{U})$, of all closed subsets of $\mathbb{U}$ with the Effros Borel structure, as the space of all Polish metric spaces. It is of course a standard Borel space (see Kechris [1995, 12.C]). On $F(\mathbb{U})$ we then consider the following equivalence relation

$$C \cong_i D \Leftrightarrow C \text{ is isometric to } D.$$

It is clear that $\cong_i$ is analytic. Our goal in this section is to compute the precise complexity of $\cong_i$ in the hierarchy of analytic equivalence relations under Borel reducibility.

2B. Recall from §1 that there is a universal (under $\leq_B$) equivalence relation induced by a Borel action of a Polish group. It is of course unique up to $\sim_B$.

Theorem 2.1. *Isometry of Polish metric spaces, $\cong_i$, is Borel bireducible with the universal equivalence relation induced by a Borel action of a Polish group.*

In particular, this shows that $\cong_i$ is not Borel, a fact which also follows, in a much simpler way, from the results we prove in §3.

We will devote the rest of this section to the proof of 2.1. As in Uspenskiĭ [1990], our main tool will be Katětov's construction of the Urysohn space, see Katětov [1988]. It will be therefore important to review this construction and establish some notation. For technical reasons, we will actually use a slight variant of this construction.

2C. Fix a Polish metric space (X, d), with $X \neq \emptyset$. Denote by $E(X, \omega)$ the set of all $f : X \to \mathbb{R}$ such that

(i) $|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$, for $x, y \in X$,

(ii) for some *finite* $Y \subseteq X$, called a *support* of f, we have

$$f(x) = \inf\{d(x, y) + f(y) : y \in Y\},$$

for all $x \in X$. Note that if Y is a support of f, so is any $Z \supseteq Y$. We identify $x \in X$ with the function $f_x \in E(X, \omega)$ given by $f_x(y) = d(x, y)$ (a support for f_x is for example $Y = \{x\}$). On $E(X, \omega)$ we define the sup metric

$$d_E(f, g) = \sup\{|f(x) - g(x)| : x \in X\}.$$

Since $d(x, y) = d_E(f_x, f_y)$, it is legitimate to view, via the identification $x \mapsto f_x$, the space (X, d) as a subspace of $(E(X, \omega), d_E)$. Note here that for $f \in E(X, \omega)$ and $x \in X$ we have that

$$f(x) = d_E(f, x)(= d_E(f, f_x)).$$

Finally, denote, for each $n \geq 1$, by $E(X, n)$ the subspace of all $f \in E(X, \omega)$ which have support of cardinality $\leq n$. Thus $X \subseteq E(X, 1) \subseteq E(X, 2) \subseteq \dots$ and $\bigcup_{n \geq 1} E(X, n) = E(X, \omega)$. Let also

$$(\overline{E}(X, \omega), \overline{d_E}) = \text{the completion of } (E(X, \omega), d_E).$$

Now define inductively,

$$X_0 = X,$$

$$X_{n+1} = \overline{E}(X_n, \omega),$$

so that $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$, and let

$$X_\infty = \bigcup_n X_n.$$

Then X_∞ is a Urysohn space, and so its completion $X^* = \overline{X_\infty}$ is isometric to $\mathbb{U}$. The main point to check here is that X_∞ is separable, and this follows from the fact that $E(X, \omega)$ is separable. To see this, note that the $f \in E(X, \omega)$ that have finite support in some fixed countable dense subset of X, and take rational values on their support, form a countable dense subset of $E(X, \omega)$.

For definiteness, in the following we will identify $\mathbb{U}$ with $\mathbb{R}^*$:

$$\mathbb{U} = \mathbb{R}^*.$$

Now consider two Polish spaces X, Y and an isometric embedding $\varphi : X \to Y$. Then φ extends to an isometric embedding $E(\varphi, \omega) : E(X, \omega) \to E(Y, \omega)$, given by

$$E(\varphi, \omega)(f)(y) = \inf\{d(y, z) + f(\varphi^{-1}(z)) : z \in \varphi(X)\},$$

for $y \in Y$, and thus to an isometric embedding $\overline{E}(\varphi, \omega) : \overline{E}(X, \omega) \to \overline{E}(y, \omega)$. Define inductively,

$$\varphi_0 = \varphi$$

$$\varphi_{n+1} = \overline{E}(\varphi_n, \omega),$$

so that $\varphi_0 \subseteq \varphi_1 \subseteq \ldots$, and let

$$\varphi_\infty = \bigcup_n \varphi_n.$$

Then φ_∞ is an isometric embedding of X_∞ into Y_∞, and so extends to an isometric embedding

$$\varphi^* : X^* \to Y^*.$$

It is easy to check that if $\varphi : X \to Y$, $\psi : Y \to Z$ are isometric embeddings, then $E(\psi \circ \varphi, \omega) = E(\psi, \omega) \circ E(\varphi, \omega)$ and so

$$(\psi \circ \varphi)^* = \psi^* \circ \varphi^*.$$

For each separable metric space X denote by $\mathrm{Iso}(X)$ its group of isometries with the pointwise convergence topology. If X is Polish, then $\mathrm{Iso}(X)$ is a Polish group (see Kechris [1995, p. 60]). The map $\varphi \in \mathrm{Iso}(X) \mapsto \varphi^* \in \mathrm{Iso}(X^*)$ is a topological group isomorphism and $\mathrm{Iso}(X)^* = \{\varphi^* : \varphi \in \mathrm{Iso}(X)\}$ is therefore a closed subgroup of $\mathrm{Iso}(X^*)$. The main point to check here is that $\varphi \mapsto E(\varphi, \omega)$ from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(E(X, \omega))$ is continuous, which follows from the fact that each $f \in E(X, \omega)$ has finite support, and $E(\varphi, \omega)(f) = f \circ \varphi^{-1}$.

2D. We will prove here the first half of 2.1, i.e., we will show that there is a Polish group G and a Borel G-space W such that $(\cong_i) \leq_B E_G^W$.

If X, Y are two Polish metric spaces and $\varphi : X \to Y$ an isometry, then $\varphi^* : X^* \to Y^*$ is an isometry extending φ, so $\varphi^*(X) = Y$.

Now take $G = \text{Iso}(\mathbb{U})$ and $W = F(\mathbb{U})$. The group G acts in a Borel way on $F(\mathbb{U})$ in the obvious fashion:

$$g \cdot F = g(F).$$

We will use the above observation to show that

$$(\cong_i) \leq_B E_G^W.$$

We will need first the following technical lemma. Its proof is a routine, although somewhat cumbersome, calculation, based on the proof that any two Polish Urysohn spaces are isometric, see, e.g., Gromov [1999, p. 79].

Lemma 2.2. *There is a Borel function $f : F(\mathbb{U}) \to F(\mathbb{U})$ such that $f(\emptyset) = \emptyset$, and for $C \neq \emptyset$, $C \in F(\mathbb{U})$, there is an isometry $\varphi_C : C^* \to \mathbb{U}$ with $\varphi_C(C) = f(C)$.*

Using this lemma, we verify that for $C, D \in F(\mathbb{U})$,

$$C \cong_i D \Leftrightarrow f(C) E_G^W f(D),$$

which shows that $(\cong_i) \leq_B E_G^W$. This is clear if one of C, D is empty. So assume that $C, D \neq \emptyset$. If $C \cong_i D$, let $\varphi : C \to D$ be an isometry. Then $\varphi^* : C^* \to D^*$ is an isometry with $\varphi^*(C) = D$. Now $g = \varphi_D \circ \varphi^* \circ \varphi_C^{-1} \in \text{Iso}(\mathbb{U})$ and $g \cdot f(C) = f(D)$, so $f(C) E_G^W f(D)$. Conversely, if $f(C) E_G^W f(D)$, then $f(C)$, $f(D)$ are isometric, so as C is isometric to $f(C)$ and D to $f(D)$, C, D are isometric.

2E. We will now embark on the longer argument that, for every Polish group G and Borel G-space X, we have that $E_G^X \leq_B (\cong_i)$.

For each Polish group G consider its action on $F(G)$ by left-translation.

$$g \cdot C = gC.$$

Denote by $E(G) = E_G^{F(G)}$ the corresponding equivalence relation. By Theorem 3.5.3 of Becker-Kechris [1996] and the remarks following it (notice that the Polish group G is a closed subgroup of $H = G \times \mathbb{Z}^{<\omega}$ and $H \times \mathbb{Z} \cong H$), we see that it is enough to show that for each Polish group G, $E(G) \leq_B (\cong_i)$.

So fix G and a left-invariant compatible metric d_G on G. Let $(\overline{G}, \overline{d_G})$ be the completion of (G, d_G). For each $g \in G$ and $x = \lim_n h_n \in \overline{G}$, where (h_n) is a Cauchy sequence in G, let

$$\tilde{g}(x) = \lim_n g(h_n).$$

It is easy to see that this is well-defined and $\tilde{g}$ is an isometry of $(\overline{G}, \overline{d_G})$. Moreover $g \mapsto \tilde{g}$ is topological group isomorphism of G and $\tilde{G} = \{\tilde{g} : g \in G\} \subseteq \mathrm{Iso}(\overline{G})$, so in particular $\tilde{G}$ is a closed subgroup of $\mathrm{Iso}(\overline{G})$.

Consider the action of $\tilde{G}$ on $F(\overline{G})$ given by

$$\tilde{g} \cdot C = \tilde{g}(C).$$

Then the Borel map

$$C \mapsto \overline{C},$$

where $\overline{C}$ is the closure of C in $\overline{G}$, from $F(G)$ to $F(\overline{G})$, shows that $E(G) \leq_B E_{\tilde{G}}^{F(\overline{G})}$, so it is enough to show that $E_{\tilde{G}}^{F(\overline{G})} \leq_B (\cong_i)$.

More generally, it is enough to show that if X is a Polish space, H a closed subgroup of $\mathrm{Iso}(X)$ and H acts on $F(X)$ by

$$f \cdot C = f(C),$$

then $E_H^{F(X)} \leq_B (\cong_i)$.

Consider X^* and $H^* = \{h^* : h \in H\}$, so that H^* is a closed subgroup of $\mathrm{Iso}(X^*)$. We will first need the following lemma characterizing closed subgroups of isometry groups of Polish metric spaces.

Lemma 2.3. *Let Y be a Polish metric space, $K \subseteq \mathrm{Iso}(Y)$ a closed subgroup of its group of isometries. Then there is a sequence of closed sets $R_n \subseteq Y^n, n \geq 2$, such that*

$$K = \{\varphi \in \mathrm{Iso}(Y) : \varphi(R_n) = R_n, \forall n \geq 2\},$$

where $\varphi(R_n) = \{(\varphi(y_1), \ldots, \varphi(y_n)) : (y_1, \ldots, y_n) \in R_n\}$.

Proof. Fix a dense sequence $\{q_i\}$ in Y. Let

$$R_n = \overline{\{(\varphi(q_1), \ldots, \varphi(q_n)) : \varphi \in K\}},$$

which is a closed subset of Y^n. We claim that this works. Clearly, for $\varphi \in K$, $\varphi(R_n) = R_n$, for each n. Let $\varphi \in \mathrm{Iso}(Y)$ be such that for each n, $\varphi(R_n) = R_n$.

To show that $\varphi \in K$, it is enough to show that for each $\epsilon > 0$, and each $n \geq 2$, there is $\psi \in K$ with $d(\psi(q_i), \varphi(q_i)) < \epsilon$, for $i = 1, \ldots, n$. Since $\varphi(R_n) = R_n$, and $(q_1, \ldots, q_n) \in R_n$, $(\varphi(q_1), \ldots, \varphi(q_n)) \in R_n$, so, by the definition of R_n, there is $\psi \in K$ with $d(\varphi(q_i), \psi(q_i)) < \epsilon$, for $i = 1, \ldots, n$. ⊣

Consider now the action of $\mathrm{Iso}(\mathbb{U})$ on $\prod_{n\geq 1} F(\mathbb{U}^n)$ given by

$$\varphi \cdot (R_n) = (\varphi(R_n)).$$

Call $E^\infty(\mathbb{U})$ the corresponding equivalence relation.

Lemma 2.4. $E_H^{F(X)} \leq_B E^\infty(\mathbb{U})$.

Proof. We can of course replace here $\mathbb{U}$ by X^*. Consider the closed subgroup H^* of $\mathrm{Iso}(X^*)$ and let $R_n \in F((X^*)^n)$ be such that

$$H^* = \{\varphi \in \mathrm{Iso}(X^*) : \varphi(R_n) = R_n, \forall n \geq 2\}.$$

Viewing X as a closed subset of X^*, consider now the Borel map

$$C \in F(X) \mapsto (C, R_2, R_3, \ldots) \in \prod_{n\geq 1} F((X^*)^n).$$

We claim that this is a reduction of $E_H^{F(X)}$ to $E^\infty(\mathbb{U})$. First if $h \in H$ is such that $h(C_1) = C_2$ ($C_i \in F(X)$), then clearly $h^*(C_1) = C_2$ and $h^*(R_n) = R_n$ for $n \geq 2$, so $h^* \cdot (C_1, R_2, \ldots) = (C_2, R_2, \ldots)$. Conversely, let $\varphi \in \mathrm{Iso}(X^*)$ be such that $\varphi \cdot (C_1, R_2, R_3, \ldots) = (C_2, R_2, R_3, \ldots)$. Then $\varphi(C_1) = C_2$ and $\varphi(R_n) = R_n$ for $n \geq 2$, so $\varphi \in H^*$. Say $\varphi = h^*$, with $h \in H$. Then clearly $h(C_1) = C_2$. ⊣

2F. It is therefore enough to show that $E^\infty(\mathbb{U}) \leq_B (\cong_i)$.

For any Polish metric space X, consider the action of $\mathrm{Iso}(X)$ on $F(X)^{\mathbb{N}}$ given by

$$\varphi \cdot (C_1, C_2, \ldots) = (\varphi(C_1), \varphi(C_2), \ldots),$$

and let $E^1(X)$ be the corresponding equivalence relation. We will break-up the proof that $E^\infty(\mathbb{U}) \leq_B (\cong_i)$ in two steps.

Step 1. For any Polish metric space X, $E^1(X) \leq_B (\cong_i)$.

Step 2. For some Polish metric space X, $E^\infty(\mathbb{U}) \leq_B E^1(X)$.

We will first deal with Step 1.

2G. We prove here

Lemma 2.5. *For any Polish metric space X, $E^1(X) \leq_B (\cong_i)$.*

Proof. Let d be the metric of X, and assume without loss of generality that X has at least 2 elements. Consider then the equivalent metric

$$\delta = \frac{d}{1+d}.$$

Then (X, δ) is a Polish metric space and $\delta(x,y) < 1$ for all $x, y \in X$. Given $\vec{C} = (C_0, C_1, \dots) \in F(X)^{\mathbb{N}}$ consider the Polish metric space $(X_{\vec{C}}, d_{\vec{C}})$ defined as follows: For each n for which $C_n \neq \emptyset$, choose a point $x_n^{\vec{C}}$ not in X and assume that all these points are distinct. Let $X_{\vec{C}} = X \cup \{x_n^{\vec{C}}\}$. Define the metric $d_{\vec{C}}$ as follows: $d_{\vec{C}}$ agrees with δ on X. The distance between any two distinct $x_n^{\vec{C}}, x_m^{\vec{C}}$ is equal to $|n-m|+1$. Finally, if $u \in X$ we define

$$d_{\vec{C}}(x_n^{\vec{C}}, u) = (n+2) + \delta(u, C_n),$$

where $\delta(u, C_n)$ is the δ-distance of u from C_n.

We claim that

$$\vec{C} E^1(X) \vec{D} \Leftrightarrow X_{\vec{C}}, X_{\vec{D}} \text{ are isometric.}$$

The direction $\Rightarrow$ is obvious. Conversely, assume $\varphi : X_{\vec{C}} \to X_{\vec{D}}$ is an isometry. Then it is easy to check that $\varphi(X) = X$ and $\psi = \varphi|X$ is an isometry of X. Moreover $C_n \neq \emptyset \Leftrightarrow D_n \neq \emptyset$, and $\varphi(x_n^{\vec{C}}) = x_n^{\vec{D}}$. Since for any n for which $C_n \neq \emptyset$, we have that $C_n = \{u \in X : d_{\vec{C}}(x_n^{\vec{C}}, u) = n+2\}$ and $D_n = \{u \in X : d_{\vec{D}}(x_n^{\vec{D}}, u) = n+2\}$, it follows that $\psi(C_n) = D_n$, so $\vec{C} E^1(X) \vec{D}$.

Finally observe that one can easily construct a Borel function $f : F(X)^{\mathbb{N}} \to F(\mathbb{U})$ such that $f(\vec{C})$ is isometric to $X_{\vec{C}}$. Thus

$$\vec{C} E^1(X) \vec{D} \Leftrightarrow f(\vec{C}) \cong_i f(\vec{D}),$$

so $E^1(X) \leq_B (\cong_i)$. $\dashv$

2H. It now remains to show that for an appropriately chosen Polish metric space X,

$$E^\infty(\mathbb{U}) \leq_B E^1(X).$$

First, and for technical reasons that will be apparent in a moment, we will replace $E^\infty(\mathbb{U})$ by a slight variant, $\tilde{E}^\infty(\mathbb{U})$. Consider the action of $\mathrm{Iso}(\mathbb{U})$ on $\prod_{n\geq 1} F(\mathbb{U}^{(3^n)})$ given as usual by

$$\varphi \cdot (C_1, C_2, \dots) = (\varphi(C_1), \varphi(C_2), \dots),$$

and let $\tilde{E}^\infty(\mathbb{U})$ be the corresponding equivalence relation. The map

$$(C_1, C_2, \dots) \mapsto (\tilde{C}_1, \tilde{C}_2, \dots)$$

from $\prod_{n\geq 1} F(\mathbb{U}^n)$ into $\prod_{n\geq 1} F(\mathbb{U}^{(3^n)})$, given by

$$\tilde{C}_n = \{(x_1, x_2, \dots, x_n, x_1, x_1, \dots, x_1) : (x_1, \dots, x_n) \in C_n\},$$

is clearly a Borel reduction of $E^\infty(\mathbb{U})$ into $\tilde{E}^\infty(\mathbb{U})$, so it is enough to show that for some Polish metric space X,

$$\tilde{E}^\infty(\mathbb{U}) \leq_B E^1(X).$$

Endow each $\mathbb{U}^N$, $N \geq 1$ with the metric

$$d_N(\vec{x}, \vec{y}) = \frac{1}{N} \sum_{i=1}^{N} d(x_i, y_i),$$

so it becomes a Polish metric space. We can also identify any $\vec{x} \in \mathbb{U}^{(3^n)}$ with $(\vec{x}, \vec{x}, \vec{x}) \in \mathbb{U}^{(3^{n+1})}$ (allowing also here the case $n = 0$). This is consistent with the definition of the metric, as

$$d_{3^{n+1}}((\vec{x}, \vec{x}, \vec{x}), (\vec{y}, \vec{y}, \vec{y})) = d_{3^n}(\vec{x}, \vec{y}),$$

so we have

$$\mathbb{U} \subseteq \mathbb{U}^3 \subseteq \mathbb{U}^9 \subseteq \dots \subseteq \mathbb{U}^{(3^n)} \subseteq \dots,$$

as metric spaces. Put

$$\mathbb{U}^\infty = \bigcup_n \mathbb{U}^{(3^n)},$$

and

$$X = \text{the completion of } \mathbb{U}^\infty.$$

We will show that

$$\tilde{E}^\infty(\mathbb{U}) \leq E^1(X).$$

For each isometry $\varphi \in \mathrm{Iso}(\mathbb{U})$, let φ^N be the isometry on $\mathbb{U}^N$ given by

$$\varphi^N(x_1, \ldots, x_N) = (\varphi(x_1), \ldots, \varphi(x_N)).$$

We clearly have that

$$\varphi = \varphi^1 \subseteq \varphi^3 \subseteq \varphi^9 \subseteq \ldots,$$

so

$$\varphi^\infty = \bigcup_n \varphi^{(3^n)}$$

is an isometry of $\mathbb{U}^\infty$ and therefore extends to an isometry $\overline{\varphi}^\infty$ on X. Put

$$\varphi^+ = (\overline{\varphi}^\infty) \in \mathrm{Iso}(X),$$

and let

$$\mathrm{Iso}(\mathbb{U})^+ = \{\varphi^+ : \varphi \in \mathrm{Iso}(\mathbb{U})\}.$$

The main lemma here is now the following:

Lemma 2.6. *In the preceding notation, there is a sequence (D_n) of closed subsets of X such that*

$$\mathrm{Iso}(\mathbb{U})^+ = \{\Phi \in \mathrm{Iso}(X) : \forall n(\Phi(D_n) = D_n)\}.$$

Granting this, we can easily complete the proof that $\tilde{E}^\infty(\mathbb{U}) \leq_B E^1(X)$. Consider the Borel map

$$(C_1, C_2, C_3, \ldots) \mapsto (C_1, D_1, C_2, D_2, \ldots)$$

from $\prod_{n\geq 1} F(\mathbb{U}^{(3^n)})$ into $F(X)^{\mathbb{N}}$. It is clearly a Borel reduction of $\tilde{E}^\infty(\mathbb{U})$ into $E^1(X)$.

So it only remains to give the

Proof of Lemma 2.6. For each $N \geq 1$, view $\mathbb{U}$ as a subset of $\mathbb{U}^N$ identifying $x \in \mathbb{U}$ with $(x, \ldots, x) \in \mathbb{U}^N$. For each $\varphi \in \mathrm{Iso}(\mathbb{U})$, let $\varphi^N \in \mathrm{Iso}(\mathbb{U}^N)$ be defined by $\varphi^N(x_1, \ldots, x_n) = (\varphi(x_1), \ldots, \varphi(x_n))$. Let also $\mathrm{Iso}(\mathbb{U})^N = \{\varphi^N : \varphi \in \mathrm{Iso}(\mathbb{U})\} \subseteq \mathrm{Iso}(\mathbb{U}^N)$. The proof of 2.6 will follow easily from the following:

Sublemma 2.7. *For each $N \geq 3$, there is a sequence of closed sets (K_i) contained in $\mathbb{U}^N$ such that*

$$\mathrm{Iso}(\mathbb{U})^N = \{\Phi \in \mathrm{Iso}(\mathbb{U}^N) : \forall i(\Phi(K_i) = K_i)\}.$$

Granting this, we can complete the proof of 2.6 as follows: For each $n \geq 1$, fix a sequence $(K_{i,n})_{i \in \mathbb{N}}$ which satisfies 2.7 for $\mathbb{U}^{(3^n)}$. Let (D_n) be an enumeration of $\{\mathbb{U}^{(3^n)} : n \geq 0\} \cup \{K_{i,n} : i \in \mathbb{N}, n \geq 1\}$. We claim that this satisfies 2.6. It is clear that if $\Phi = \varphi^+$, where $\varphi \in \mathrm{Iso}(\mathbb{U})$, then $\Phi(D_n) = D_n$, for all n. Conversely, let $\Phi \in \mathrm{Iso}(X)$ be such that $\Phi(D_n) = D_n$ for all n. First $\Phi(\mathbb{U}^{(3^n)}) = \mathbb{U}^{(3^n)}$ for all $n \geq 0$. In particular, $\Phi(\mathbb{U}) = \mathbb{U}$. Let $\varphi = \Phi | U$. By 2.7, $\Phi | \mathbb{U}^{(3^n)} = \varphi^{(3^n)}$ for all $n \geq 1$, so $\Phi | \mathbb{U}^\infty = \varphi^\infty$ and thus $\Phi = \overline{\varphi}^\infty = \varphi^+$.

So it only remains to give the

Proof of Sublemma 2.7. For each tuple $\overline{p} = (p_{i,j})_{1 \leq i,j \leq N}$ of positive rationals, let

$$K_{\overline{p}} = \{(x_1, \dots, x_N) \in \mathbb{U}^N : d(x_i, x_j) \leq p_{i,j}, \forall 1 \leq i, j \leq N\}.$$

We will show that

$$\mathrm{Iso}(\mathbb{U})^N = \{\Phi \in \mathrm{Iso}(\mathbb{U}^N) : \forall \overline{p}(\Phi(K_{\overline{p}}) = K_{\overline{p}})\}.$$

Call the right hand side of this equation G. Then it is clear that

$$\begin{aligned} G = \{\Phi \in \mathrm{Iso}(\mathbb{U}^N) : \forall (x_1, \dots x_N) \in \mathbb{U}^N, \text{ if } \\ \Phi(x_1, \dots, x_N) = (y_1, \dots, y_N), \text{ then } \\ d(x_i, x_j) = d(y_i, y_j), \forall 1 \leq i, j \leq N\}. \end{aligned}$$

It is also clear that $\mathrm{Iso}(\mathbb{U})^N \subseteq G$. Assume now that $\Phi \in G$. It follows that $\Phi(\mathbb{U}) = \mathbb{U}$ (recall that $\mathbb{U}$ is identified with $\{(x, \dots, x) : x \in \mathbb{U}\}$). Put $\varphi = \Phi | \mathbb{U}$. We will show that $\Phi = \varphi^N$. First notice that $\Phi \circ (\varphi^N)^{-1} | \mathbb{U} = id$, so we may as well assume that $\varphi = id$, and show that $\Phi = id$. It is clearly enough to find a dense subset of $\mathbb{U}^N$ on which Φ is the identity.

Claim 1. The set of $(x_1, \dots, x_N) \in \mathbb{U}^N$ for which the x_i are distinct and the distances $d(x_i, x_j)$, $1 \leq i < j \leq N$, are all distinct, is dense in $\mathbb{U}^N$.

Proof. It is clearly enough to approximate any $(y_1, \dots, y_N) \in \mathbb{U}^N$ in which all y_i are distinct by such a $(x_1, \dots, x_N)$. Using the definition of $\mathbb{U}$, it is then enough to prove the following:

For each finite metric space $M = \{p_1, \dots, p_m\}$ and $\epsilon > 0$ there is a metric space $M' = \{p_1, \dots, p_m, q_1, \dots, q_m\}$ extending M such that $d_{M'}(p_i, q_i) < \epsilon$, for all i, and all the distances $d_{M'}(q_i, q_j)$, $1 \leq i < j \leq m$, are distinct.

We prove this by induction on $m \geq 1$. For $m = 1$, this is obviously true. So assume it is true for $m - 1$ and consider $M = \{p_1, \dots, p_m\}$. By the induction hypothesis, find $q_1, \dots, q_{m-1}$ that work for $\{p_1, \dots, p_{m-1}\}$ and put $M_1 = \{p_1, \dots, p_{m-1}, q_1, \dots, q_{m-1}\}$. We can of course assume that $p_m \neq q_i$, $\forall 1 \leq i \leq m-1$. Let $M_2 = \{p_1, \dots, p_{m-1}, p_m, q_1, \dots, q_{m-1}\}$ and define a metric on it by extending the metric on M_1, and defining for $x \in M_1$:

$$d_{M_2}(p_m, x) = \min\{d_M(p_m, p_i) + d_{M_1}(p_i, x) : 1 \leq i \leq m-1\}.$$

This also extends M. Enumerate the points in M_2 in a sequence $M_2 = \{r_i : 1 \leq i \leq 2m-1\}$ in such a way that the numbers $\alpha_i = d_{M_2}(p_m, r_i)$ are in non-decreasing order:

$$\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{2m-1}.$$

In particular $r_1 = p_m$ and $\alpha_1 = 0 < \alpha_2$. We also have for $1 \leq i < j \leq 2m-1$:

$$\alpha_j - \alpha_i = |\alpha_i - \alpha_j| \leq d(r_i, r_j) \leq \alpha_i + \alpha_j.$$

Choose now numbers $\epsilon_1 > \epsilon_2 > \dots > \epsilon_{2m-1} > 0$ such that
(i) $\epsilon_1 < \epsilon$,
(ii) $\epsilon_1 < \min\{d(r_i, r_j) : 1 \leq i < j \leq 2m-1\}$,
(iii) $\epsilon_1 < \min\{\alpha_j - \alpha_i : 1 \leq i < j \leq 2m-1, \alpha_j > \alpha_i\}$,
and, if we put

$$\alpha'_i = \alpha_i + \epsilon_i,$$

we also have
(iv) all the α'_j are distinct from each other and from any of $d_{M_2}(q_i, q_j), 1 \leq i < j \leq m-1$.

Then for $1 \leq i < j \leq 2m-1$, we have

$$|\alpha'_i - \alpha'_j| \leq d(r_i, r_j) \leq \alpha'_i + \alpha'_j.$$

Define then the metric space $M' = \{p_1, \dots, p_m, q_1, \dots q_m\}$, so that q_m is a point not in M_2, by extending the distance of M_2 and letting

$$d_{M'}(q_m, r_i) = \alpha'_i.$$

Clearly all $d_{M'}(q_i, q_j)$ are distinct, if $1 \leq i < j \leq m$, and $d_{M'}(q_m, p_m) = d_{M'}(q_m, r_1) = \alpha'_1 = 0 + \epsilon_1 < \epsilon$.

This concludes the proof of Claim 1.

So it is enough to show that if $(x_1, \dots, x_N) \in \mathbb{U}^N$ is such that the x_i are distinct and all the distances $d(x_i, x_j), 1 \leq i < j \leq N$ are distinct, then $\Phi(x_1, \dots x_N) = (x_1, \dots, x_N)$. Put

$$\Phi(x_1, \dots, x_N) = (y_1, \dots, y_N).$$

We have that for $1 \leq i < j \leq N$,

$$d(x_i, x_j) = d(y_i, y_j),$$

so that in particular all the y_i are also distinct.

We now claim the following

Claim 2. $\{x_1, \ldots, x_N\} = \{y_1, \ldots, y_N\}$.

Granting this, it is easy to conclude that $y_i = x_i, \forall 1 \leq i \leq N$. Otherwise there is $i \neq j$ such that $y_i = x_j$. Since $N \geq 3$, choose $k \notin \{i, j\}$. Then $d(x_i, x_k) = d(y_i, y_k) = d(x_j, x_\ell)$, where $y_k = x_\ell$, which is absurd, as $\{x_i, x_k\} \neq \{x_j, x_\ell\}$.

So it only remains to give the

Proof of Claim 2. Fix an arbitrary $u \in \mathbb{U}$. Then we have

$$d_N((x_1, \ldots, x_N), (u, \ldots, u)) = d_N((y_1, \ldots y_N), (u, \ldots, u)),$$

i.e.,

$$\sum_{i=1}^{N} d(x_i, u) = \sum_{i=1}^{N} d(y_i, u).$$

Assuming, towards a contradiction, that $\{x_1, \ldots, x_N\} \neq \{y_1, \ldots, y_N\}$, let

$$A = \{x_1, \ldots x_N\} \cap \{y_1, \ldots, y_N\},$$

$$\{x'_1, \ldots x'_t\} = \{x_1, \ldots, x_N\} \setminus A,$$

$$\{y'_1, \ldots y'_t\} = \{y_1, \ldots, y_N\} \setminus A,$$

for some $t \geq 1$. Then for all $u \in \mathbb{U}$,

$$\sum_{i=1}^{t} d(x'_i, u) = \sum_{i=1}^{t} d(y'_i, u).$$

We will contradict this by finding a $u \in \mathbb{U}$ that fails to satisfy this equation.

First notice that for any finite metric space $M = \{p_1, \ldots, p_\ell\}$ we can find a sequence $a_1, \ldots, a_\ell$ of reals such that

(i) $|a_i - a_j| < d(p_i, p_j), \forall 1 \leq i \neq j \leq \ell$,

(ii) $d(p_i, p_j) < a_i + a_j, \forall 1 \leq i, j \leq \ell$.

(Start for example with $a'_i = d(p_1, p_i)$ and apply an argument similar to that of the proof of Claim 2, to increase a'_i slightly so that (i), (ii) hold.)

Now let

$$z_1 = x'_1, \dots, z_t = x'_t, z_{t+1} = y'_1, \dots, z_{2t} = y'_t$$

and consider the set $W \subseteq (0,\infty)^{2t}$ consisting of all $(\alpha_1, \dots, \alpha_{2t})$ such that

(a) $|\alpha_i - \alpha_j| < d(z_i, z_j), \forall 1 \leq i \neq j \leq 2t$,

(b) $d(z_i, z_j) < \alpha_i + \alpha_j, \forall 1 \leq i, j \leq 2t$.

This is clearly open and non-empty by the preceding remarks. Now consider the set

$$C = \{(\alpha_1, \dots, \alpha_{2t}) \in (0,\infty)^{2t} : \sum_{i=1}^{t} \alpha_i = \sum_{i=1}^{t} \alpha_{t+i}\}.$$

This is clearly closed and has no interior, so $W \not\subseteq C$. Fix $(\alpha_i)_{1 \leq i \leq 2t} \in W \setminus C$. By the definition of $\mathbb{U}$, there is $u \in \mathbb{U}$ such that $d(u, z_i) = \alpha_i$, so that

$$\sum_{i=1}^{t} d(x'_i, u) \neq \sum_{i=1}^{t} d(y'_i, u),$$

which is the desired contradiction. ⊣

The proof of Theorem 2.1 is thus complete. The following corollary follows immediately from this proof.

Corollary 2.8. *The equivalence relation induced by the action of the isometry group,* Iso($\mathbb{U}$)*, of the Urysohn space, on the set* $F(\mathbb{U})$ *of its closed subsets is universal for equivalence relations induced by Borel actions of Polish groups.*

3 Characterizing the isometry groups of Polish metric spaces

As an application of the ideas used in §2, we will characterize here the isometry groups of Polish metric spaces. The following notation will be convenient:

Given a Polish metric space X and a sequence of closed sets $R_n \subseteq X^{p(n)}$, where $p(n) \geq 1$, denote by

$$\mathrm{Iso}(X, (R_n)_{n \in \mathbb{N}})$$

the closed subgroup of Iso(X) defined by

$$\mathrm{Iso}(X,(R_n)_{n\in\mathbb{N}})=\{\varphi\in\mathrm{Iso}(X):\forall n(\varphi(R_n)=R_n)\}.$$

Then we have:

Theorem 3.1. *i) Up to (topological group) isomorphism, the isometry groups of Polish metric spaces are exactly the Polish groups.*

ii) Every Polish group is isomorphic to a group of the form

$$\mathrm{Iso}(\mathbb{U},(C_n)_{n\in\mathbb{N}}),$$

where (C_n) *is a sequence of closed subsets of* $\mathbb{U}$.

Proof. i) Let G be a Polish group. We will find a Polish metric space X, so that G is isomorphic to Iso(X). By Uspenskiĭ [1990] we can assume that G is a closed subgroup of Iso($\mathbb{U}$). By Lemma 2.3 (and making the simple modifications as in the beginning of 2H), we can find a sequence of closed sets $C_n\subseteq\mathbb{U}^{(3^n))}, n\geq 0$, such that

$$G=\mathrm{Iso}(\mathbb{U},(C_n)_{n\in\mathbb{N}}).$$

Now consider, as in 2H, the space $X=$ the completion of $\mathbb{U}^\infty$, where $\mathbb{U}^\infty=\bigcup_n\mathbb{U}^{(3^n)}$. Let also for each $\varphi\in$Iso($\mathbb{U}$), φ^+ be its canonical extension to X. Then $\varphi\mapsto\varphi^+$ is a topological group isomorphism, so it is enough to check that

$$G^+=\{\varphi^+:\varphi\in G\}$$

is of the required form. By Lemma 2.6, we see that for some sequence (D_n) of closed subsets of X,

$$\mathrm{Iso}(\mathbb{U})^+=\mathrm{Iso}(X,(D_n)_{n\in\mathbb{N}}),$$

and therefore

$$G^+=\mathrm{Iso}(X,(E_n)_{n\in\mathbb{N}}),$$

where (E_n) enumerates $\{C_n\}\cup\{D_n\}$ (we view of course here C_n as a closed subset of X). Now, going back to the proof of 2.5, let $X_{\vec{E}}$ be the Polish metric space defined there from X and $\vec{E}=(E_n)_{n\in\mathbb{N}}$. Then every isometry $\varphi\in G^+$ extends to a unique isometry φ' of $X_{\vec{E}}$, by defining it to be the identity on the additional points, and conversely every isometry of $X_{\vec{E}}$ is of that form for some $\varphi \in G^+$. Thus the map $\varphi \mapsto \varphi'$ is an isomorphism of G^+ with $\mathrm{Iso}(X_{\vec{E}})$, and the proof is complete.

ii) It is enough to prove the following, where we recall from 2C that $\varphi \mapsto \varphi^*$ is a topological group isomorphism of $\mathrm{Iso}(X)$ with $\mathrm{Iso}(X)^* = \{\varphi^* : \varphi \in \mathrm{Iso}(X)\}$, for any Polish metric space X, and X^* is isometric to $\mathbb{U}$.

Lemma 3.2. *Let X be a Polish metric space. Then there is a sequence (C_n) of closed subsets of X^* such that*

$$\mathrm{Iso}(X)^* = \mathrm{Iso}(X^*, (C_n)_{n\in\mathbb{N}}).$$

Proof. The proof follows immediately from the following two sublemmas:

Sublemma 3.3. (Katětov [1988], 1.6) *Given $\varphi \in \mathrm{Iso}(X), E(\varphi, \omega)$ is the unique isometry Φ of $E(X, \omega)$ such that $\Phi(X) = X$ and $\Phi|X = \varphi$. Moreover $E(\varphi, \omega)(E(X, n)) = E(X, n)$, for each $n \geq 1$.*

Sublemma 3.4. *$(E(X, n), d_E)$ is complete, for each $n \geq 1$.*

Proof of 3.3. Let $\Phi \in \mathrm{Iso}(E(X, \omega))$ be such that $\Phi(X) = X$ and $\Phi|X = \varphi$. We have to show that $\Phi = E(\varphi, \omega)$. Recall that $E(\varphi, \omega)(f) = f \circ \varphi^{-1}$, for $f \in E(X, \omega)$. So we have to show that $\Phi(f) = f \circ \varphi^{-1}$. Put $\Phi(f) = g$ in order to show that $f = g \circ \varphi$. We compute, using the fact that $g \in E(X, \omega)$ and the observation that for $h \in E(X, \omega)$ and $x \in X$ we have $h(x) = d_E(h, x)$ $(= d_E(h, f_x))$:

$$\begin{aligned} g(\varphi(x)) &= d_E(g, \varphi(x)) \\ &= d_E(\Phi(f), \Phi(x)) \\ &= d_E(f, x) \\ &= f(x). \end{aligned}$$

The second part of 3.3 is obvious. ⊣

Proof of 3.4. We will prove by induction on $N \geq 1$ that $(E(X, N), d_E)$ is complete. First consider the case $N = 1$.

Let (f_n) be a Cauchy sequence in $(E(X, 1), d_E)$. Clearly $(f_n(x))$ is Cauchy for each $x \in X$, so let $f(x) = \lim_n f_n(x)$. It is clear that $\sup\{|f_n(x) - f(x)| : x \in X\} \to 0$ as well. It remains to show that $f \in E(X, 1)$. Fix $y_n \in X$ so that

$$f_n(x) = f_n(y_n) + d(x, y_n),\ \forall x \in X. \qquad (*)$$

Then for any m, n,

$$f_n(y_m) = f_n(y_n) + d(y_m, y_n),$$
$$f_m(y_n) = f_m(y_m) + d(y_m, y_n),$$

so

$$(f_n(y_m) - f_m(y_m)) + (f_m(y_n) - f_n(y_n)) = 2d(y_m, y_n).$$

Thus $d(y_m, y_n) \to 0$ as $m, n \to \infty$, i.e., (y_n) is Cauchy. Say $y_n \to y \in X$. By letting $n \to \infty$ in (*), we get

$$f(x) = f(y) + d(x, y), \ \forall x \in X,$$

and therefore $f \in E(X, 1)$.

Now assume that the result is true for all integers $< N$ and fix a Cauchy sequence (f_n) in $E(X, N)$. Let $Y_n \subseteq X$ be a support for f_n of cardinality $\leq N$. By induction hypothesis and by going to a subsequence, if necessary, we can assume that $\text{card}(Y_n) = N$ for all n. Say

$$Y_n = \{p_n^1, p_n^2, \ldots, p_n^N\}.$$

Fix a pair $m < n$. For each $1 \leq i \leq N$, pick $1 \leq j(i) \leq N$ such that

$$f_n(p_m^i) = f_n(p_n^{j(i)}) + d(p_m^i, p_n^{j(i)}).$$

Similarly, for each $1 \leq j \leq N$, pick $1 \leq i(j) \leq N$ such that

$$f_m(p_n^j) = f_m(p_m^{i(j)}) + d(p_n^j, p_m^{i(j)}).$$

This defines a directed bipartite graph on (Y_m, Y_n), where there is an edge from p_m^i to $p_n^{j(i)}$ and an edge from p_n^j to $p_m^{i(j)}$.

Notation. (i) For $I \subseteq \{1, \ldots, N\}$, let $Y_n | I = \{p_n^i : i \in I\}$.

(ii) d_H denotes the *Hausdorff distance* on nonempty bounded subsets of X:

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\},$$

where

$$\rho(A, B) = \sup\{d(x, B) : x \in A\}.$$

Claim. For each fixed $m < n$, there are $I, J \subseteq \{1, \ldots, N\}$ with

$$\text{card}(I) = \text{card}(J) > 0,$$

such that

(i) $d_H(Y_m|I, Y_n|J) \leq 2N d_E(f_m, f_n)$;

(ii) For each $p \in Y_m$, the oriented path starting from p reaches, in at most $2N$ steps, the set $(Y_m|I) \cup (Y_n|J)$ and similarly for any $q \in Y_n$.

We will assume this temporarily and proceed to complete the proof.

By Ramsey's Theorem we can find an infinite subset $A \subseteq \mathbb{N}$ and a pair $I_0, J_0 \subseteq \{1, \dots, N\}$, with $\mathrm{card}(I_0) = \mathrm{card}(J_0) > 0$, such that if $m < n$, and $m, n \in A$, then the (I, J) corresponding to $m < n$ in the claim is equal to (I_0, J_0).

Fix $\epsilon > 0$. We will find $n(\epsilon)$ such that if $n(\epsilon) < m < n$, and $m, n \in A$, then

$$d_H(Y_m|(I_0 \cup J_0), Y_n|(I_0 \cup J_0)) < \epsilon. \tag{**}$$

Indeed, let $n(\epsilon) \in A$ be such that if $n(\epsilon) \leq m < n$, then $4N d_E(f_m, f_n) < \epsilon$. Now take any $n(\epsilon) < m < n$ with $m, n \in A$ and fix $m_0, n_0 \in A$ with $n(\epsilon) \leq m_0 < m < n < n_0$.

If $p^i_m \in Y_m|I_0$, clearly $d(p^i_m, Y_n|(I_0 \cup J_0)) \leq 2N d_E(f_m, f_n) < \epsilon$ by part (i) of the claim. If $p^i_m \in Y_m|J_0$, then, as $d_H(Y_{m_0}|I_0, Y_m|J_0) \leq 2N d_E(f_{m_0}, f_m) < \epsilon/2$, we have some $p^k_{m_0} \in Y_{m_0}|I_0$ with $d(p^k_{m_0}, p^i_m) < \epsilon/2$. But also $d_H(Y_{m_0}|I_0, Y_n|J_0) \leq 2N d_E(f_{m_0}, f_n) < \epsilon/2$, so there is $p^j_n \in Y_n|J_0$ with $d(p^k_{m_0}, p^j_n) < \epsilon/2$, therefore $d(p^i_m, p^j_n) < \epsilon$, so $d(p^i_m, Y_n|(I_0 \cup J_0)) < \epsilon$. Similarly we deal with each $p^j_n \in Y_n|(I_0 \cup J_0)$, using n_0 now. This completes the proof of (**).

It follows that $(Y_m|(I_0 \cup J_0))_{m \in A}$ is Cauchy in the Hausdorff metric d_H, so it converges in this metric to some $Y \subseteq X$ with cardinality $\leq N$. Let as usual $f : X \to \mathbb{R}$ be defined by

$$f(x) = \lim_n f_n(x),$$

so that also $\sup\{|f_n(x) - f(x)| : x \in X\} \to 0$. We will show that $f \in E(X, N)$ by checking that Y is a support for f, i.e.

$$f(x) = \min\{f(y) + d(x, y) : y \in Y\}, \ \forall x \in X.$$

So fix any $x \in X$. Choose $p^{t(m)}_m \in Y_m$ with

$$f_m(x) = f_m(p^{t(m)}_m) + d(x, p^{t(m)}_m).$$

Also fix $\epsilon > 0$, and choose $m(\epsilon)$ so that $m(\epsilon) < m < n \Rightarrow d_E(f_m, f_n) < \epsilon$.

Finally, fix $m \in A$, $m(\epsilon) < m$, let n be the least element of A bigger than m. In the bipartite graph for $m < n$, fix a path of length $\leq 2N$ starting from $p_m^{t(m)}$ and ending in $Y_m|(I_0 \cup J_0) \cup Y_n|(I_0 \cup J_0)$. For definiteness assume that it ends in $Y_n|(I_0 \cup J_0)$ and its end is the point $p(m)$, the other case being similar. Denote this path by $p_m^{t(m)}$, $q_1, q_2, \ldots, q_{\ell-1}, p(m)$. Then we have

$$\begin{aligned} f_m(x) &= f_m(p_m^{t(m)}) + d(x, p_m^{t(m)}) \\ f_n(p_m^{t(m)}) &= f_n(q_1) + d(p_m^{t(m)}, q_1) \\ f_m(q_1) &= f_m(q_2) + d(q_1, q_2) \\ &\ldots \\ f_n(q_{\ell-1}) &= f_n(p(m)) + d(q_{\ell-1}, p(m)) \\ f_n(p(m)) &= f_m(p(m)) + (f_n(p(m)) - f_m(p(m))). \end{aligned}$$

Adding these up we get

$$f_m(x) - f_m(p(m)) \geq d(x, p(m)) - 2N\epsilon.$$

Since $Y_m \to Y$ is the Hausdorff metric, there is a subsequence of $(p(m))_{m \in A}$, which converges to some $p \in Y$. Letting $m \to \infty$ on this subsequence we obtain

$$f(x) - f(p) \geq d(x, p) - 2N\epsilon.$$

Since $f_m(x) - f_m(p) \leq d(x, p)$ for all m, we also have $f(x) - f(p) \leq d(x, p)$, thus

$$d(x, p) \geq f(x) - f(p) \geq d(x, p) - 2N\epsilon,$$

so letting $\epsilon \to 0$ we have

$$f(x) = f(p) + d(x, p),$$

so that

$$f(x) = \min\{f(y) + d(x, y) : y \in Y\},$$

and the proof is complete.

It then only remains to give the

Proof of the Claim. Call a pair $\overline{I}$, $\overline{J}$ of subsets of $\{1, \ldots, N\}$ *good* if $\text{card}(\overline{I}) = \text{card}(\overline{J}) = r$, for some $1 \leq r \leq N$, and there is an oriented path $p_1, q_1, p_2, q_2, \ldots, p_r, q_r, p_1$ in the graph, where $\{p_1, \ldots, p_r\} = Y_m|\overline{I}$ and $\{q_1, \ldots, q_r\} = Y_n|\overline{J}$. It is not hard to see that there is a sequence

$$(I_1, J_1), \ldots, (I_k, J_k)$$

of good pairs with $I_a \cap I_b = \emptyset$, $J_a \cap J_b = \emptyset$, if $a \neq b$, and such that if $I = I_1 \cup \ldots \cup I_k$, $J = J_1 \cup \ldots \cup J_k$, then (ii) of the claim holds. So it is enough to prove (i) for these I, J. For this it suffices to show that (i) holds for each good pair $\overline{I}, \overline{J}$. We have

$$f_n(p_1) = f_n(q_1) + d(p_1, q_1)$$

$$f_m(q_1) = f_m(p_2) + d(q_1, p_2)$$
$$f_n(p_2) = f_n(q_2) + d(p_2, q_2)$$
$$\ldots$$
$$f_n(p_r) = f_n(q_r) + d(p_r, q_r)$$
$$f_m(q_r) = f_m(p_1) + d(q_r, p_1),$$

and adding these up we have

$$\begin{aligned} &d(p_1, q_1) + d(q_1, p_2) + d(p_2, q_2) + \ldots + d(p_r, q_r) + d(q_r, p_1) \\ &\quad = (f_n(p_1) - f_m(p_1)) + (f_m(q_1) - f_n(q_1)) + (f_n(p_2) - f_m(p_2)) + \\ &\quad \cdots + (f_m(q_r) - f_n(q_r)), \end{aligned}$$

so for any $p \in Y_m | \overline{I},\ q \in Y_n | \overline{J}$,

$$d(p, q) \leq 2r d_E(f_m, f_n) \leq 2N d_E(f_m, f_n),$$

thus

$$d_H(Y_m | \overline{I},\ Y_n | \overline{J}) \leq 2N d_E(f_m, f_n),$$

which completes the proof. $\dashv$

The following is an open problem: Can every Polish group be represented, up to isomorphism, by a group of the form $\mathrm{Iso}(\mathbb{U}, F)$, for a single closed subset $F \subseteq \mathbb{U}$?

4 Some special cases

We will now look at the complexity of the isometric classification of some special classes of Polish metric spaces.

The following result is contained in Gromov [1999, $3.11\frac{1}{2}_+$ or 3.27].

Theorem 4.1. (Gromov) *Isometry of compact metric spaces is concretely classifiable.*

More explicitly, this means that there is a Borel function $f : F(\mathbb{U}) \to X$, where X is some Polish space, such that for *compact* $K, L \subseteq \mathbb{U}$,

$$K \cong_i L \Leftrightarrow f(K) = f(L).$$

The next interesting case along these lines is that of *locally compact* Polish metric spaces. Here we do not know the precise answer, although it is easy to derive a lower bound. Indeed considering a connected graph as a metric space in the usual way, we see that isomorphism of connected graphs (on $\mathbb{N}$) can be Borel reduced to isometry of discrete Polish metric spaces. Conversely any discrete Polish space can be viewed as a countable structure in such a way that isometries correspond to isomorphisms. So we have

Proposition 4.2. *Isometry of discrete Polish spaces is Borel bireducible with graph isomorphism. In particular, graph isomorphism is Borel reducible to isometry of locally compact Polish metric spaces.*

We can obtain an exact classification if we restrict attention to the class of 0-dimensional locally compact spaces. (A space is *0-dimensional* if it has a clopen basis.)

Theorem 4.3. *Isometry of 0-dimensional locally compact Polish metric spaces is Borel bireducible with graph isomorphism.*

Proof. Given a 0-dimensional locally compact Polish metric space X, consider the countable structure

$$\mathcal{B}_X = \langle B_X, R_X, S_X, T_X^n \rangle_{n \in \mathbb{N}},$$

with

$$\begin{aligned} \mathcal{B}_X &= \{K \subseteq X : K \text{ is compact open}\}, \\ R_X(K_1, K_2) &\Leftrightarrow K_1 \subseteq K_2, \\ S_X(K_1, K_2) &\Leftrightarrow K_1 \cap K_2 = \emptyset, \\ T_X^n(K) &\Leftrightarrow \text{diam}(K) < q_n, \end{aligned}$$

where $\{q_n\}$ enumerates the positive rationals. It is enough to check that

$$X, Y \text{ are isometric} \Leftrightarrow \mathcal{B}_X \cong \mathcal{B}_Y.$$

The direction $\Rightarrow$ is trivial. For the other direction, assume $\varphi : \mathcal{B}_X \to \mathcal{B}_Y$ is an isomorphism. Then for any $x \in X, \bigcap\{\varphi(K) : K \in B_X \ \& \ x \in K\}$ is a singleton, say $\{\psi(x)\}$. It is easy to check that $\psi : X \to Y$ is 1-1 and onto. To see that it is an isometry, fix $x, y \in X$ and say $d_X(x,y) < a$. Then there is compact open $K \supseteq \{x,y\}$ with $\text{diam}(K) < a$. Since $\{\psi(x), \psi(y)\} \subseteq \varphi(\dot{K})$ and $\text{diam}(\varphi(K)) = \text{diam}(K)$, it follows that $d_Y(\psi(x), \psi(y)) < a$. $\dashv$

In a different direction, we can compute exactly the complexity of the isometric classification of ultrametric Polish spaces. Recall that (X, d) is *ultrametric* if

$$d(x,y) \leq \max\{d(x,z), d(y,z)\}.$$

Theorem 4.4. *Isometry of ultrametric Polish spaces is Borel bireducible with graph isomorphism.*

Proof. (A) We will first show that graph isomorphism is Borel reducible to isometry of ultrametric Polish spaces. We will find a Borel map $f : X_L \to F(\mathbb{U})$, where $L = \{R\}$, R a binary relation symbol, such that for each $x \in X_L, f(x)$ is ultrametric and

$$x \cong y \Leftrightarrow f(x) \cong_i f(y).$$

We will do this in two steps: first we will find a Borel map $f_1 : X_L \to \mathcal{T}$, where $\mathcal{T}$ is the space of all nonempty trees on $\mathbb{N}$ (see Kechris [1995]), such that

$$x \cong y \Leftrightarrow f_1(x) \cong f_1(y),$$

where two trees T_1, T_2 on $\mathbb{N}$ are isomorphic, in symbols $T_1 \cong T_2$, iff there is a bijection $\varphi : T_1 \to T_2$ with $\varphi(\emptyset) = \emptyset$ and $s \subseteq t \Leftrightarrow \varphi(s) \subseteq \varphi(t)$. This is already done in Friedman-Stanley [1989] but we will present below a simpler construction, which also has the additional feature that each tree $f_1(x)$ has the property that every $s \in f_1(x)$ has at least 2 immediate extensions in $f_1(x)$ (this property will prove useful in the second step). Denote by $\mathcal{T}_1$ the set of all trees on $\mathbb{N}$ with this property, so that $f_1 : X_L \to \mathcal{T}_1$. In the second step, we will find a Borel map $f_2 : \mathcal{T}_1 \to F(\mathbb{U})$ such that

$$T \cong S \Leftrightarrow f_2(T) \cong_i f_2(S).$$

Finally, we put $f = f_2 \circ f_1$.

We now give the details.

Step 1. There is a Borel function $f_1 : X_L \to \mathcal{T}_1$ such that $x \cong y \Leftrightarrow f(x) \cong f(y)$.

Every $x \in X_L$ represents a structure of the form $\langle \mathbb{N}, R \rangle$, where $R \subseteq \mathbb{N}^2$. For each $(a_1, \ldots, a_n) \in \mathbb{N}^{\mathbb{N}}$ denote by

$$tp_R(a_1, \ldots, a_n)$$

the atomic type of $(a_1, \ldots, a_n)$ in $\langle \mathbb{N}, R \rangle$. There are only countably many atomic types, so let $\tau \mapsto \langle \tau \rangle$ be an injection of the set of such types into $\mathbb{N} \setminus \{0, 1\}$. Now associate to each $\langle \mathbb{N}, R \rangle$ a tree T_R defined as follows: T_R consists of all the initial segments of the sequences of the form

$$(a_0, n_0, a_1, n_1, \ldots, a_k, n_k)$$

that have the following property:

$$n_0 < \langle tp_R(a_0) \rangle$$

$$n_1 < \langle tp_R(a_0, a_1) \rangle$$

$$\ldots$$

$$n_k < \langle tp_R(a_0, \ldots, a_k) \rangle.$$

(Thus $tp_R(a_0, \ldots, a_i)$ is coded in the number of immediate successors of the sequence $\langle a_0, n_0, a_1, n_1, a_2, \ldots, a_i \rangle$.) Note that every $s \in T_R$ has at least two immediate extensions, since $\langle \tau \rangle \geq 2$ for every atomic type τ. So $T_R \in \mathcal{T}_1$.

Now an easy back-and-forth argument shows that

$$\langle \mathbb{N}, R \rangle \cong \langle \mathbb{N}, R' \rangle \Leftrightarrow T_R \cong T_{R'}.$$

So we let $T_1(x) = T_R$, if x represents $\langle \mathbb{N}, R \rangle$.

Step 2. There is a Borel map $f_2 : \mathcal{T}_1 \to F(\mathbb{U})$ such that $f_2(T)$ is ultrametric, and $T \cong S \Leftrightarrow f_2(T) \cong_i f_2(S)$.

On the Baire space $\mathbb{N}^{\mathbb{N}}$ consider the usual ultrametric

$$d(x, y) = 2^{-n-1},$$

where for $x \neq y$, n is the least number such that $x_n \neq y_n$. For a tree T on $\mathbb{N}$ let

$$[T] = \{X \in \mathbb{N}^{\mathbb{N}} : \forall n (x|n \in T)\}$$

be the body of T. This is a closed subset of $\mathbb{N}^{\mathbb{N}}$ and we view it as an ultrametric space restricting d to $[T]$. Clearly there is a Borel map $f_2' : \mathcal{T} \to F(\mathbb{U})$ such that $[S]$ is isometric to $f_2'(S)$. We take $f_2 = f_2'|\mathcal{T}_1$. It remains therefore to show that for $S, T \in \mathcal{T}_1$:

$$S \cong T \Leftrightarrow [S], [T] \text{ are isometric.}$$

It is clear that if $S \cong T$, then $[S], [T]$ are isometric. Now assume that $[S], [T]$ are isometric, in order to show that $S \cong T$. Fix an isometry $\varphi : [S] \to [T]$.

For each $u \in \mathbb{N}^{<\mathbb{N}}$, let $N_u = \{x \in \mathbb{N}^{\mathbb{N}} : x|\text{length}(u) = u\}$ and put $[S_u] = [S] \cap N_u$, $[T_u] = [T] \cap N_u$. Then notice that for $u \in S$, diam $([S_u]) = 2^{-\text{length}(u)-1}$, since u has at least two immediate extensions in S. So diam$(\varphi([S_u]) = 2^{-\text{length}(u)-1}$, from which it follows that all $\varphi(x)$, $x \in [C_u]$, agree in their first length(u) many coordinates. Let $\psi(u) = \varphi(x)|\text{length}(u) \in T$, for any $x \in [S_u]$. Thus

$$\varphi([S_u]) \subseteq [T_{\psi(u)}].$$

If $y \in [T_{\psi(u)}]$ and we choose any $x \in S_u$, we see that $d(y, \varphi(x)) \leq 2^{-\text{length}(u)-1}$, so $d(\varphi^{-1}(y), x) \leq 2^{-\text{length}(u)-1}$, thus $\varphi^{-1}(y), x$ agree in their first length(u) many coordinates, so $\varphi^{-1}(y) \in [S_u]$. Thus

$$\varphi([S_u]) = [T_{\psi(u)}].$$

It is now easy to check that $\psi : S \to T$ is an isomorphism.

(B) Finally, we will show that isometry of ultrametric spaces is Borel reducible to isomorphism of (countable) structures in some appropriate language.

Let (X, d) be an ultrametric space. Consider the set of all open balls

$$B_r(x) = \{y \in X : d(y, x) < r\},$$

with rational $r > 0$. Since d is ultrametric

$$y \in B_r(x) \Rightarrow B_r(x) = B_r(y),$$

so $\{B_r(x) : x \in X, r > 0 \text{ rational}\} = \{B_r(d) : d \in D, r > 0 \text{ rational}\}$, for each countable dense subset D of X. Thus this set of balls is countable.

Moreover, each such ball is actually clopen (see Kechris [1995, 7.1]). Consider now the countable structure

$$\mathcal{A}_X = \langle A_X, R_X, S_X^n \rangle_{n\in\mathbb{N}},$$

where

$$\begin{aligned} A_X &= \{B(x,r) : x \in X, r > 0 \text{ rational}\}, \\ R_X(B_1, B_2) &\Leftrightarrow B_1 \subseteq B_2, \\ S_X^n(B) &\Leftrightarrow \mathrm{diam}(B) < q_n, \end{aligned}$$

with $\{q_n\}$ enumerating the positive rationals. We claim that

$$X, Y \text{ are isometric} \Leftrightarrow \mathcal{A}_X \cong \mathcal{A}_Y.$$

The direction $\Rightarrow$ is obvious. Conversely assume $\mathcal{A}_X \cong \mathcal{A}_Y$ and fix an isomorphism $\varphi : \mathcal{A}_X \to \mathcal{A}_Y$. We will use this to define an isometry between X and Y. Fix $x \in X$ and consider the balls $B_{1/n}(x)$. If $\varphi(B_{1/n}(x)) = B_n$, then $B_1 \supseteq B_2 \supseteq \dots$, and the diameters of $B_n \to 0$, so $\bigcap_n B_n$ is a singleton, say $\{\psi(x)\}$.

We claim that $\psi : X \to Y$ is an isometry. That ψ is 1-1 follows from the fact that for $B_1, B_2 \in \mathcal{A}_X$, $B_1 \cap B_2 \neq \emptyset \Leftrightarrow B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

Reversing the roles of X, Y in the above, we see that ψ is onto. Finally, if $d_X(x,y) < q_n$, then $y \in B_{q_n}(x)$, so $B_{q_n}(y) = B_{q_n}(x)$. Now $\varphi(B_{q_n}(y)) = \varphi(B_{q_n}(x))$ has diameter q_n and contains both $\varphi(x), \varphi(y)$, so $d_Y(\varphi(x), \varphi(y)) \leq q_n$. It follows that $d_Y(\varphi(x), \varphi(y)) = d_X(x,y)$. $\dashv$

Every ultrametric Polish space is 0-dimensional. However, we do not know the exact complexity of the isometric classification of 0-dimensional Polish metric spaces.

We can finally obtain some information about isometry groups.

Theorem 4.5. *Up to isomorphism, the isometry groups of 0-dimensional locally compact Polish metric spaces are exactly the closed subgroups of* S_∞.

Proof. Let X be such a space and let $I = \{K \subseteq X : K \text{ is compact open}\}$, so that I is a countable set. Every element $\varphi \in \mathrm{Iso}(X)$ induces a permutation φ^* of I and it is easy to see that $\varphi \mapsto \varphi^*$ is an isomorphism of $\mathrm{Iso}(X)$ with a closed subgroup of the group of permutations of I (with the pointwise convergence topology), so $\mathrm{Iso}(X)$ is isomorphic to a closed subgroup of S_∞.

Conversely, let G be a closed subgroup of S_∞. Then (see, e.g., Becker-Kechris [1996]) there is a sequence of relations $R_n \subseteq \mathbb{N}^n$ such that $G = \mathrm{Aut}(\langle \mathbb{N}, R_n \rangle_{n\in\mathbb{N}})$ (the automorphism group of the structure $\langle \mathbb{N}, R_n \rangle_{n\in\mathbb{N}}$). Now it is well-known that for any countable structure $\mathcal{A}$ in a countable language there is a countable graph $\mathcal{G}$ such that $\mathrm{Aut}(\mathcal{A})$ is isomorphic to $\mathrm{Aut}(\mathcal{G})$. We can assign now to $\mathcal{G}$ a discrete metric space $X_\mathcal{G}$, whose points are the vertices of $\mathcal{G}$ and whose metric is defined by: $d(x,y) = \frac{n}{n+1}$, if n is the length of the shortest path between x, y, if such exists; $d(x,y) = 1$, otherwise. Clearly $\mathrm{Aut}(\mathcal{G}) = \mathrm{Iso}(X_\mathcal{G})$, so this shows that G is isomorphic to the isometry group of a 0-dimensional locally compact Polish metric space (in fact a discrete one).

For the convenience of the reader, we sketch the construction of $\mathcal{G}$ from $\mathcal{A}$. Assume without loss of generality that $\mathcal{A} = \langle A, R_i \rangle_{i\in I}$, when R_i is an n_i-any relation symbol. First we will replace $\mathcal{A}$ by a countable structure $\mathcal{B}$ which has only unary and binary relations and $\mathrm{Aut}(\mathcal{A})$ is isomorphic to $\mathrm{Aut}(\mathcal{B})$. We simply take the universe of B to be the disjoint union $A \sqcup A^2 \sqcup A^3 \sqcup \dots$ and define the relations of $\mathcal{B}$ to be those of $\mathcal{A}$ (which now become unary) together with the following:

$$\begin{aligned} T_n(x) &\Leftrightarrow x \in A^n \quad (1 \le n) \\ P_{i,n}(x,y) &\Leftrightarrow x \in A, y \in A^n, \text{ say } y = \langle y_1, \dots, y_n \rangle, \\ &\qquad \text{and } x = y_i \ (1 \le i \le n). \end{aligned}$$

Next we replace $\mathcal{B}$ by a countable structure $\mathcal{C}$ which has only finitely many relation symbols and $\mathrm{Aut}(\mathcal{B})$ is isomorphic to $\mathrm{Aut}(\mathcal{C})$. Here we enumerate as $\{T_n\}$, resp. $\{P_n\}$ the unary, resp., binary relations of $\mathcal{B}$, define the universe of $\mathcal{C}$ to be the disjoint union $B \sqcup \mathbb{N}$, and define the relations of $\mathcal{C}$ as follows:

$$\begin{aligned} U_1(x) &\Leftrightarrow x \in B \\ U_2(x) &\Leftrightarrow x \in \mathbb{N} \\ V_1(x,y) &\Leftrightarrow x, y \in \mathbb{N} \text{ and } x < y \\ V_2(x,y) &\Leftrightarrow x \in \mathbb{N} \text{ and } y \in B \text{ and } T_x(y) \\ V_3(x,y,z) &\Leftrightarrow x \in \mathbb{N} \text{ and } y, z \in B \text{ and } P_x(y,z). \end{aligned}$$

Finally, there is a standard procedure which replaces $\mathcal{C}$ by a countable graph $\mathcal{G}$, with $\mathrm{Aut}(\mathcal{C})$ isomorphic to $\mathrm{Aut}(\mathcal{G})$, see, e.g., Hodges [1993], pp. 228–229. $\dashv$

By a similar argument it can be seen that the isometry group of an ultrametric Polish space is isomorphic to a closed subgroup of S_∞, but we do not know how to characterize exactly these isometry groups.

5 Isometries of locally compact spaces, I: The pseudo-connected case

We will analyze in this and the next section the isometry groups of locally compact Polish metric spaces and their actions on the underlying space.

It turns out that this analysis does not require the completeness of the given metric. So from now on we will assume that (X, d) is just locally compact separable (with d not necessarily complete).

For each $x \in X$, we define its *radius of compactness*, $\rho(x)$, by

$$\rho(x) = \sup\{r > 0 : B_r^{c\ell}(x) \text{ is compact}\},$$

where

$$B_r^{c\ell}(x) = \{y \in X : d(x, y) \leq r\}$$

(recall that $B_r(x) = \{y \in X : d(x, y) < r\}$). So $0 < \rho(x) \leq \infty$.

Note that if $\rho(x) = \infty$ for *some* $x \in X$, then $\rho(x) = \infty$ for *all* $x \in X$. If this happens, then d is called a *Heine-Borel* metric, and (X, d) a *Heine-Borel space*. The standard example is of course $\mathbb{R}^n$.

We first record the following simple fact, where in the formula below we agree that $\infty - \infty = 0$.

Proposition 5.1. *$\rho(x)$ is a Lipschitz function, i.e., $|\rho(x) - \rho(y)| \leq d(x, y)$.*

Proof. We check that $\rho(x) \leq \rho(y) + d(x, y)$. If $\rho(x) \leq d(x, y)$, we are done. Else choose any r such that $d(x, y) < r < \rho(x)$. Then $B_r^{c\ell}(x)$ is compact. But $B_{r-d(x,y)}^{c\ell}(y) \subseteq B_r^{c\ell}(x)$, so $B_{r-d(x,y)}^{c\ell}(y)$ is compact, thus $r - d(x, y) \leq \rho(y)$, or $r \leq \rho(y) + d(x, y)$, and, since r was arbitrary, $\rho(x) \leq \rho(y) + d(x, y)$. $\dashv$

It follows that ρ is continuous and if $K \subseteq X$ is compact, there is $\rho > 0$ such that $\rho(x) \geq \rho$ for all $x \in K$.

We will first note that, with the pointwise convergence topology, the group $\mathrm{Iso}(X, d)$ of a locally compact separable metric space is Polish. The main point of course is that we do not assume that the metric d is complete.

Proposition 5.2. *Let (X, d) be a locally compact separable metric space. Then* $\mathrm{Iso}(X)$*, equipped with the pointwise convergence topology, is Polish.*

Proof. Fix a dense sequence (x_n) in X. Then put on $\mathrm{Iso}(X)$ the metric

$$\delta(\varphi, \psi) = \sum_n \frac{1}{2^n} \left[\frac{d(\varphi(x_n), \psi(x_n))}{1 + d(\varphi(x_n), \psi(x_n))} + \frac{d(\varphi^{-1}(x_n), \psi^{-1}(x_n))}{1 + d(\varphi^{-1}(x_n), \psi^{-1}(x_n))} \right].$$

This is compatible with the topology of $\mathrm{Iso}(X)$, so it is enough to show that δ is complete.

Let (φ_i) be δ-Cauchy. Then it is clear that for each fixed n, $(\varphi_i(x_n))$ is d-Cauchy. Moreover, $\rho(\varphi_i(x_n)) = \rho(x_n), i \in \mathbb{N}$, from which it follows that for some large enough N and positive $\epsilon > 0, d(\varphi_i(x_n), \varphi_N(x_n)) \leq \epsilon < \rho(\varphi_N(x_n))$, for all $i \geq N$, thus $(\varphi_i(x_n))_{i \in \mathbb{N}}$ has a convergent subsequence and, since it is d-Cauchy, $(\varphi_i(x_n))_{i \in \mathbb{N}}$ converges, say

$$\varphi_i(x_n) \to y_n, \text{ as } i \to \infty.$$

It follows that for each $x \in X, (\varphi_i(x))_{i \in \mathbb{N}}$ converges. Indeed, for any n, $d(\varphi_i(x), \varphi_j(x)) \leq d(\varphi_i(x), \varphi_i(x_n)) + d(\varphi_i(x_n), \varphi_j(x_n)) + d(\varphi_j(x_n), \varphi_j(x)) = 2d(x, x_n) + d(\varphi_i(x_n), \varphi_j(x_n))$, which shows that $(\varphi_i(x))$ is d-Cauchy, so exactly as before it converges. Let

$$\varphi(x) = \lim_{i \to \infty} \varphi_i(x).$$

Similarly $(\varphi_i^{-1}(x))$ converges and let

$$\psi(x) = \lim_{i \to \infty} \varphi_i^{-1}(x).$$

It is enough to show that $\varphi \in \mathrm{Iso}(X)$.

Clearly, $d(\varphi(x), \varphi(y)) = d(x, y)$. Also note that

$$\begin{aligned} d(\varphi(\psi(x)), x) &= \lim_{i \to \infty} d(\varphi_i(\psi(x)); x) \\ &= \lim_{i \to \infty} d(\psi(x), \varphi_i^{-1}(x)) \\ &= 0 = d(\psi(\varphi(x)), x), \end{aligned}$$

so $\varphi = \psi^{-1}$ and this completes the proof. ⊣

We will now introduce the concept of a pseudo-connected locally compact separable metric space, which plays a crucial role in our analysis.

Let (X, d) be a locally compact separable metric space. Define a directed graph R on X by

$$xRy \Leftrightarrow x \neq y \text{ and } d(x,y) < \rho(x).$$

Let R^* be the transitive closure of R, i.e.,

$$xR^*y \Leftrightarrow \text{for some } u_0 = x, u_1, \ldots, u_n = y$$

$$\text{we have } \forall i < n(u_i R u_{i+1}).$$

Finally, define the following equivalence relation E on X

$$xEy \Leftrightarrow x = y \text{ or } (xR^*y \text{ and } yR^*x).$$

We call the E-equivalence class of x the *pseudo-component* of x, and denote it by $C(x)$. We call X *pseudo-connected* if it has only one pseudo-component. We have

Proposition 5.3 *i) Each pseudo-component is clopen, so there are only countably many pseudo-components.*

ii) If X is either connected or the metric d is Heine-Borel, then X is pseudo-connected.

Proof. It is of course enough to show that each pseudo-component is open. So fix $x \in X$ and $y \in C(x)$. Fix $0 < r_0 < \rho(y)$. Then $B^{c\ell}_{r_0}(y)$ is compact, so there is $r_0 > \rho_0 > 0$ with $\rho(z) > \rho_0$, for all $z \in B^{c\ell}_{r_0}(y)$. Then it is easy to check that $B_{\rho_0}(y) \subseteq C(x)$.

ii) The first statement follows from i). The second is obvious from the definition of pseudo-components. ⊣

We will first analyze the isometry groups of pseudo-connected spaces.

The main facts below generalize results of van Dantzig-van der Waerden [1928] (see also Strantzalos [1974], [1989]), who dealt with the case where X is *connected*. Strantzalos [1989] contains another generalization in the case where the space of connected components of X is compact.

Recall that an action $(g, x) \mapsto g \cdot x$ of a topological group G on a topological space X is *proper* if for every $x, y \in X$ there are open nbhds U_x, U_y of x, y, resp., such that $\{g \in G : g \cdot U_x \cap U_y \neq \emptyset\}$ is precompact (i.e., has compact closure).

We now have

Theorem 5.4. *Let (X,d) be a pseudo-connected locally compact separable metric space. Then the action of* $\mathrm{Iso}(X)$ *(by evaluation) on X is proper.*

Proof. We will make use of the following lemma:

Lemma 5.5. *Let (X,d) be a locally compact separable metric space. Suppose that d is not Heine-Borel, $N > 0, 0 < \epsilon_0, \dots, \epsilon_{N-1} < 1$ and for each $i \in \mathbb{N}, (x_0^i, x_1^i, \dots, x_N^i) \in X^{N+1}$ are such that $d(x_k^i, x_{k+1}^i) \leq \epsilon_k \rho(x_k^i)$ for each $k \leq N-1$, and $(x_0^i)_{i\in\mathbb{N}}$ has a convergent subsequence. Then $(x_N^i)_{i\in\mathbb{N}}$ has a convergent subsequence. Similarly, if d is Heine-Borel and $d(x_k^i, x_{k+1}^i) \leq M_k$, for some given $M_0, \dots, M_{N-1} > 0$.*

Proof. Consider the case where d is not Heine-Borel, the other case being similar. We show by induction on $k \leq N$ that $(x_k^i)_{i\in\mathbb{N}}$ has a convergent subsequence. This is given for $k = 0$. By induction hypothesis, assume then, without loss of generality, that $(x_k^i)_{i\in\mathbb{N}}$ converges to some x_k, where $k < N$. We will then find a convergent subsequence of (x_{k+1}^i). We have that $d(x_k^i, x_{k+1}^i) \leq \epsilon_k\rho(x_k^i)$ and $\rho(x_k^i) \to \rho(x_k)$, so if $\delta_k > 0$ is such that $\epsilon_k + 2\delta_k < 1$, then for i large enough

$$\begin{aligned} d(x_k, x_{k+1}^i) &\leq d(x_k^i, x_k) + d(x_k^i, x_{k+1}^i) \\ &\leq \delta_k\rho(x_k) + \epsilon_k\rho(x_k) + \delta_k\rho(x_k) \\ &= (\epsilon_k + 2\delta_k)\rho(x_k) < \rho(x_k), \end{aligned}$$

so (x_{k+1}^i) has a convergent subsequence. $\dashv$

Now fix $x, y \in X$ and let $0 < r < \frac{1}{2}\min\{\rho(x), \rho(y)\}, U_x = B_r(x), U_y = B_r(y)$. Then for $g \in \mathrm{Iso}(X)$,

$$g(U_x) \cap U_y \neq \emptyset \Rightarrow d\big(g(x), y\big) < 2r.$$

So it is enough to show that the closed set

$$K = \{g \in \mathrm{Iso}(X) : d\big(g(x), y\big) \leq 2r\}$$

is compact. We could use here the Arzela-Ascoli Theorem (see Dugundji [1966], 6.4) but we prefer to employ a direct elementary argument.

It is enough to show that if $\{g_i\} \subseteq K$, then (g_i) has a convergent (in $\mathrm{Iso}(X)$) subsequence. For that it is enough, by the argument in the proof of 5.2, to show that there is a subsequence (g_{n_i}) such that $(g_{n_i}(z)), (g_{n_i}^{-1}(z))$ converge, for every $z \in X$. Fix a dense set $\{x_n\}$ in X. Again as in the proof of 5.2, it is enough to find (g_{n_i}) so that $(g_{n_i}(x_n))_{i\in\mathbb{N}}, (g_{n_i}^{-1}(x_n))_{i\in\mathbb{N}}$ converge for each n, and, by the usual diagonal argument, it is enough to show that for each *fixed* n, we can find (g_{n_i}) so that $(g_{n_i}(x_n))_{i\in\mathbb{N}}, (g_{n_i}^{-1}(x_n))_{i\in\mathbb{N}}$ converge. So finally we reduced our problem to showing that, for each fixed $z \in X$, there is (g_{n_i}) so that $(g_{n_i}(z)), (g_{n_i}^{-1}(z))$ converge, and, noticing that for $g \in K$ we also have that $d(g^{-1}(y), x) \leq 2r < \rho(x)$, it is enough to show that we can find (g_{n_i}) such that $(g_{n_i}(z))$ converges, the argument for $(g_{n_i}^{-1}(z))$ being similar. We will also assume that d is not Heine-Borel, the proof in the other case being analogous.

Fix a finite sequence $z_0 = x, z_1, \dots, z_N = z$ such that $d(z_k, z_{k+1}) < \rho(z_k)$. Let $0 < \epsilon_k < 1$ be such that $d(z_k, z_{k+1}) \leq \epsilon_k \rho(z_k)$. Put $y_k^i = g_i(z_k)$, for $k \leq N$. Then $y_0^i = g_i(x) \in B_{2r}^{c\ell}(y)$, so it has a convergent subsequence, and $d(y_k^i, y_{k+1}^i) = d(g_i(z_k), g_i(z_{k+1})) = d(z_k, z_{k+1}) \leq \epsilon_k \rho(z_k) = \epsilon_k \rho(y_k^i)$ for $k \leq N-1$. So, by 5.5, $(y_N^i) = (g_i(z))$ has a convergent subsequence. ⊣

The properness of the action has the following standard implications (see, e.g., Strantzalos [1989]), whose straightforward arguments we include for the reader's convenience.

Corollary 5.6. *Let X be a pseudo-connected locally compact separable metric space and $G \subseteq \mathrm{Iso}(X)$ a closed subgroup. Then we have:*

(i) $\mathrm{Iso}(X)$ is locally compact.

(ii) The action of G on X is proper, each stabilizer $G_x = \{g \in G : g(x) = x\}$ is compact, and the orbit equivalence relation

$$xE_G^X y \Leftrightarrow \exists g \in G\big(g(x) = y\big)$$

is closed (as a subset of X^2), so, in particular, each orbit $G(x) = \{g(x) : g \in G\}$ is closed. Moreover, the map $g \mapsto g(x)$ from G onto $G(x)$ is open.

(iii) The statements in part (ii) hold as well for the action of G on $X^n, n \geq 1$, by coordinatewise evaluation: $g\cdot(x_1, \dots, x_n) = (g(x_1), \dots, g(x_n))$.

Proof. (i) By the definition of properness, for some open nbhd U_x of x, $\{g \in \mathrm{Iso}(X) : g(U_x) \cap U_x \neq \emptyset\}$ is a precompact open nbhd of the identity of $\mathrm{Iso}(X)$, so $\mathrm{Iso}(X)$ is locally compact.

(ii) The first assertion is obvious. For the second, notice that for some open nbhd U_x of x, the set $\{g \in G : g(U_x) \cap U_x \neq \emptyset\}$ is precompact and the closed set G_x is contained in it, so G_x is compact. Finally, assume that $x_n E_G^X y_n, x_n \to x, y_n \to y$. Fix $g_n \in G$ with $g_n(x_n) = y_n$. Let also U_x, U_y be open nbhds of x, y, resp., with $K = \{g \in G : g(U_x) \cap U_y \neq \emptyset\}$ precompact.

Then, for large enough n, $x_n \in U_x, y_n = g_n(x_n) \in U_y$, so $g_n \in K$, thus we can find a convergent subsequence $g_{n_i} \to g$. Then $g(x) = \lim g_{n_i}(x_{n_i}) = \lim y_{n_i} = y$, so $xE_G^X y$.

Finally we check that $g \mapsto g(x)$ is open from G onto $G(x)$. Let U be open in G, in order to show that $\{g(x) : g \in U\}$ is open in $G(x)$, or, equivalently, $G(x) \setminus \{g(x) : g \in U\}$ is closed. Let $g_n(x) \in G(x) \setminus \{g(x) : g \in U\}$, and $g_n(x) \to y \in \{g(x) : g \in U\}$, towards a contradiction. Then $y = g(x)$ for some $g \in U$ but $g_n \notin UG_x$ for each n. Also, by properness, some subsequence (g_{n_i}) converges, say to h. Clearly, as UG_x is open, $h \notin UG_x$. But $g_{n_i}(x) \to h(x) = y = g(x)$, so $g^{-1}h \in G_x$ and $h \in gG_x \subseteq UG_x$, a contradiction.

(iii) The action of G on X^n is proper by (ii). $\dashv$

Remarks. As we will see in the next section, Iso(X) is still locally compact if X has only finitely many pseudo-components. This of course fails for arbitrary locally compact X, as S_∞ is the isometry group of a countable discrete space. The compactness of the stabilizers though can fail even if X has two pseudo-components. For example, let G be a connected non-compact second countable locally compact group and d a left-invariant compatible metric on G. We can assume that $d < 1$. Let $x_0 \notin G$ and define $d(x_0, g) = 1, \forall g \in G$. Then if $X = G \cup \{x_0\}, (X, d)$ has two pseudo-components, namely G and $\{x_0\}$, but the stabilizer of x_0 is not compact, since it contains an isomorphic copy of G. (This example is essentially the same as 4.1 in Strantzalos [1989].)

This shows that the action of Iso(X) on X may fail to be proper even if X has two pseudo-components. Finally, we do not know if the orbits of the isometry group acting on X are closed, when X has only finitely many pseudo-components. This should fail if X is arbitrary, but we do not actually know a counterexample.

In the sequel, if X, Y are metric spaces, $\vec{x} \in X^n, \vec{y} \in Y^n, n \geq 0$, then $(X, \vec{x}) \cong_i (Y, \vec{y})$ means that there is an isometry $\varphi : X \to Y$ with $\varphi(x_i) = y_i, \forall i \leq n$.

We note here an application of the preceding analysis to the isometry problem for pointed pseudo-connected locally compact *Polish* metric spaces (X, d), i.e., where d is now complete. This can be viewed as the equivalence relation $(F, x) \cong_i (H, y)$ on the following subset of the standard Borel space $F(\mathbb{U}) \times \mathbb{U}$:

$$F_0^{plc}(\mathbb{U}) = \{(F, x) : F \in F(\mathbb{U}), F \text{ is pseudo-connected locally compact}, x \in F\}.$$

¿From this we also obtain the complexity of the isometry equivalence relation of homogeneous pseudo-connected locally compact Polish metric spaces.

Theorem 5.7. *The equivalence relation of isometry of pointed pseudo-connected locally compact Polish metric spaces is concretely classifiable.*

Proof. Fix a pseudo-connected locally compact separable metric space (X, d), and a point $x \in X$. Assume first that we are dealing with non-Heine-Borel spaces. For $n \geq 1$, let

$$K_n(X,x) = \{y : \exists x_0 = x, x_1, \ldots, x_n = y, \forall i < n(d(x_i, x_{i+1}) \leq \frac{n}{n+1}\rho(x_i))\}.$$

Then, by 5.5, we see that $K_n(X, x)$ is compact. Moreover $K_n(X, x) \subseteq K_{n+1}(X, x)$ and $\bigcup_n K_n(X, x) = X$.

For each $n \geq 1, m \geq 1$, define now the following compact subset $K_{n,m}(X, x)$ of $\mathbb{R}^{m^2+m}$

$$K_{n,m}(X,x) = \{(d(x_i, x_j))_{0 \leq i,j \leq m-1}\hat{\ }(\rho(x_i))_{0 \leq i \leq m-1} :$$

$$x_1, \ldots, x_{m-1} \in K_n(X,x), x_0 = x\}.$$

Finally, let

$$L(X,x) = (K_{n,m}(X,x))_{n \geq 1, m \geq 1} \in \prod_{n \geq 1, m \geq 1} K(\mathbb{R}^{m^2+m}),$$

where $K(\mathbb{R}^n)$ is the Polish space of compact subsets of $\mathbb{R}^n$ with the Hausdorff metric. Now we claim that

$$(X,x) \cong_i (Y,y) \Leftrightarrow L(X,x) = L(Y,y),$$

which completes the proof in this case, as it is easy to check that the function L restricted to $F_0^{plc}(\mathbb{U})$ is Borel. The Heine-Borel case is similar, replacing in the definition of $K_n(X, x)$ above $\frac{n}{n+1}\rho(x_i)$ by n and omitting the $\rho(x_i)$ in $K_{n,m}(X, x)$.

Direction $\Rightarrow$ is obvious. For $\Leftarrow$ note that by an argument similar to that in Gromov [1999, 3.27], for each $n \geq 1$, there is an isometry φ_n : $K_n(X, x) \to K_n(Y, y)$, with $\varphi_n(x) = y$ and $\rho(u) = \rho(\varphi_n(u))$, for each $u \in K_n(X, x)$, so that in particular $\varphi_n | K_m(X, x)$ is an isometry from $K_m(X, x)$ onto $K_m(Y, y)$, for all $m \leq n$. Consider then $\varphi_n | K_1(X, x), n = 1, 2, \ldots$ These are isometries from $K_1(X, x)$ onto $K_1(Y, y)$, so there is a subsequence (n_i) such that $(\varphi_{n_i} | K_i(X, x))$ converges (pointwise). Similarly there is a subsequence of (n_i), say (n_{i_j}), such that $(\varphi_{n_{i_j}}|K_2(X,x))$ converges, etc. By a standard diagonal argument, we can finally find a subsequence (φ_{i_n}) of (φ_n) such that for each n, $(\varphi_{i_j}|K_n(X,x))_{j\geq n}$ converges. Let $\varphi(u) = \lim_{n\to\infty}\varphi_{i_n}(u)$. Then φ is an isometry between X and Y with $\varphi(x) = y$. $\dashv$

Corollary 5.8. *The equivalence relation of isometry of homogeneous pseudo-connected locally compact Polish metric spaces is concretely classifiable.*

(Recall that a metric space is *homogenous* if its isometry group acts transitively on the space.)

Proof. There is a Borel function S assigning to each $F \in F(\mathbb{U}), F \neq \emptyset$ an element $S(F) \in F$. Then note that for $F, H \in F(\mathbb{U})$ pseudo-connected locally compact and homogeneous:

$$\begin{aligned} F \cong_i H &\Leftrightarrow (F, S(F)) \cong_i (H, S(H)) \\ &\Leftrightarrow L(F, S(F)) = L(H, S(H)). \end{aligned}$$

Since the function $F \mapsto L(F, S(F))$ is Borel, we are done. $\dashv$

We remark that 5.7 fails for arbitrary locally compact spaces as we can easily Borel reduce isomorphism of trees on $\mathbb{N}$ (see the proof of 4.3) to isometry of pointed locally compact Polish metric spaces. We do not know if 5.7 is true in the case of locally compact spaces with only finitely many pseudo-components.

We finally give an analysis of the isometry types of n-tuples in pseudo-connected locally compact separable metric spaces, reminiscent of the Scott analysis in model theory (or equivalently Ehrenfeucht-Fraissé games).

Suppose $(X, d_X), (Y, d_Y)$ are locally compact separable metric spaces and $\vec{x} \in X^n, \vec{y} \in Y^n, n \geq 1$. For each ordinal α, we define the notion of α-equivalence, $\equiv^\alpha$, by induction, as follows:

$$\begin{aligned} (X,\vec{x}) \equiv^0 (Y,\vec{y}) \Leftrightarrow\ & \forall i,j \leq n(d_X(x_i,x_j) = d_Y(y_i,y_j))\ \& \\ & \forall i \leq n(\rho_X(x_i) = \rho_Y(y_i)). \end{aligned}$$

If λ is limit:

$$(X,\vec{x}) \equiv^\lambda (Y,\vec{y}) \Leftrightarrow \forall \alpha < \lambda((X,\vec{x}) \equiv^\alpha (Y,\vec{y})).$$

Finally, we let

$$\begin{aligned}(X,\vec{x}) \equiv^{\alpha+1} (Y,\vec{y}) \Leftrightarrow {} & (X,\vec{x}) \equiv^{\alpha} (Y,\vec{y}) \ \& \ \forall r \in \mathbb{Q}^{+} \forall x_{n+1} \in X[d_X(x_n, x_{n+1}) < r \\ & < \rho_X(x_n) \Rightarrow \exists y_{n+1} \in Y(d_Y(y_n, y_{n+1}) \leq r \ \& \\ & (X, \vec{x}\,\hat{}\,x_{n+1}) \equiv^{\alpha} (Y, \vec{y}\,\hat{}\,y_{n+1})] \\ & \& \text{ vice versa.}\end{aligned}$$

Sometimes we will simply write $\vec{x} \equiv^{\alpha} \vec{y}$, instead of $(X,\vec{x}) \equiv^{\alpha} (Y,\vec{y})$.

We now have, letting for each ordinal α and $n \geq 1$,

$$\Phi_n^{\alpha}(X,Y) = \Phi_n^{\alpha} = \{(\vec{x},\vec{y}) \in X^n \times Y^n : (X,\vec{x}) \equiv^{\alpha} (Y,\vec{y})\}.$$

Proposition 5.9. *For any locally compact separable metric spaces (X, d_X), (Y, d_Y), any ordinal α, and any $n \geq 1$, the set $\Phi_n^{\alpha}(X,Y)$ is closed in $X^n \times Y^n$.*

Proof. We will omit explicitly indicating X, Y, when they are understood.

We prove this proposition by induction on α. Note that for $\alpha = 0$,

$$\begin{aligned}(\vec{x},\vec{y}) \in \Phi_n^0 \Leftrightarrow & \forall r \in \mathbb{Q}^{+}[\forall i, j \leq n(d(x_i, x_j) > r \Rightarrow d(y_i, y_j) \geq r) \\ & \& \text{ vice versa } \& \ \forall i(\rho(x_i) > r \Rightarrow \rho(y_i) \geq r) \ \& \text{ vice versa }],\end{aligned}$$

so clearly Φ_n^0 is closed.

The limit case is obvious, as $\Phi_n^{\lambda} = \bigcap_{\alpha<\lambda} \Phi_n^{\alpha}$. So assume that Φ_m^{α} is closed, for each m, in order to show that $\Phi_n^{\alpha+1}$ is closed.

Notice that $\gamma < \delta \Rightarrow \Phi_n^{\gamma} \supseteq \Phi_n^{\delta}$, so if $(\vec{x},\vec{y}) \in \Phi_n^{\alpha}$, we also have $(\vec{x},\vec{y}) \in \Phi_n^0$, thus $\rho(x_n) = \rho(y_n)$. Now

$$\begin{aligned}(\vec{x},\vec{y}) \in \Phi_n^{\alpha+1} \quad \Leftrightarrow \quad & (\vec{x},\vec{y}) \in \Phi_n^{\alpha} \ \& \ \forall r \in \mathbb{Q}^{+} \forall x_{n+1}[d(x_n, x_{n+1}) < r < \rho(x_n) \Rightarrow \\ & \exists y_{n+1}(d(y_n, y_{n+1}) \leq r \ \& \ (\vec{x}\,\hat{}\,x_{x+1}, \vec{y}\,\hat{}\,y_{n+1}) \in \Phi_{n+1}^{\alpha})] \\ & \& \text{ vice versa.}\end{aligned}$$

So assume that $\vec{x}^i \to \vec{x}, \vec{y}^i \to \vec{y}$ and $(\vec{x}^i, \vec{y}^i) \in \Phi_n^{\alpha+1}$, in order to show that $(\vec{x},\vec{y}) \in \Phi_n^{\alpha+1}$. Since $(\vec{x}^i, \vec{y}^i) \in \Phi_n^{\alpha}$, by induction hypothesis it follows that $(\vec{x},\vec{y}) \in \Phi_n^{\alpha}$. Since $(\vec{x}^i, \vec{y}^i) \in \Phi_n^0$, we have in particular that $\rho(x_n^i) = \rho(y_n^i)$.

Now fix $r \in \mathbb{Q}^{+}, x_{n+1}$ with $d(x_n, x_{n+1}) < r < \rho(x_n)$. We will find y_{n+1} with $d(y_n, y_{n+1}) \leq r$ and $(\vec{x}\,\hat{}\,x_{n+1}, \vec{y}\,\hat{}\,y_{n+1}) \in \Phi_{n+1}^{\alpha}$. For some $\epsilon \in \mathbb{Q}^{+}$ and i large enough we have that $d(x_n^i, x_{n+1}) < r - \epsilon < r < r + \epsilon < \rho(x_n^i)$, so there is y^i_{n+1} with $d(y^i_n, y^i_{n+1}) \leq r - \epsilon$ and $(\vec{x}^i\,\hat{}\,x_{n+1}, \vec{y}^i\,\hat{}\,y^i_{n+1}) \in \Phi^\alpha_{n+1}$. Now $d(y^i_{n+1}, y_n) \leq d(y^i_{n+1}, y^i_n) + d(y^i_n, y_n) \leq r - \epsilon + d(y^i_n, y_n)$, so, if i is large enough $d(y^i_{n+1}, y_n) \leq r$. Also $\rho(y_n) = \lim_i \rho(y^i_n) = \lim_i \rho(x^i_n) \geq r + \epsilon > r$, thus, for i large enough, $d(y^i_{n+1}, y_n) \leq r < \rho(y_n)$, so (y^i_{n+1}) has a convergent subsequence $y^{i_k}_{n+1} \to_k y_{n+1}$. Then $d(y_n, y_{n+1}) \leq r$ and, as Φ^α_{n+1} is closed, $(\vec{x}\,\hat{}\,x_{n+1}, \vec{y}\,\hat{}\,y_{n+1}) \in \Phi^\alpha_{n+1}$. ⊣

We will now reformulate the definition of $\Phi^\alpha_n(X, Y)$ in terms of a game of the Ehrenfeucht-Fraissé type.

Fix locally compact separable metric spaces $(X, d_X), (Y, d_Y)$ and $n \geq 1, \vec{x} \in X^n, \vec{y} \in Y^n$. We define the game $G^\alpha(\vec{x}, X; \vec{y}, Y) = G^\alpha$ as follows:

Let $a_0 = x_n, b_0 = y_n, \alpha_0 = \alpha$. In the ith round of the game, $i \geq 1$, player I chooses $Z_i = X$ or Y and plays some $a_i \in Z_i$, and an ordinal $\alpha_i < \alpha_{i-1}$. Then, player II responds by playing $b_i \in Z'_i$, where $Z'_i = X$, if $Z_i = Y$, and $Z'_i = Y$, if $Z_i = X$. The game ends when a round m is reached, where $\alpha_m = 0$. Let $\vec{z} = (z_1, z_2, \ldots, z_m) \in X^m, \vec{w} = (w_1, \ldots, w_m) \in Y^m$ be the elements played by both players in this run of the game, in the order played. The players must observe the following rules:

For $1 \leq i \leq m$, $d_X(z_{i-1}, z_i) < \rho_X(z_{i-1})$, $d_Y(w_{i-1}, w_i) < \rho_Y(w_{i-1})$, where $z_0 = x_n$ and $w_0 = y_n$.

Finally II wins this run of the game iff

$$(X, \vec{x}\,\hat{}\,\vec{z}) \equiv^0 (Y, \vec{y}\,\hat{}\,\vec{w}).$$

Proposition 5.10. *Player II has a winning strategy in $G^\alpha(\vec{x}, X; \vec{y}, Y)$ iff $(X, \vec{x}) \equiv^\alpha (Y, \vec{y})$.*

Proof. Again we do not indicate explicitly X, Y, when they are understood. The proof is by induction on α. The case $\alpha = 0$ is vacuously true, as there are no moves in the game.

Now assume α is limit. If II wins G^α, II wins each $G^\beta, \beta < \alpha$, so, by induction hypothesis, $\vec{x} \equiv^\beta \vec{y}$, for all $\beta < \alpha$, so $\vec{x} \equiv^\alpha \vec{y}$. Conversely, if $\vec{x} \equiv^\alpha \vec{y}$, then, reversing this, we have that II wins each $G^\beta, \beta < \alpha$. Then it follows that II wins G^α: When I starts by playing $\alpha_1 < \alpha$, II follows his winning strategy for G^{α_1+1}.

Next consider the successor case. Assume first II wins $G^{\alpha+1}$. Then notice that II also wins G^α, so, by induction hypothesis, $\vec{x} \equiv^\alpha \vec{y}$. We have to show that $\vec{x} \equiv^{\alpha+1} \vec{y}$. Fix $r \in \mathbb{Q}^+, x_{n+1}$ such that $d(x_n, x_{n+1}) < r < \rho(x_n)$. Let y_{n+1} be the first move in II's winning strategy in $G^{\alpha+1}$, when I plays x_{n+1},

α in his first move. Then clearly II wins $G^\alpha(\vec{x}\hat{\ }x_{n+1};\vec{y}\hat{\ }y_{n+1})$, so, by induction hypothesis, $\vec{x}\hat{\ }x_{n+1} \equiv^\alpha \vec{y}\hat{\ }y_{n+1}$. In particular $d(y_n, y_{n+1}) = d(x_n, x_{n+1}) \leq r$. Next fix $r \in \mathbb{Q}^+, r < \rho(x_n) = \rho(y_n)$, and let y_{n+1} be such that $d(y_n, y_{n+1}) < r$. By a similar argument as above, we can find x_{n+1} with $d(x_n, x_{n+1}) \leq r$ and $\vec{x}\hat{\ }x_{n+1} \equiv^\alpha \vec{y}\hat{\ }y_{n+1}$.

Finally, assume that $\vec{x} \equiv^{\alpha+1} \vec{y}$. We need to find a winning strategy for II in $G^{\alpha+1}$. Suppose I starts with $\alpha_1 \leq \alpha, a_1 = x_{n+1} \in X$. Then let $r \in \mathbb{Q}^+$ be such that $d(x_n, x_{n+1}) < r < \rho(x_n) = \rho(y_n)$. Then there is y_{n+1} with $d(y_n, y_{n+1}) \leq r < \rho(y_n)$ and $\vec{x}\hat{\ }x_{n+1} \equiv^\alpha \vec{y}\hat{\ }y_{n+1}$. Thus, by induction hypothesis, II has a winning strategy in the game $G^\alpha(\vec{x}\hat{\ }x_{n+1};\vec{y}\hat{\ }y_{n+1})$. II then simply follows this strategy in the rest of the game $G^{\alpha+1}$. This is a winning strategy for II in $G^{\alpha+1}$. The argument is similar if I starts with $\alpha_1 \leq \alpha$ and $a_1 = y_{n+1} \in Y$. ⊣

The next result is the main fact concerning the notion of α-equivalence.

Theorem 5.11. *For any pseudo-connected locally compact separable metric spaces $X, Y, \vec{x} \in X^n, \vec{y} \in Y^n, n \geq 1$, the following are equivalent:*

(i) $(X, \vec{x}) \equiv^\omega (Y, \vec{y})$;

(ii) $(X, \vec{x}) \cong_i (Y, \vec{y})$.

In particular, $(X, \vec{x}) \equiv^\omega (Y, \vec{y}) \Leftrightarrow [(X, \vec{x}) \equiv^\alpha (Y, \vec{y})$, *for any* $\alpha \geq \omega]$.

Proof. Clearly $(X, \vec{x}) \cong_i (Y, \vec{y}) \Rightarrow (X, \vec{x}) \equiv^\alpha (Y, \vec{y})$, for any ordinal α.

Assume now that $(X, \vec{x}) \equiv^\omega (Y, \vec{y})$, in order to show that $(X, \vec{x}) \cong_i (Y, \vec{y})$. For notational simplicity we will take $n = 1$, i.e., we assume that $x \in X, y \in Y, (X, x) \equiv^\omega (Y, y)$, and we will show that there is an isometry $\varphi : X \to Y$ with $\varphi(x) = y$. By 5.10, player II has a winning strategy in the game $G^\omega(x; y)$ and we fix such a strategy $\mathcal{S}$ from now on.

To find our isometry we will make use of the following general lemma:

Lemma 5.12. *Let $(X, d_X), (Y, d_Y)$ be locally compact separable metric spaces, $D_X \subseteq X, D_Y \subseteq Y$ dense subsets of X, Y resp. and $\psi : D_X \to D_Y$ an isometry such that moreover $\rho_X(x) = \rho_Y(\psi(x))$ for every $x \in X$. Then ψ extends (uniquely) to an isometry $\varphi : X \to Y$.*

Proof. Fix $x \in X$ and let $x_n \in D_X$ be such that $x_n \to x$. Then (x_n) is d_X-Cauchy, so if $y_n = \psi(x_n), (y_n)$ is d_Y-Cauchy. We will find a convergent subsequence (y_{n_i}) of (y_n). This implies that (y_n) converges, say to y. We put $\varphi(x) = y$. This clearly defines the desired extension.

To find the desired subsequence, notice that $\rho_X(x_n) \to \rho_X(x) > 0$, so there is $\rho_0 > 0$ such that $\rho_X(x_n) \geq \rho_0$, for all n. So $\rho_Y(y_n) \geq \rho_0$, for all n. Choose then N_0 large enough, so that for $m \geq n \geq N_0, d_Y(y_m, y_n) \leq \rho_0$. Then $d_Y(y_{N_0}, y_n) \leq \rho_0 < \rho_Y(y_{N_0})$, $\forall n \geq N_0$. Since $B^{c\ell}_{\rho_0}(y_{N_0})$ is compact, this gives a convergent subsequence (y_{n_i}). ⊣

We fix from now on a dense sequence $p_1, p_2, \dots$ in X, and a dense sequence $r_1, r_2, \dots$ in Y. Our goal will be to find a sequence $q_1, q_2, \dots$ in X and a sequence $s_1, s_2, \dots$ in Y such that the map $x \mapsto y, p_i \mapsto s_i, q_i \mapsto r_i$ is an isometry and moreover $\rho(x) = \rho(y), \rho(p_i) = \rho(s_i), \rho(q_i) = \rho(r_i)$. By 5.12, this will complete the proof.

Before we start out construction, we want to record the following obvious fact: If in $G^\omega(x; y), (k_1, a_1), b_1, (k_2, a_2), \dots, (k_n, a_n), b_n$ is a sequence of moves alternatively played by I, II, with II following $\mathcal{S}$ and $(z_1, z_2, \dots, z_n) \in X^n, (w_1, \dots, w_n) \in Y^n$ are the elements played by both players in this sequence, in the order played, then $x \mapsto y, z_i \mapsto w_i$ is an isometry and $\rho(x) = \rho(y), \rho(z_i) = \rho(w_i)$.

Below we will only consider runs of $G^\omega(x; y)$ in which I starts with (k, a_1), for some $k \in \mathbb{N}, k > 0$, and then in his subsequent moves $(k_2, a_2), (k_3, a_3), \dots$ he simply plays $k_2 = k - 1, k_3 = k - 2, \dots, 0$. Since the moves $k_2, k_3, \dots$ are completely determined by k, we will simply ignore them and pretend that I only plays (k, a_1) to start with and the plays $a_2, a_3, \dots$.

We now start our construction: Our plan is to construct by induction the following:

(i) A sequence $A_1 \supseteq A_2 \supseteq \dots$ of infinite subsets of $\mathbb{N}$,

(ii) A sequence $k_0, \ell_0, k_1, \ell_1, k_2, \ell_2, \dots$ of positive integers,

(iii) A sequence $q_1, q_2, \dots$ of elements of X and a sequence $s_1, s_2, \dots$ of elements of Y,

(iv) Finite sequences $(q_{n,1}, \dots, q_{n,\ell_n - 1}), (s_{n,1}, \dots s_{n,k_n - 1})$, in X^{ℓ_n}, Y^{k_n}, resp.,

(v) Positive reals $\epsilon_1, \epsilon_2, \dots, \delta_1, \delta_2, \dots$,

with the following properties:

(a) If I starts with (k, p_1), where $k \in A_1$, and II, using $\mathcal{S}$, plays $s^k_1 \in Y$, then s^k_1 for $k \in A_1, k \to \infty$ converges to s_1. Moreover, $d(s^k_1, s_1) \leq \epsilon_1 < \rho(s^k_1)$, for any $k \in A_1$.

(b) For each $n \geq 1, \min(A_n) > 1 + (k_1 + 1) + (\ell_1 + 1) + \dots + (k_{n-1} + 1) + (\ell_{n-1} + 1)$, and for each $k \in A_n$ the sequence of moves indicated in the following diagram is a legal sequence of moves in which II follows $\mathcal{S}$ (arrows indicate applications of $\mathcal{S}$):

$$\begin{array}{lcccccccccccc}
X : x & p_1 & p^k_{1,0} & p^k_{1,1} & \cdots & p^k_{1,k_1-1} & q^k_1 & q_1 & q_{1,1} & \cdots & q_{1,l_1-1} & p_2 & \\
 & \Downarrow & \Uparrow & \Uparrow & \cdots & \Uparrow & \Uparrow & \Downarrow & \Downarrow & \cdots & \Downarrow & \Downarrow & \\
Y : y & s^k_1 & s_1 & s_{1,1} & \cdots & s_{1,k_1-1} & r_1 & r^k_{1,0} & r^k_{1,1} & \cdots & r^k_{1,l_1-1} & s^k_2 & \cdots
\end{array}$$

$$\begin{array}{cccccc}
p_{n-1} & p^k_{n-1,0} & p^k_{n-1,1} & \cdots & p^k_{n-1,k_{n-1}-1} & q^k_{n-1} \\
\Downarrow & \Uparrow & \Uparrow & \cdots & \Uparrow & \Uparrow \\
s^k_{n-1} & s_{n-1} & s_{n-1,1} & \cdots & s_{n-1,k_{n-1}-1} & r_{n-1}
\end{array}$$

$$\begin{array}{ccccc}
q_{n-1} & q_{n-1,1} & \cdots & q_{n-1,l_{n-1}-1} & p_n \\
\Downarrow & \Downarrow & \cdots & \Downarrow & \Downarrow \\
r^k_{n-1,0} & r^k_{n-1,1} & \cdots & r^k_{n-1,l_{n-1}-1} & s^k_n
\end{array}$$

Moreover, we have

$$q^k_1 \to q_1, s^k_2 \to s_2, \ldots, q^k_{n-1} \to q_{n-1}, s^k_n \to s_n$$

for $k \to \infty, k \in A_n$, and

$$\begin{aligned} d(q^k_i, q_i) &\leq \delta_i < \rho(q^k_i), \\ d(s^k_i, s_i) &\leq \epsilon_i < \rho(s^k_i), \end{aligned}$$

for $k \in A_n, i \leq n-1$.

Granting (a), (b) we now verify that $q_1, q_2, \ldots, s_1, s_2, \ldots$ have the required properties. By (b) and for each $k \in A_n$, clearly

$$x \mapsto y, p_i \mapsto s^k_i, q^k_i \mapsto r_i, i \leq n-1,$$

is an isometry, and for all $i \leq n-1$.

$$\rho(x) = \rho(y), \rho(p_i) = \rho(s^k_i), \rho(q^k_i) = \rho(r_i).$$

Letting $k \to \infty, k \in A_n$, we conclude that

$$x \mapsto y, p_i \mapsto s_i, q_i \mapsto r_i, i \leq n-1,$$

is an isometry and for all $i \leq n-1$,

$$\rho(x) = \rho(y), \rho(p_i) = \rho(s_i), \rho(q_i) = \rho(r_i),$$

so we are done.

For the first stage of our construction, it is enough to show that when I starts with (k, p_1) and II plays, by following $\mathcal{S}, s_1^k$, then (s_1^k) has a convergence subsequence. But this is clear as $d(s_1^k, y) = d(p_1, x) < \rho(x) = \rho(y)$. So choose infinite A_1' and s_1 such that for $k \in A_1', k \to \infty, s_1^k \to s_1$. As $\rho(s_1^k) \to \rho(s_1) > 0$ and $d(s_1^k, s_1) \to 0$, it follows that we can also choose $\epsilon_1 > 0$ and infinite $A_1 \subseteq A_1'$ with $d(s_1^k, s_1) \leq \epsilon_1 < \rho(s_1^k)$ for all $k \in A_1$, and $\min(A_1) > 3$. Also let $k_0 = \ell_0 = 1$.

Assume now $n \geq 1$ and $A_1 \supseteq \cdots \supseteq A_n, k_1, \ldots, k_{n-1}, \ell_1, \ldots \ell_{n-1}, s_1, \ldots, s_n, q_1, \ldots, q_{n-1}, (s_{i,1}, \ldots, s_{i,k_i-1}), (q_{i,1}, \ldots, q_{i,\ell_i-1})$, for $i \leq n-1$, and $\epsilon_1, \ldots \epsilon_{n-1}, \delta_1, \ldots, \delta_{n-1}$ have been constructed, satisfying (b) above. We will construct $A_{n+1} \subseteq A_n, k_n, \ell_n, s_{n+1}, q_n, (s_{n,1}, \ldots, s_{n,k_n-1}), (q_{n,1}, \ldots q_{n,\ell_n-1}), \epsilon_n, \delta_n$, so that (b) is still satisfied at $n+1$.

First as $s_n^k \to s_n$, for $k \in A_n$, we choose infinite $A_n' \subseteq A_n$ and $\epsilon_n > 0$ so that for $k \in A_n', d(s_n^k, s_{n-1}) \leq \epsilon_n < \rho(s_n^k)$. Since Y is pseudo-connected we pick k_n and a sequence $(s_{n,1}, \ldots, s_{n,k_n-1})$ such that $d(s_{n,1}, s_n) < \rho(s_n)$ and $d(s_{n,i}, s_{n,i+1}) < \rho(s_{n,i})$, for $i \leq k_n - 1$, and $d(s_{n,k_n-1}, r_n) < \rho(s_{n,k_n-1})$. Then for any $k \in A_n'$, with $k > 1 + (k_1+1) + \cdots + (\ell_{n-1}+1) + (k_n+1)$, continue the play given by (b) above, to the following moves, where II follows $\mathcal{S}$ (see the diagram below):

$$I: s_n, II: p_{n,0}^k, I: s_{n,1}, II: p_{n,1}^k, \ldots, I: s_{n,k_n-1}, II: p_{n,k_n-1}^k, I: r_n, II: q_n^k.$$

$$\begin{array}{cccccccccccccc}
 & & & & & & & \leq \delta_1 & & & & & & \\
x & p_1 & & p_{1,0}^k & p_{1,1}^k & \cdots & p_{1,k_1-1}^k & q_1^k & \longrightarrow & q_1 & q_{1,1} & \cdots & q_{1,l_1-1} & p_2 \\
 & \Downarrow & & \Uparrow & \Uparrow & \cdots & \Uparrow & \Uparrow & & \Downarrow & \Downarrow & \cdots & \Downarrow & \Downarrow \\
y & s_1^k & \longrightarrow & s_1 & s_{1,1} & \cdots & s_{1,k_1-1} & r_1 & & r_{1,0}^k & r_{1,1}^k & \cdots & r_{1,l_1-1}^k & s_2^k \quad \cdots \\
 & & \leq \epsilon_1 & & & & & & & & & & &
\end{array}$$

$$\begin{array}{cccccccc}
 & & & & & & & \leq \delta_{n-1} \\
p_{n-1} & & p_{n-1,0}^k & p_{n-1,1}^k & \cdots & p_{n-1,k_{n-1}-1}^k & q_{n-1}^k & \longrightarrow \\
\Downarrow & & \Uparrow & \Uparrow & \cdots & \Uparrow & \Uparrow & \\
s_{n-1}^k & \longrightarrow & s_{n-1} & s_{n-1,1} & \cdots & s_{n-1,k_{n-1}-1} & r_{n-1} & \\
 & \leq \epsilon_{n-1} & & & & & &
\end{array}$$

$$\begin{array}{ccccccc}
\leq \delta_{n-1} & & & & & & \\
\longrightarrow & q_{n-1} & q_{n-1,1} & \cdots & q_{n-1,l_{n-1}-1} & p_n & \\
 & \Downarrow & \Downarrow & \cdots & \Downarrow & \Downarrow & \\
 & r^k_{n-1,0} & r^k_{n-1,1} & \cdots & r^k_{n-1,l_{n-1}-1} & s^k_n & \longrightarrow \\
 & & & & & & \leq \epsilon_n
\end{array}$$

$$\begin{array}{cccccccccccc}
 & & & & & & \leq \delta_n & & & & & \\
 & p^k_{n,0} & p^k_{n,1} & \cdots & p^k_{n,k_n-1} & q^k_n & \longrightarrow & q_n & q_{n,1} & \cdots & q_{n,l_n-1} & p_{n+1} \\
 & \Uparrow & \Uparrow & \cdots & \Uparrow & \Uparrow & & \Downarrow & \Downarrow & \cdots & \Downarrow & \Downarrow \\
\longrightarrow & s_n & s_{n,1} & \cdots & s_{n,k_n-1} & r_n & & r^k_{n,0} & r^k_{n,1} & \cdots & r^k_{n,l_n-1} & s^k_{n+1} \ \cdots \\
\leq \epsilon_n & & & & & & & & & & &
\end{array}$$

Now we want to argue that there is an infinite subset $A''_n \subseteq A'_n$ on which the sequence $(q^k_n), k \in A''_n$, converges. Noticing that $d(p_n, p^k_{n,0}) \leq \epsilon_n < \rho(s^k_n) = \rho(p_n)$, $d(p^k_{n,i}, p^k_{n,i+1}) \leq d(s_{n,i}, s_{n,i+1}) < \rho(s_{n,i}) = \rho(p^k_{n,i})$, and $d(p^k_{n,k_n-1}, q^k_n) = d(s_{n,k_n-1}, r_n) < \rho(s_{n,k_n-1}) = \rho(p^k_{n,k_n-1})$, while moreover $\rho(p^k_{n,i}) = \rho(s_{n,i}), \rho(q^k_n) = \rho(r_n)$, we obtain this conclusion from 5.5.

So fix an infinite subset $A''_n \subseteq A_n$ with $\min(A''_n) > 1 + (k_1 + 1) + \cdots + (\ell_{n-1} + 1) + (k_n + 1)$, such that $(q^k_n), k \in A''_n, k \to \infty$, converges to some value which we call q_n. Then choose $\delta_n > 0$ and infinite $A'''_n \subseteq A''_n$, so that for $k \in A'''_n, d(q^k_n, q_n) < \delta_n \leq \rho(q^k_n)$.

Next pick ℓ_n and a sequence $(q_{n,1}, \ldots q_{n,\ell_n-1})$ such that $d(q_n, q_{n,1}) < \rho(q_n), d(q_{n,i}, q_{n,i+1}) < \rho(q_{n,i})$, for $i \leq \ell_n - 1$, and $d(q_{n,\ell_n-1}, p_{n+1}) < \rho(q_{n,\ell_n-1})$. Then for any $k \in A'''_n, k > 1 + (k_1 + 1) + \cdots + (k_n + 1) + (\ell_n + 1)$, continue to play the following moves, where II follows $\mathcal{S}$ (see again the preceding diagram):

$$I : q_n, II : r^k_{n,0}, I : q_{n,1}, II : r^k_{n,1}, \ldots, I : q_{n,\ell_n-1}, II : r^k_{n,\ell_n-1}, I : p_{n+1}, II : s^k_{n+1}.$$

Then, again by Lemma 5.5, we can find an infinite subset $A_{n+1} \subseteq A'''_n$ with $\min(A_{n+1}) > 1 + (k_1 + 1) + (\ell_1 + 1) + \cdots + (k_n + 1) + (\ell_n + 1)$ such that $(s^k_{n+1}), k \in A_{n+1}, k \to \infty$ converges to a value which we call s_{n+1}. This completes the inductive construction, and the proof of the theorem. $\dashv$

6 Isometries of locally compact spaces, *II*: The general case

We will now use the analysis of isometries of pseudo-connected locally compact separable metric spaces to characterize the isometry groups of general locally compact separable metric spaces. We first need to establish some notation.

For any countable set I, we denote by S_I the Polish group of permutations of I with the pointwise convergence topology. Thus, up to isomorphism, S_I is either a finite group S_n, or S_∞. Let now G be a Polish group and consider the power G^I, which is isomorphic either to a finite power G^n, or $G^{\mathbb{N}}$, thus it is a Polish group. The group S_I acts by homomorphisms on G^I as follows:

$$g \cdot x(i) = x(g^{-1}(i)).$$

Consider then the semidirect product

$$S_I \ltimes G^I,$$

with underlying set $S_I \times G^I$, in which the group operation is given as usual by

$$(g, x)(h, y) = (gh, (h^{-1} \cdot x)y).$$

Equipped with the product topology this becomes a Polish group.

We now have the following

Theorem 6.1. *Let X be a locally compact separable metric space. Then there is a finite or infinite sequence $(I_n)_{n\in N}$ of non-empty countable sets and a sequence of locally compact Polish groups $(G_n)_{n\in N}$, where $N = \{0, \dots, m\}$ or $N = \mathbb{N}$, such that* $\mathrm{Iso}(X)$ *is isomorphic to a closed subgroup of the product*

$$\prod_{n\in N} (S_{I_n} \ltimes G_n^{I_n}).$$

Proof. Let $C_0, C_1, \dots$ (a finite or infinite list) enumerate without repetition the pseudo-components of X. Let $(X_n)_{n\in N}$ be a finite or infinite list of pseudo-connected locally compact separable metric spaces with $m \neq n \Rightarrow X_m \not\cong_i X_n$ and $\forall m \exists n (C_m \cong_i X_n), \forall m \exists n (X_m \cong_i C_n)$. For each n, let $I_n = \{k : C_k \cong_i X_n\}$. Let $G_n = \mathrm{Iso}(X_n)$, which is Polish locally compact by 5.6 i). We will show that $\mathrm{Iso}(X)$ is isomorphic to a closed subgroup of $\prod_n S_{I_n} \ltimes G_n^{I_n}$.

First note that if $\varphi \in \mathrm{Iso}(X)$, then for each $k, \varphi(C_k)$ is also a pseudo-component of X, so $\varphi(C_k) = C_j$ for some (unique) j. Moreover as $\varphi|C_k : C_k \to C_j$ is an isometry, $C_k \cong_i C_j$. Thus for each n, φ induces a permutation $\pi_n(\varphi) \in S_{I_n}$, given by $\varphi(C_k) = C_{\pi_n(\varphi)(k)}$. Clearly $\pi_n : \mathrm{Iso}(X) \to S_{I_n}$ is a homomorphism.

Now for each n and $k \in I_n$, fix an isometry $\theta_n^k : X_n \to C_k$. Then define $\rho_n : \mathrm{Iso}(X) \to G_n^{I_n}$ by

$$\rho_n(\varphi)(k) = (\theta_n^{\pi_n(\varphi)(k)})^{-1} \circ (\varphi|C_k) \circ \theta_n^k.$$

Then we claim that the map

$$\pi : \mathrm{Iso}(X) \to \prod_n S_{I_n} \ltimes G_n^{I_n},$$

given by

$$\pi(\varphi)(n) = (\pi_n(\varphi), \rho_n(\varphi)),$$

is an isomorphism, thus $\mathrm{Iso}(X)$ is isomorphic to a closed subgroup of $\prod_n S_{I_n} \ltimes G_n^{I_n}$.

To check that it is a homomorphism, it is enough to check that for each n,

$$\varphi \mapsto (\pi_n(\varphi), \rho_n(\varphi))$$

is a homomorphism of $\mathrm{Iso}(X)$ into $S_{I_n} \ltimes G_n^{I_n}$, and this is a routine computation. It is also trivial to check that π, π^{-1} are continuous. Finally, π is injective, since $\varphi|C_k = (\theta_n^{\pi_n(\varphi)(k)}) \circ \rho_n(\varphi)(k) \circ (\theta_n^k)^{-1}$, if $k \in I_n$, i.e., φ is completely determined by $\pi(\varphi)$. $\dashv$

It should be pointed out that, in the notation of 6.1, $\sum_n \mathrm{card}(I_n) =$ cardinality of the pseudo-components of X, and so if X has only finitely many pseudo-components, the index set N and each I_n in 6.1 is finite, and thus the product $\prod_{n\in N}(S_{I_n} \ltimes G_n^{I_n})$ is locally compact. So we have:

Corollary 6.2. *The isometry group of a locally compact separable metric space with only finitely many pseudo-components is locally compact.*

Since it is clear that for any countable set I, and locally compact group G, $S_I \ltimes G^I$ is isomorphic to a closed subgroup of $S_\infty \ltimes G^{\mathbb{N}}$, we can rewrite

6.1 by saying that every $\mathrm{Iso}(X)$, for X a locally compact separable metric space, is, up to isomorphism, a closed subgroup of a Polish group of the form

$$\prod_{n\in\mathbb{N}}(S_\infty \ltimes G_n^{\mathbb{N}}),$$

for some sequence (G_n) of locally compact Polish groups. We will next prove the converse of this theorem, thereby characterizing exactly the isometry groups of locally compact separable metric spaces. Moreover, our proof will also yield that the same characterization works as well for the locally compact Polish metric spaces, the σ-compact Polish metric spaces and the almost locally compact Polish metric spaces. (Recall that a *σ-compact space* is a union of countably many compact sets and we define an *almost locally compact space* to be a space in which the points that have a compact nbhd form a dense set.)

Theorem 6.3. *Up to (topological group) isomorphism the following five classes of groups are the same:*

i) The isometry groups of locally compact separable metric spaces.

ii) The isometry groups of locally compact Polish metric spaces.

iii) The isometry groups of σ-compact Polish metric spaces.

iv) The isometry groups of almost locally compact Polish metric spaces.

v) The closed subgroups of groups of the form

$$\prod_{n\in\mathbb{N}}(S_\infty \ltimes G_n^{\mathbb{N}}),$$

where (G_n) is a sequence of locally compact Polish groups.

Proof. We will first note some simple facts about almost locally compact Polish metric spaces.

Let (X, d) be a metric space. We define its *core* by

$$x \in \mathrm{Core}(X) \Leftrightarrow \exists r > 0(B_r^{c\ell}(x) \text{ is compact}).$$

Clearly, X is almost locally compact iff $\mathrm{Core}(X)$ is dense in X. Also $\mathrm{Core}(X)$ is open and $(\mathrm{Core}(X), d)$ is locally compact.

Next we have

Lemma 6.4. *Every σ-compact Polish metric space X is almost locally compact.*

Proof. To check that X is almost locally compact we have to verify that Core(X) is dense. Otherwise, let $U \subseteq X$ be nonempty open with $U \cap \text{Core}(X) = \emptyset$. Let K_n be compact with $X = \bigcup_n K_n$. Then, by the Baire Category Theorem, there is nonempty open $V \subseteq U$ and n with $V \subseteq K_n$, so $V \subseteq \text{Core}(X)$, a contradiction. $\dashv$

Denote now by $\mathcal{G}_1 - \mathcal{G}_5$ the classes of groups in i)-v) of 6.3 (up to isomorphism). It is clear that $\mathcal{G}_2 \subseteq \mathcal{G}_3 \subseteq \mathcal{G}_4$.

We will next check that $\mathcal{G}_4 \subseteq \mathcal{G}_2$ (so that $\mathcal{G}_2 = \mathcal{G}_3 = \mathcal{G}_4 \subseteq \mathcal{G}_1$).

Consider an almost locally compact Polish metric space (X, d). Let $X_0 = \text{Core}(X)$ and put the metric d on X_0. Then (X_0, d) is locally compact and separable but not necessarily complete. Consider next the equivalent metric

$$\delta = \frac{d}{1+d}$$

on X_0, denote by ρ its associated ρ-function and define the pseudo-metric

$$\delta_\rho(x,y) = \frac{\left|\frac{1}{\rho(x)} - \frac{1}{\rho(y)}\right|}{1 + \left|\frac{1}{\rho(x)} - \frac{1}{\rho(y)}\right|}$$

on X_0.

Define then the metric d' on X_0 by

$$d'(x,y) = \frac{1}{4}\delta(x,y) + \frac{\text{diam}(\delta)}{8}\delta_\rho(x,y).$$

Then d' is a compatible metric on X_0 which is now complete: if (x_n) is a d'-Cauchy sequence, then it is δ-Cauchy and $(\frac{1}{\rho(x_n)})$ is a Cauchy sequence, so $\rho(x_n) \to r > 0$. Then if N is large enough, $\rho(x_N) > r/2$ and $\delta(x_N, x_n) < r/2$ if $n > N$, so (x_n) has a converging subsequence and so converges. Moreover $\text{diam}(X_0, d') \leq \frac{3}{8}\text{diam}(\delta)$.

Let ρ' be the ρ-function of the metric d'. Let

$$\delta_{\rho'} = \frac{\left|\frac{1}{\rho'(x)} - \frac{1}{\rho'(y)}\right|}{1 + \left|\frac{1}{\rho'(x)} - \frac{1}{\rho'(y)}\right|},$$

and let

$$d''(x,y) = \frac{1}{2}\delta(x,y) + \frac{1}{4}\delta_{\rho'}(x,y).$$

Then, by a similar argument, d'' is a compatible complete metric on X_0 and $\text{diam}(X_0, d'') \geq \frac{1}{2}\text{diam}(\delta) > \text{diam}(X_0, d')$ (provided that X_0 is not a singleton).

Define now a metric space (Y, σ) as follows: $Y = (\{0\} \times X_0) \cup (\{1\} \times X_0)$. The metric σ is defined as follows:

$$\begin{aligned} \sigma((0,x),(0,y)) &= d'(x,y) \\ \sigma((1,x),(1,y)) &= d''(x,y) \\ \sigma((0,x),(1,y)) &= \sigma((1,y),(0,x)) \\ &= 1 + \frac{1}{4}\delta(x,y). \end{aligned}$$

It is routine to check that this is a metric and that (Y, σ) is a locally compact Polish metric space.

We verify that $\text{Iso}(X, d)$ is isomorphic to $\text{Iso}(Y, \sigma)$, which will show that $\mathcal{G}_4 \subseteq \mathcal{G}_2$.

First notice that the map $\varphi \mapsto \varphi | X_0$ is an isomorphism of $\text{Iso}(X, d)$ with $\text{Iso}(X_0, d) = \text{Iso}(X_0, \delta)$, so it is enough to show that $\text{Iso}(X_0, \delta)$ is isomorphic to $\text{Iso}(Y, \sigma)$. Let $\varphi \in \text{Iso}(X_0, \delta)$ and define $\varphi^* : Y \to Y$ by $\varphi^*((i, x)) = (i, \varphi(x))$. Clearly $\varphi^* \in \text{Iso}(Y, \sigma)$ and $\varphi \mapsto \varphi^*$ is an isomorphism, so it is enough to show that it is onto.

Let $\psi \in \text{Iso}(Y, \sigma)$. It is clear that ψ maps $\{i\} \times X_0$ onto $\{j\} \times X_0$. Since the σ-diameter of $\{0\} \times X_0$ = d'-diameter of X_0 < d''-diameter of X_0 = σ-diameter of $\{1\} \times X_0$, it follows that $i = j$, i.e., ψ maps $\{0\} \times X_0$ onto $\{0\} \times X_0$ and $\{1\} \times X_0$ onto $\{1\} \times X_1$. Thus it gives two isometries ψ_0, ψ_1 of $(X_0, d'), (X_0, d'')$, resp., defined by $(i, \psi_i(x)) = \psi((i, x))$. It is then enough to show that $\psi_0 = \psi_1$ $(= \varphi)$ and φ is a δ-isometry (since then $\varphi^* = \psi$). To see that $\psi_0 = \psi_1$, notice that if $y \in X_0$, $(1, y)$ is the unique point in $\{1\} \times X_0$ which has σ-distance 1 from $(0, y)$. So if $\psi_0(x) = y, \psi_1(x) = z$, so that $\psi((0, x)) = (0, y)$ and $\psi((1, x)) = (1, z)$, we must have $\sigma((0, y)), (1, z)) = \sigma((0, x), (1, x)) = 1$, so $z = y$.

Finally, we check that φ is a δ-isometry.

Since $\varphi = \psi_0$, clearly φ is a d'-isometry, so, in particular, it preserves the

ρ'-function. Since $\varphi = \psi$, it is also a d''-isometry, so

$$\begin{aligned}\delta(\varphi(x), \varphi(y)) &= 2d''(\varphi(x), \varphi(y)) - \frac{1}{2}\delta_{\rho'}(\varphi(x), \varphi(y)) \\ &= 2d''(x, y) - \frac{1}{2}\delta_{\rho'}(x, y) \\ &= \delta(x, y)\end{aligned}$$

and we are done.

We have already seen that $\mathcal{G}_1 \subseteq \mathcal{G}_5$, so we complete the proof by showing that $\mathcal{G}_5 \subseteq \mathcal{G}_3$.

First let us note the following standard fact.

Lemma 6.5. *Let G be a locally compact Polish group, and let d be a left-invariant compatible metric. Then d is complete, and G is isomorphic to a closed subgroup of* $\mathrm{Iso}(G, d)$.

Proof. Consider the space (G, d). By homogeneity, it is clear that the function $\rho(x)$ is constant, from which it easily follows that any d-Cauchy sequence has a convergent subsequence, hence converges. Every element $g \in G$ induces a d-isometry $\varphi_g(x) = gx$, and it is easy to check that the map $g \mapsto \varphi_g$ is an isomorphism of G with a closed subgroup of $\mathrm{Iso}(G, d)$. $\dashv$

So given a locally compact Polish group G, we can view G as a closed subgroup of $\mathrm{Iso}(X)$, for some locally compact Polish metric space X, and thus $G^{\mathbb{N}}$ as a closed subgroup of $\mathrm{Iso}(X)^{\mathbb{N}}$. It follows that $S_\infty \ltimes G^{\mathbb{N}}$ is a closed subgroup of $S_\infty \ltimes \mathrm{Iso}(X)^{\mathbb{N}}$.

For each sequence of metric spaces $X_n = (X_n, d_n)$, with $d_n < 1$, define their *direct sum* $(\bigsqcup_n X_n, d)$ by

$$\bigsqcup_n X_n = \text{the disjoint union of the } X_n\text{'s},$$

$$d(x, y) = \begin{cases} d_n(x, y), & \text{if } x, y \in X_n; \\ 1, & \text{if } x \in X_n, y \in X_m, n \neq m. \end{cases}$$

If $X_n = X$ for all n, we write X_∞ instead of $\bigsqcup_n X_n$. It is clear that if each X_n is a Polish metric space, so is $\bigsqcup_n X_n$ and if each X_n is also locally compact, so is $\bigsqcup_n X_n$.

Now if (X, d) is a locally compact Polish metric space, and we assume that $d < 1$ (which we can without changing its isometry group, by replacing

d by $\frac{d}{1+d}$), then it is easy to see that $S_\infty \ltimes \mathrm{Iso}(X)^{\mathbb{N}}$ is isomorphic to a closed subgroup of $\mathrm{Iso}(X_\infty)$. Indeed, let X_n be the nth copy of X in $X_\infty = \bigsqcup_n X_n$. Then if $(g, (\varphi_n)) \in S_\infty \ltimes \mathrm{Iso}(X)^{\mathbb{N}}$, assign to it $\pi(g, (\varphi_n)) \in \mathrm{Iso}(X_\infty)$, defined as follows: If $x \in X_n$, then $\pi(g, (\varphi_n))$ sends x to $\varphi_n(x)$ but located in the $g(n)$ copy of X, i.e., in $X_{g(n)}$. It is easy to check that this is an isomorphism.

It follows that for each sequence (G_n) of locally compact Polish groups, there is a sequence (X_n) of locally compact Polish metric spaces such that $\prod_{n\in\mathbb{N}} S_\infty \ltimes G_n^{\mathbb{N}}$ is a isomorphic to a closed subgroup of $\prod_{n\in\mathbb{N}} \mathrm{Iso}(X_n)$. Finally, it is easy to check that $\prod_{n\in\mathbb{N}} \mathrm{Iso}(X_n)$ is isomorphic to a closed subgroup of $\mathrm{Iso}(\bigsqcup_n X_n)$: send $(\varphi_n) \in \prod_{n\in\mathbb{N}} \mathrm{Iso}(X_n)$ to $\bigsqcup_n \varphi_n \in \mathrm{Iso}(\bigsqcup_n X_n)$.

We have thus concluded that any group in $\mathcal{G}_5$ is isomorphic to a closed subgroup of $\mathrm{Iso}(X)$, for some locally compact Polish metric space (X, d) (we emphasize here that d is complete). It thus remains to show that every closed subgroup of such an $\mathrm{Iso}(X)$ is isomorphic to $\mathrm{Iso}(Y)$, where Y is a σ-compact Polish metric space.

Our strategy for proving this is similar to that used in the proof of 3.1. First we claim that it is enough to show that, given a closed subgroup G of $\mathrm{Iso}(X)$, there is a σ-compact Polish space Z such that G is isomorphic to $\mathrm{Iso}(Z, (E_n))$, for some sequence (E_n) of closed subsets of Z: Indeed if such $Z, (E_n)$ exist, going back to the proof of 2.5, let $Z_{\vec{E}}$ be the Polish metric space defined there from Z and $\vec{E} = (E_n)$. Every isometry $\varphi \in \mathrm{Iso}(Z, (E_n))$ extends to a unique isometry φ' of $Z_{\vec{E}}$, by defining it to be the identity on the additional points, and conversely every isometry of $Z_{\vec{E}}$ is equal to φ' for some $\varphi \in \mathrm{Iso}(Z, (E_n))$. Thus the map $\varphi \mapsto \varphi'$ is an isomorphism of $\mathrm{Iso}(Z, (E_n))$ with $\mathrm{Iso}(Z_{\vec{E}})$. Since $Y = Z_{\vec{E}}$ is clearly a σ-compact Polish metric space, this completes the proof.

So, to summarize, our proof is reduced to the following problem: Given a locally compact Polish metric space X, and a closed subgroup $G \subseteq \mathrm{Iso}(X)$, find a σ-compact Polish space Z and a sequence of closed subsets (E_n) of Z such that G is isomorphic to $\mathrm{Iso}(Z, (E_n))$.

First we will introduce some convenient terminology. Let us call a metric space (M, δ) *rich* if it satisfies the following two properties:

(i) (Rich 1) For each $n \geq 1$, the set of all $(x_1, \dots, x_n) \in M^n$ for which the x_i are distinct and the distances $d(x_i, x_j), 1 \leq i < j \leq n$, are distinct is dense in M^n.

(ii) (Rich 2) For all $n \geq 1$, given $2n$ distinct points $x_1, \dots, x_n, y_1, \dots, y_n$

in M, there is a point $u \in M$ such that

$$\sum_{i=1}^{n} \delta(x_i, u) \neq \sum_{i=1}^{n} \delta(y_i, u).$$

We now claim the following:

Lemma 6.6. *Let X be a locally compact Polish metric space. There is a σ-compact Polish space (M, δ) which is rich, and is such that* $\mathrm{Iso}(X)$ *is isomorphic to a closed subgroup of* $\mathrm{Iso}(M)$.

We will assume this temporarily and proceed to the rest of the proof.

It follows from 2.3, and the remarks at the beginning of 2H, that there is a sequence of closed sets $R_n \subseteq M^{(3^n)}$ such that G is isomorphic to

$$\mathrm{Iso}(M, (R_n)).$$

Consider then the space $M_n = M^{(3^n)}, n \geq 0$, with the metric

$$d_n(\vec{x}, \vec{y}) = \delta_{3^n}(\vec{x}, \vec{y}) = \frac{1}{3^n} \sum_{i=1}^{3^n} \delta(x_i, y_i).$$

M_n embeds isometrically to M_{n+1} by $j_{n,n+1}(\vec{x}) = (\vec{x}, \vec{x}, \vec{x})$. Let for each $m < n, j_{m,n} = j_{n-1,n} \circ \cdots \circ j_{m,m+1}$. Define finally the required space (Z, ρ) as follows: Z is the disjoint union of the M_n's,

$$Z = \bigsqcup_n M_n,$$

Let $\rho | M_n = d_n$. Finally, if $x, y \in Z$ with $x \in M_m, y \in M_n$ and $m < n$, let

$$\rho(x, y) = 1 + d_n(j_{m,n}(x), y).$$

Clearly Z is a σ-compact Polish metric space.

For each $n \geq 0$ and $\varphi \in \mathrm{Iso}(M)$, let $\varphi_n = \varphi^{(3^n)}$ be the induced isometry on $M_n, \varphi_n(x_1, \ldots, x_{3^n}) = (\varphi(x_1), \ldots, \varphi(x_{3^n}))$. Then the argument in the proof of Sublemma 2.7, using now the richness of M, shows that there is a sequence of closed sets $(K_{i,n})_{i \in \mathbb{N}}$ such that if $\mathrm{Iso}(M)_n = \{\varphi_n : \varphi \in \mathrm{Iso}(M)\}$, then

$$\mathrm{Iso}(M)_n = \mathrm{Iso}(M_n, (K_{i,n})).$$

We can of course view each R_n as a closed subset of M_n and thus of Z. We finally check that if (E_n)-enumerates $\{M_n\} \cup \{j_{0n}(M)\} \cup \{K_{i,n}\} \cup \{R_n\}$, then $\text{Iso}(M, (R_n))$ is isomorphic to $\text{Iso}(Z, (E_n))$.

Indeed, consider the map

$$\varphi \mapsto \varphi' = \bigsqcup_n \varphi_n,$$

from $\text{Iso}(M)$ into $\text{Iso}(Z)$. It is clearly an isomorphism, so it only remains to show that it maps $\text{Iso}(M, (R_n))$ onto $\text{Iso}(Z, (E_n))$. Clearly, it maps $\text{Iso}(M, (R_n))$ into $\text{Iso}(Z, (E_n))$. Conversely, let $\Phi \in \text{Iso}(Z, (E_n))$. Let $\varphi = \Phi|M_0 = \Phi|M \in \text{Iso}(M)$. Let also for $n \geq 1, \tilde{\varphi}_n = \Phi|j_{0n}(M)$, so that up to identifying M with $j_{0n}(M), \tilde{\varphi}_n \in \text{Iso}(M)$ as well. Moreover as $\Phi|M_n$ preserves $(K_{i,n})$, it follows that $\tilde{\varphi}_n = (\varphi_n^0)_n$, for some $\varphi_n^0 \in \text{Iso}(M)$. We check that $\varphi_n^0 = \varphi$. Indeed, given $x \in M, \rho(x, j_{0n}(x)) = 1$, so $\rho(\Phi(x), \Phi(j_{0n}(x))) = 1$. But $\Phi(x) = \varphi(x)$ and $\Phi(j_{0n}(x)) = j_{0n}(\varphi_n^0(x))$, so $\rho(\varphi(x), j_{0n}(\varphi_n^0(x))) = 1$. But the only point in M_n that has distance exactly 1 from $\varphi(x)$ is $j_{0n}(\varphi(x))$, so $\varphi(x) = \varphi_n^0(x)$. Thus we have that $\Phi|M_n = \varphi_n$ and since $\Phi(R_n) = R_n$, it follows that $\varphi(R_n) = R_n$, so $\varphi \in \text{Iso}(M, (R_n))$ and $\Phi = \varphi'$.

So it only remains to prove Lemma 6.6.

Proof of 6.6. We start with the locally compact Polish metric space (X, d). Recall the definition (in 2C) of $E(W, n), n \geq 1$, for each Polish metric space (W, σ). We embed isometrically W into $E(W, n)$ via $i_n(w) = f_w$. Define then the metric space $E'(W, n)$ as follows: $E'(W, n)$ is the disjoint union of W and $E(W, n), E'(W, n) = W \sqcup E(W, n)$. The metric $d_{E'(W,n)}$ is equal to σ on W, to $\sigma_E(f, g) = \sup\{|f(w) - g(w)| : w \in W\}$ on $E(W, n)$, and for $w \in W, f \in E(W, n), d_{E'(W,n)}(w, f) = 1 + \sigma_E(i_n(w), f)$.

Now define inductively

$$\begin{aligned} M_0 &= E(X, 1), \\ M_{n+1} &= E'(M_n, n+1). \end{aligned}$$

Up to the obvious identifications, each M_n is, as a metric space, a subspace of M_{n+1},

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots,$$

and finally we define

$$M = \bigcup_n M_n,$$

with distance say δ.

Every $\varphi \in \mathrm{Iso}(X)$ has an obvious extension $\varphi^* \in \mathrm{Iso}(M)$, and $\varphi \mapsto \varphi^*$ is an isomorphism, so it is enough to check that M is a σ-compact Polish metric space, which is moreover rich.

This will follow from the following two sublemmas:

Sublemma 6.7. *If (W, σ) is a σ-compact Polish space, then $E(W, n)$ is a σ-compact Polish metric space.*

Remark. It is not true that if W is locally compact, then $E(W, n)$ is locally compact, if $n \geq 2$. A counterexample is, for example, $E(\mathbb{R}, 2)$: The function $f(x) = |x| + a_0, a_0 > 1$, is in $E(\mathbb{R}, 2)$ and has support $\{0\}$. For each $1 > \epsilon > 0, n \geq 1$, the elements $f_n \in E(\mathbb{R}, 2)$ with support $\{0, n\}$, and $f_n(0) = a_0, f_n(n) = a_0 + n - \epsilon$, are in the ϵ-nbhd of f but they have no uniformly convergent subsequence. So although we start with a *locally compact* space X, we only end up with a *σ-compact* space M.

Sublemma 6.8. *For any Polish metric space $(W, \sigma), E(W, 1)$ is Rich 1. If (W, σ) is Rich 1, so is $E'(W, n)$, for any $n \geq 1$.*

Granting these two sublemmas, it is clear that M is a σ-compact Polish space, being a discrete sum of such spaces, and it is also Rich 1 being an increasing union of Rich 1 spaces. To check that it is also Rich 2, fix distinct points $x_1, \ldots, x_n, y_1, \ldots, y_n \in M$. Then for some $m > 2n, x_1, \ldots, x_n, y_1, \ldots, y_n \in M_m$. Then, by the argument in the proof of Sublemma 2.7, Claim 2, it is easy to see that there is $u \in E(M_m, m) \subseteq E'(M_m, m) = M_{m+1} \subseteq M$, so that $\sum_{i=1}^n u(x_i) \neq \sum_{i=1}^n u(y_i)$. But then, as $\delta(u, x_i) = 1 + u(x_i)$, it follows that $\sum_{i=1}^n \delta(u, x_i) \neq \sum_{i=1}^n \delta(u, y_i)$, and we are done.

Proof of Sublemma 6.7. We have seen in 3.4 that $E(W, n)$ is a Polish metric space. For $K \subseteq W, K$ compact, let $E(K, n) = \{f \in E(W, n) : f$ is supported by $K\}$. It is clearly enough then to show that if for $g \in E(W, n), \epsilon > 0, B_\epsilon^{c\ell}(g)$ denotes the closed ball of radius ϵ with center g in $E(W, n)$, then each $B_\epsilon^{c\ell}(g) \cap E(K, n)$ is compact. By the argument in the proof of 3.4, it is closed. Now fix a sequence (f_i) of elements of $B_\epsilon^{c\ell}(g) \cap E(K, n)$ with supports (Y_i), where $\mathrm{card}(Y_i) \leq n$ and $Y_i \subseteq K$, in order to find a convergent subsequence. Since the space of subsets of K of cardinality $\leq n$, with the Hausdorff metric, is also compact, we can assume that $Y_i \to Y, Y \subseteq K$, $\mathrm{card}(Y) \leq n$, where convergence is in the Hausdorff metric. We claim also that we can assume, by going to a subsequence if necessary, that $(f_i(x))$ converges for each x. To see this note that for each $x \in W, |f_i(x) - g(x)| \leq \epsilon$, so there is a subsequence of (f_i) which converges pointwise at x. Then, by the usual diagonalization argument, there is a subsequence, still called (f_i), such that $(f_i(x))$ converges for all x in a countable dense set $D \subseteq W$. Then we claim that $(f_i(x))$ converges for all x. Indeed, fix $\delta > 0$ and $y \in D$, with $\sigma(x,y) < \delta$. Then $|f_i(x) - f_j(x)| \leq |f_i(x) - f_i(y)| + |f_i(y) - f_j(y)| + |f_j(y) - f_j(x)| \leq 2\sigma(x,y) + |f_i(y) - f_j(y)|$. So if $n(\delta)$ is such that $i, j > n(\delta) \Rightarrow |f_i(y) - f_j(y)| < \delta$, we have $|f_i(x) - f_j(x)| < 3\delta$, for $i, j > n(\delta)$.

Let $f(x) = \lim f_i(x)$. Then $|f(x) - f(y)| \leq \sigma(x,y) \leq f(x) + f(y)$ for all $x, y \in W$. If we can show that f has support Y and $f_i \to f$ uniformly, then we will be done.

To see that f has support Y, fix any $x \in W$. Then find $y_i(x) \in Y_i$ such that $f_i(x) = f_i(y_i(x)) + \sigma(x, y_i(x))$. By going to a subsequence, we can assume that $y_i(x) \to y \in Y$. Then

$$\begin{aligned} f_i(x) &= f_i(y_i(x)) + \sigma(x, y_i(x)) \\ &= f_i(y_i(x)) - f_i(y) + f_i(y) - \sigma(x, y_i(x)). \end{aligned}$$

But $|f_i(y_i(x)) - f_i(y)| \leq \sigma(y_i(x), y)$, so, letting $i \to \infty$, we get

$$f(x) = f(y) + \sigma(x,y).$$

Finally, we check that $f_i \to f$ uniformly: Fix $\delta > 0$ and let $N(\delta)$ be such that $i > N(\delta) \Rightarrow (|f_i(y) - f(y)| < \delta, \forall y \in Y$, and the Hausdorff distance of Y_i, Y is $< \delta$). Fix now any $x \in W, i > N(\delta)$. Then

$$f_i(x) - f(x) = \inf_{y \in Y_i}\{f_i(y) + \sigma(x,y)\} - \inf_{y \in Y}\{f(y) + \sigma(x,y)\}.$$

If the second inf is realized by $y \in Y$, choose $y_i \in Y_i$ with $\sigma(y_i, y) < \delta$. Then

$$\begin{aligned} f_i(x) - f(x) &\leq (f_i(y_i) + \sigma(x, y_i)) - (f(y) + \sigma(x,y)) \\ &= f_i(y_i) - f_i(y) + f_i(y) - f(y) + \sigma(x, y_i) - \sigma(x,y) \\ &< 2\sigma(y_i, y) + \delta < 3\delta. \end{aligned}$$

Also switching the roles of f_i, f in the above argument we have $f(x) - f_i(x) < 3\delta$, and the proof is complete. ⊣

Proof of Sublemma 6.8. It is routine to verify that $E(W,1)$ is Rich 1, since $E(W,1)$ is isometric to $W \times \mathbb{R}$ with the metric

$$\tau((x,a),(y,b)) = |a - b| + \sigma(x,y).$$

Now assume (W,σ) is Rich 1, and let $(x_1,\ldots,x_m)\in E'(W,n)^m$. Then each x_i is in the copy of W or in the copy of $E(W,n)$. By renumbering (which does not affect the argument below), we can assume that for some $0\le i\le m, x_1,\ldots x_i\in W$ and $x_{i+1},\ldots,x_m\in E(W,n)$. Write $f_i=x_{i+j}, 1\le j\le m-i$. Since W is Rich 1 we can assume that the $x_1,\ldots,x_i$ are distinct and have distinct distances $\sigma(x_k,x_\ell), 1\le k<\ell\le i$. Then, given $\epsilon>0$, by induction on $j=1,\ldots,m-i$ we will construct $f'_j, f'_j\ne f'_\ell, \forall\ell<j$, so that its distance from f_j is $<\epsilon$, and the distances $d_{E'(W,n)}(x_k,f'_j), d_{E'(W,n)}(f'_\ell,f'_j), 1\le k\le i,\ell<j$, are all distinct and different from the distances $\sigma(x_k,x_\ell), 1\le k<\ell\le i, \sigma_E(f'_\ell,f'_p), 1\le\ell<p<j$, and $\sigma(x_k,f'_p), 1\le k\le i, p<j$. Let $Y=\{y_1,\ldots,y_r\}, r\le n$, be a support of f_j. Put $f_j(y_t)=\alpha_t, t\le r$. We will take as a support of f'_j the set Y. If we call $f'_j(y_t)=\alpha'_t$, then the above conditions amount to choosing α'_t, so that $|\alpha'_t-\alpha_t|<\epsilon$,

$$|\alpha'_t-\alpha'_s|\le\sigma(y_t,y_s)\le\alpha'_t+\alpha'_s,$$

and moreover $\alpha'_1,\ldots,\alpha'_r$ avoid some finite set of reals (we are using here that each $f'_\ell,\ell<j$, has finite support). This is easily achieved, for example by an argument similar to that used for Claim 1 in the proof of Sublemma 2.7. ⊣

We will now use this characterization to determine the complexity of the orbit equivalence induced by a Borel action of Iso(X), where X is any locally compact Polish metric space.

Theorem 6.9. *Let H be the isometry group of a locally compact separable metric space. Let Y be a Borel H-space with orbit equivalence relation E^Y_H. Then E^Y_H is Borel reducible to graph isomorphism.*

Proof. This proof requires detailed knowledge of Hjorth's theory of turbulence (see Hjorth [1999]). We will find it more convenient here to use the exposition of Hjorth's theory in Kechris [2000]. We will assume familiarity with this paper and use its notation and terminology. Since we will make repeated references to it we will abbreviate it below as K2.

Recall that a Polish group is called a *GE group* if any continuous and minimal action (i.e., one in which orbits are dense) of this group on a Polish space which has the property that it contains a G_δ orbit, is transitive (i.e., has only one orbit).

Hjorth [2000] has shown that any countable product $\prod_n G_n$ with each G_n a locally compact Polish group is a GE group. Moreover, he has shown that every GE group satisfies the statement of Theorem 13.18 of K2.

Now consider any Polish group K acting continuously by homomorphisms, on a Polish group G via $(k, g) \mapsto k \cdot g$. Then we can define as usual the semidirect product $K \ltimes G$ by taking as the underlying set the product $K \times G$ and defining the group operation by

$$(k_1, g_1)(k_2, g_2) = (k_1 k_2, (k_2^{-1} \cdot g_1) g_2).$$

Equipped with the product topology this becomes a Polish group.

Now the proof of Theorem 6.1 shows that $H = \mathrm{Iso}(X)$ is isomorphic to a closed subgroup of a semidirect product $K \ltimes G$ where K is (isomorphic to) a closed subgroup of S_∞ and G is a countable product of locally compact Polish groups, so G satisfies Theorem 13.18 of K2.

Moreover, the action of K on G has a technical property that will be important later on: There is a countable dense subgroup $G_0 \subseteq G$ such that for any $k \in K, k \cdot G_0 = G_0$. To see this notice that in the notation of 6.1, $K = \prod_{n \in N} S_{I_n}$ and $G = \prod_{n \in N} G_n^{I_n}$ and K is acting on G as follows: $(k_n)_{n \in N} \cdot (x_n)_{n \in N} = (k_n \cdot x_n)_{n \in N}$, where $k_n \cdot x_n(i) = x_n(k_n^{-1}(i))$. If we let then G_n^0 be a countable dense subgroup of G_n we see that we can take $G_0 = \{(g_n) \in \prod_{n \in N} G_n^{I_n} : g_n = 1$ for all but finitely many n, and $g_n = (g_i^n)_{i \in I_n}$, where $g_i^n = 1$ for all but finitely many i and $g_i^n \in G_n^0$, for all $i\}$.

Since, by Becker-Kechris [1996, proof of 3.5.1], E_H^Y is then Borel reducible to some $E_{K \ltimes G}^X$, X a Borel $K \ltimes G$-space, it is enough to show that if X is a Borel $K \ltimes G$-space, then $E_{K \ltimes G}^X$ can be Borel reduced to graph isomorphism. By Becker-Kechris [1996, 5.2], we can assume that X is a Polish space on which $K \ltimes G$ acts continuously.

We will now apply the theory of Section 13 of K2. We will view K, G as usual as subgroups of $K \ltimes G$ (identified, resp., with $K \times \{1\}, \{1\} \times G$) and we will denote by $k \cdot g$ the action of K on G and by $(k, g) \cdot x$ the action of $K \ltimes G$ on X. Since G is a subgroup of $K \ltimes G$ we can also view X as a G-space. Again we write $g \cdot x$ $(= (1, g) \cdot x)$ for this action and we apply to it the theory of Section 13 of K2.

First note that K, being a closed subgroup of S_∞, has a countable basis $\mathcal{K}$ closed under left multiplication by elements of K. We also fix a countable dense subgroup $G_0 \subseteq G$ such that $k \cdot G_0 = G_0$, for all $k \in K$. Using these we can find a countable open basis $\mathcal{B}$ for X, containing X, which is closed under $U \mapsto g_0 \cdot U$ for $g_0 \in G_0$, and also under $U \mapsto k \cdot U = (k, 1) \cdot U$ for $k \in K$. To see this fix a countable open basis $\mathcal{B}_0$ for X, containing X, and let $\mathcal{B}$ consist of all open sets of the form $g_0 N \cdot U_0$ for $U_0 \in \mathcal{B}_0, g_0 \in G_0, N \in \mathcal{K}$. It is clear that $\mathcal{B}$ is closed under action by G_0. To see that it is also closed under action by K, let $k \in K$ and $g_0 N \cdot U_0 \in \mathcal{B}$. Then $k \cdot (g_0 N \cdot U_0) = kg_0 N \cdot U_0 = (k \cdot g_0)kN \cdot U_0 = g_0' N' \cdot U_0 \in \mathcal{B}$, where $g_0' = k \cdot g_0 \in G_0$ and $kN = N' \in \mathcal{K}$.

Similarly we can find a countable basis $\mathcal{N}$ of symmetric open nbhds of $1 \in G$, containing G, closed under $V \mapsto g_0 V g_0^{-1}$ for $g_0 \in G_0$ and $V \mapsto k \cdot V$, for $k \in K$. To see this take again $\mathcal{N}_0$ a countable basis of symmetric open nbhds of $1 \in G$, containing G, and let $\mathcal{N}$ consist of all sets of the form $g_0(N \cdot V_0)g_0^{-1}$ for $V_0 \in \mathcal{N}_0, g_0 \in G_0, N \in \mathcal{K}$.

We fix 1-1 enumerations $\mathcal{B} = \{U_\ell\}, \mathcal{B}_0 = \{U_{0,i}\}, G_0 = \{g_{0,j}\}, \mathcal{K} = \{N_m\}$. (We can of course assume these are all infinite, without loss of generality.) Then K acts on $G_0 \times \mathcal{K}$ by $k \cdot (g_0, N) = (k \cdot g_0, kN) = \rho_k((g_0, N))$ and it is easy to see that the map $k \mapsto \rho_k$ from K into the group of permutations of $G_0 \times \mathcal{K}$ (with the pointwise convergence topology) is an isomorphism, so its range is a closed subgroup of this group. Thus (see Becker-Kechris [1996, 1.5]) we can find a relational structure $\mathcal{T}_2 = \langle G_0 \times \mathcal{K}, - \rangle$ (in a countable language) with universe $G_0 \times \mathcal{K}$ such that $\mathrm{Aut}(\mathcal{T}_2) = \{\rho_k : k \in K\}$. Let also $\mathcal{T}_1$ be the structure $\mathcal{T}_1 = \langle \mathcal{B}_0, \prec \rangle$, where $U_{0,i} \prec U_{0,i'} \Leftrightarrow i < i'$.

Finally, let $\mathcal{T}$ be the structure with universe the disjoint union $\mathcal{B} \sqcup \mathcal{B}_0 \sqcup G_0 \times \mathcal{K}$ and the following relations: Three unary relation that define $\mathcal{B}, \mathcal{B}_0, G_0 \times \mathcal{K}$ resp.; $\prec$ on $\mathcal{B}_0$; the relations of $\mathcal{T}_2$ on $G_0 \times \mathcal{K}$; and finally the following ternary relation

$$\begin{aligned} R(U, U_0, (g_0, N)) \quad \Leftrightarrow \quad & U \in \mathcal{B}, U_0 \in \mathcal{B}_0, g_0 \in G_0, \\ & N \in \mathcal{K}, \text{ and } U = g_0 N \cdot U_0 \end{aligned}$$

(recall that $\mathcal{B}$ consists of all such $g_0 N \cdot U_0$).

Clearly any $k \in K$ gives an automorphism ψ_k of $\mathcal{T}$, where $\psi_k(U) = k \cdot U$ for $U \in \mathcal{B}, \psi_k(U_0) = U_0$, if $U_0 \in \mathcal{B}_0, \psi_k((g_0, N)) = (k \cdot g_0, kN) = \rho_k((g_0, N))$. (Recall here that $k \cdot (g_0 N \cdot U_0) = (k \cdot g_0)kN \cdot U_0$.)

Conversely, if $\psi \in \mathrm{Aut}(\mathcal{T})$, then $\psi | G_0 \times \mathcal{K} \in \mathrm{Aut}(\mathcal{T}_2)$, so $\psi = \rho_k$, for some $k \in K$. Also $\psi | \mathcal{B}_0 =$ identity. We claim now that $\psi(U) = k \cdot U$ for all $U \in \mathcal{B}$, so $\psi = \psi_k$. Indeed, given such U, write it as $U = g_0 N \cdot U_0$ for some $g_0 \in G_0, N \in \mathcal{K}, U_0 \in \mathcal{B}_0$. Then we have that

$$R(U, U_0, (g_0, N))$$

holds. Thus

$$R(\psi(U), \psi(U_0), \psi((g_0, N)))$$

holds, i.e.,

$$\psi(U) = (k \cdot g_0)kN \cdot U_0 = k \cdot U.$$

For each $x \in X$ consider now the following structure $\mathcal{T}^0(x)$ which is a modification of the structure $\mathcal{M}^0(x)$ in Section 13 of K2. The universe of $\mathcal{T}^0(x)$ is the disjoint union of the universe of $\mathcal{M}^0(x)$ (which consists of all elements of the form $(U, V, \mathcal{O}(y, U, V))$, where $U \in \mathcal{B}, V \in \mathcal{N}, y \in [x](= G \cdot x)$ and $\mathcal{O}(y, U, V)$ refers to the (U, V)-local orbit of y in the G-action) and the universe of $\mathcal{T}$. We write $\mathcal{O}(U, V)$ instead of $\mathcal{O}(y, U, V)$ if it is not important to exhibit y. The relations of $\mathcal{T}^0(x)$ are as follows: We have two unary relations that determine the universes of $\mathcal{M}^0(x), \mathcal{T}$, resp. We have the follow binary relation on the universe of $\mathcal{M}^0(x)$

$$(U, V, \mathcal{O}(U, V)) \leq (U', V', \mathcal{O}(U', V')) \Leftrightarrow U' \subseteq U, V' \subseteq V, \mathcal{O}(U, V) \subseteq \mathcal{O}(U', V').$$

Finally, we have the following binary relation (on the (universe of $\mathcal{M}^0(x)) \times \mathcal{B}$)

$$S((U, V, \mathcal{O}(U, V)), U_\ell) \Leftrightarrow U_\ell \cap \mathcal{O}(U, V) \neq \emptyset.$$

(recall that $\mathcal{B} = \{U_\ell\}$). By comparison the structure $\mathcal{M}^0(x)$ in Section 13 of K2 has the same binary relation $\leq$ but instead of S it has a sequence of unary relations R_ℓ given by the above formula for S, for each *fixed* ℓ.

Let us note now that if $k \in K$,

$$k \cdot \mathcal{O}(y, U, V) = \mathcal{O}(k \cdot y, k \cdot U, k \cdot V) \qquad (*)$$

and so k gives rise to an isomorphism $\theta_k : \mathcal{T}^0(x) \to \mathcal{T}^0(k \cdot x)$, where θ_k is defined by $\theta_k((U, V, \mathcal{O}(y, U, V)) = (k \cdot U, k \cdot V, k \cdot \mathcal{O}(y, U, V))$, and $\theta_k = \psi_k$ on the universe of $\mathcal{T}$.

Let now $x E^X_{K \ltimes G} y$. Then $x = k \cdot (g \cdot y)$ for some $k \in K, g \in G$. Since $\mathcal{M}^0(y) = \mathcal{M}^0(g \cdot y)$, it follows that $\mathcal{T}^0(x) \cong \mathcal{T}^0(y)$. Conversely, assume now that $\mathcal{T}^0(x) \cong \mathcal{T}^0(y)$, say via an isomorphism θ. Then clearly $\theta | T \in \mathrm{Aut}(\mathcal{T})$, where T = universe of $\mathcal{T}$. Thus there is $k \in K$ with $\theta | T = \psi_k$. Consider $z = k^{-1} \cdot y$. Then $\theta_{k^{-1}} : \mathcal{T}^0(y) \to \mathcal{T}^0(z)$ is an isomorphism and, as $\theta_{k^{-1}} | T = \psi_{k^{-1}}$, it follows that $\theta_{k^{-1}} \circ \theta : \mathcal{T}^0(x) \to \mathcal{T}^0(z)$ is an isomorphism with $(\theta_{k^{-1}} \circ \theta) | T = \psi_{k^{-1}} \circ \psi_k$ = identity. It follows that $\theta_{k^{-1}} \circ \theta$ restricted to the universe of $\mathcal{M}^0(x)$ is an isomorphism between $\mathcal{M}^0(x), \mathcal{M}^0(z)$, so, by Section 13 of K2 $\varphi_x = \varphi_z$ and thus, by Theorem 13.18 of K2, $x E^X_G z$, i.e., $\exists g \in G$ with $g \cdot z = x$, so $x = g \cdot z = gk^{-1} \cdot y$, thus $x E^X_{K \ltimes G} y$.

So we have shown that

$$xE^X_{K\ltimes G}y \Leftrightarrow \mathcal{T}^0(x) \cong \mathcal{T}^0(y).$$

Now in order to "encode" $\mathcal{M}^0(x)$ (up to isomorphism) by a structure with universe $\mathbb{N}$, and do that in a Borel way, one goes through a series of technical manipulations (described in Section 13 of K2), which eventually replace $\mathcal{M}^0(x)$ by a structure $\tilde{\mathcal{M}}(x)$ (see 13.8 of K2). Using ideas as before, one can replace the structure $\tilde{\mathcal{M}}(x)$ by a structure $\tilde{\mathcal{T}}(x)$ by a procedure similar to the one that produced $\mathcal{T}(x)$ from $\mathcal{M}(x)$. It follows that $x \mapsto \tilde{\mathcal{T}}(x)$ is Borel ($\tilde{\mathcal{T}}(x)$ is now a structure with universe $\mathbb{N}$) and $xE^X_{K\ltimes G}y \Leftrightarrow \tilde{\mathcal{T}}(x) \cong \tilde{\mathcal{T}}(y)$, so this shows that $E^X_{K\ltimes G}$ can be Borel reduced to isomorphism of structures with universe $\mathbb{N}$ (in some countable language) and thus to graph isomorphism. $\dashv$

We conclude this section with an application concerning a seemingly new proof of the existence and uniqueness of isometry invariant measures on locally compact separable metric spaces, which generalizes the concept of Haar measure.

The following result is known and is in fact a special case of a result of Loomis [1949], Theorem 10. Our proof below, based simply on the existence and uniqueness of Haar measure, may be though of some interest. (We would like to thank Slawek Solecki for guiding us to the relevant literature here.)

Theorem 6.10. (A special case of Loomis [1949]) *Let X be a locally compact separable metric space. For every subgroup $G \subseteq \mathrm{Iso}(X)$, the following are true:*

i) (Existence) There is a G-invariant Borel measure μ on X such that $\mu(K) < \infty$ for each compact subset $K \subseteq X$ (so, in particular, μ is σ-finite and regular).

ii) If there is a point in X whose G-orbit is dense, then

a) $\mu(V) > 0$ for each open nonempty $V \subseteq X$.

b) (Uniqueness) If ν is any Borel measure on X which is G-invariant and satisfies $\mu(K) < \infty$ for each compact $K \subseteq X$ and $\mu(V) > 0$ for each open nonempty $V \subseteq X$, then $\nu = c\mu$ for some positive real c.

Proof. i) Fix a pseudo-component C_0 of X and a point $x_0 \in C_0$. Recall then, from 5.6 i), that $\mathrm{Iso}(C_0)$ is locally compact. Denote then by G_0 the closure in $\mathrm{Iso}(C_0)$ of the group

$$\{g \in \mathrm{Iso}(C_0) : \exists\varphi \in G(\varphi(C_0) = C_0 \text{ and } \varphi|C_0 = g)\},$$

of all elements of $\text{Iso}(C_0)$ which extend to an isometry in G. Then G_0 is also locally compact, and we fix a left-invariant Haar measure μ_{G_0} on G_0. Let $\pi_0 : G_0 \to C_0$ be defined by $\pi_0(g) = g(x_0)$, so that $\pi(G_0) = G_0(x_0) = T_0$, the G_0-orbit of x_0. Use this to define the Borel measure μ_0 on C_0 by

$$\mu_0(A) = \mu_{G_0}(\pi_0^{-1}(A)).$$

Clearly μ_0 concentrates on T_0, which by 5.6 ii) is a closed subset of C_0. Note that if $K \subseteq C_0$ is compact, then $\pi_0^{-1}(K)$ is compact, by 5.6 ii), so $\mu_0(K) = \mu_{G_0}(\pi_0^{-1}(K)) < \infty$. Also if $V \subseteq C_0$ is open and $V \cap T_0 \neq \emptyset$, then $\pi_0^{-1}(V)$ is an open nonempty subset of G_0, so $\mu_0(V) = \mu_{G_0}(\pi_0^{-1}(V)) > 0$.

We will now extend μ_0 to a measure on X. First notice that if $\varphi \in G$, then $\varphi(T_0)$ is contained in a simple pseudo-component C on X and $\varphi(C_0) = C$. We now claim that if $\varphi, \psi \in G$ and $\varphi(T_0), \psi(T_0)$ are contained in the *same* pseudo-component of X, say C_1, then actually $\varphi(T_0) = \psi(T_0)$. Indeed, if $G_1 =$ the closure in $\text{Iso}(C_1)$ of $\{g \in \text{Iso}(C_1) : \exists \varphi \in G(\varphi(C_1) = C_1$ and $\varphi | C_1 = g)\}$, then it is not hard to check that $\varphi(T_0) =$ the G_1-orbit of $\varphi(x_0) = G_1(\varphi(x_0)), \psi(T_0) = G_1(\psi(x_0))$. Since $\psi(x_0) = \psi\varphi^{-1}(\varphi(x_0))$, and $\psi\varphi^{-1} \in G_1$, it follows that $G_1(\varphi(x_0)) = G_1(\psi(x_0))$, so $\varphi(T_0) = \psi(T_0)$. It follows that $\{\varphi(T_0) : \varphi \in G\}$ is countable, so let $\{\varphi_n\} \subseteq G$ (a finite or infinite set) be such that $\{\varphi_n(T_0) : n \in \mathbb{N}\}$ is a 1-1 enumeration of $\{\varphi(T_0) : \varphi \in G\}$, with φ_0 = identity.

Now define for each Borel set $A \subseteq X$,

$$\mu(A) = \sum_n \mu_0(\varphi_n^{-1}(A \cap \varphi_n(T_0))).$$

It is clear that μ is a Borel measure. If $K \subseteq X$ is compact, then K is contained in only finitely many pseudo-components of X, so the above sum defining $\mu_0(K)$ is finite. Moreover in each summand $\mu_0(\varphi_n^{-1}(K \cap \varphi_n(T_0))$ is finite, as the set $\varphi_n^{-1}(K \cap \varphi_n(T_0)) = \varphi_n^{-1}(K) \cap T_0$ is compact. Thus $\mu(K)$ is finite.

Finally, to verify G-invariance, let $A \subseteq X$ be Borel and $\varphi \in G$, in order to show that $\mu(A) = \mu(\varphi(A))$. We can clearly assume that $A \subseteq \varphi_n(T_0)$ for some n. Then $\varphi(A) \subseteq \varphi(\varphi_n(T_0)) = \varphi_m(T_0)$, for some m. Thus

$$\mu(\varphi(A)) = \mu_0(\varphi_m^{-1}(\varphi(A)).$$

Now $\varphi_m^{-1} \circ \varphi \circ \varphi_n \in G$ sends T_0 to T_0, so also C_0 to C_0, therefore $\varphi_m^{-1} \circ \varphi \circ$

$\varphi_n|C_0 = \theta \in G_0$. Thus

$$\begin{aligned}\mu(\varphi(A)) &= \mu_0(\varphi_m^{-1}(\varphi(A))) \\ &= \mu_0(\varphi_m^{-1} \circ \varphi \circ \varphi_n(\varphi_n^{-1}(A)) \\ &= \mu_0(\theta(\varphi_n^{-1}(A)) \\ &= \mu_0(\varphi_n^{-1}(A)) \\ &= \mu(A),\end{aligned}$$

which completes the proof of i).

ii) We now take, in the construction described in i), x_0 to be a point such that its G-orbit, $G_0(x_0)$ is dense in X.

a) Let $V \subseteq X$ be open-nonempty. Then for some $\varphi \in G, \varphi(x_0) \in V$. In particular, for some $n, V \cap \varphi_n(T_0) \neq \emptyset$, and so $\varphi_n^{-1}(V \cap \varphi_n(T_0)) = \varphi_n^{-1}(V) \cap T_0 \neq \emptyset$, thus $\mu(V) \geq \mu_0(\varphi_n^{-1}(V \cap \varphi_n(T_0))) > 0$.

b) Fix such a measure ν. Our hypothesis about x_0 clearly implies that for any pseudo-component C, there is $\varphi \in G$ with $\varphi(C_0) = C$. From this, and the G-invariance of μ, ν, it follows that it is enough to show that there is a positive constant c such that $(\nu|C_0) = c(\mu|C_0)$, i.e., for each Borel set $A \subseteq C_0, \nu(A) = c\mu(A) = c\mu_0(A)$.

Now clearly T_0 is dense in C_0 and, as it is also closed, we actually have that $C_0 = T_0 = G_0(x_0)$. Moreover, if $K_0 = \{g \in G_0 : g(x_0) = x_0\}$, by 5.6, K_0 is compact and the map $\pi_0(g) = g(x_0)$ from G_0 onto C_0 is both continuous and open. Fix the normalized Haar measure μ_{K_0} on K_0 (so $\mu_{K_0}(K_0) = 1$). We will use $\mu_{K_0}, \nu|C_0 = \nu_0$ to define a left-invariant measure on G_0.

Fix $B \subseteq G_0$. For each $x \in C_0$ consider $\pi_0^{-1}(x) = gK_0$, for any $g \in G_0$ with $g(x_0) = x$. Then $g^{-1}(B \cap \pi_0^{-1}(x)) = g^{-1}(B \cap gK_0) = g^{-1}B \cap K_0 \subseteq K_0$, and it is easy to check, using the K_0-invariance of μ_{K_0}, that if $gK_0 = hK_0$, then $\mu_{K_0}(g^{-1}B \cap K_0) = \mu_{K_0}(h^{-1}B \cap K_0)$, so

$$f_B(x) = \mu_{K_0}(g^{-1}(B \cap gK_0))$$

is well-defined. Clearly $f_B : C_0 \to [0,1]$ is a Borel function. Finally, define

$$\rho_0(B) = \int_{C_0} f_B(x) d\nu_0(x).$$

Clearly ρ_0 is a Borel measure on G_0. Notice also that $f_B(x) \neq 0 \Rightarrow B \cap \pi_0^{-1}(x) \neq \emptyset \Rightarrow x \in \pi_0(B)$, so actually

$$\rho_0(B) = \int_{\pi_0(B)} f_B(x) d\nu_0(x).$$

If now $K \subseteq G_0$ is compact, then $\pi_0(K)$ is compact, so $\nu_0(\pi_0(K)) < \infty$. Thus $\rho_0(K) < \infty$. If $V \subseteq G_0$ is open, then $\pi_0(V)$ is open in C_0, thus open, so $\nu_0(\pi_0(V)) > 0$. Also for each $x \in \pi_0(V), V \cap \pi_0^{-1}(X)$ is open in $\pi_0^{-1}(x)$, so if $gK_0 = \pi_0^{-1}(x), g^{-1}(V \cap \pi_0^{-1}(x))$ is open in K_0, so $f_B(x) = \mu_{K_0}(g^{-1}(V \cap \pi_0^{-1}(x))) > 0$. Thus $\rho_0(V) = \int_{\pi_0(V)} f_V(x) d\nu_0(x) > 0$.

We will finally show that ρ_0 is left-invariant. Granting this, we conclude that there is a constant $c > 0$ with $\rho_0 = c\mu_{G_0}$. Now if $A \subseteq C_0$ is Borel and $B = \pi_0^{-1}(A)$, then $c\mu_{G_0}(B) = \rho_0(B) = \int_A f_B(x) d\nu(x) = \int_A 1 d\nu(x) = \nu(A)$. Also $\mu_0(A) = \mu_{G_0}(B)$, so $\nu(A) = c\mu_0(A)$ and the proof is complete.

To verify left-invariance: Fix $g \in G_0$. By definition of G_0 there are $\varphi_n \in G$, with $\varphi_n(C_0) = C_0$, such that if $g_n = \varphi_n | C_0$, then $g_n \to g$ in G_0, i.e., $g_n \to g$ pointwise. Assume that we could show that ρ_0 is invariant under left-translation under any g_n. Then we claim that ρ_0 is invariant under left-translation by g. By regularity of ρ_0 it is enough to verify that $\rho_0(U) = \rho_0(gU)$ for each open $U \subseteq G_0$ and as $\rho_0(U) = \sup\{\int f d\rho_0 : f$ is continuous with compact support contained in $U\}$, it is enough to verify that $\int f d\rho_0 = \int f_g d\rho_0$, for every continuous function f on G_0 with compact support, where $f_g(h) = f(g^{-1}h)$. Since ρ_0 is invariant under left-translation by any g_n, it follows that $\int f d\rho_0 = \int f_{g_n} d\rho_0$ for each n, i.e., $\int f(h) d\rho_0(h) = \int f(g_n^{-1}h) d\rho_0(h)$. Now as $n \to \infty, g_n^{-1}h \to g^{-1}h$, so $f(g_n^{-1}h) \to f(g^{-1}h)$ for each h. Moreover, if $f_n(h) = f(g_n^{-1}h)$, then the support of f_n is g_nK, and $L = \bigcup_n (g_nK) \cup gK$ is compact, so $|f_n| \leq |f|\chi_L$ and $\int |f|\chi_L d\rho_0 < \infty$, thus by Lebesgue Dominated Convergence, $\int f(g_n^{-1}h) d\rho_0(h) \to \int f(g^{-1}h) d\rho_0(h)$, so $\int f d\rho_0 = \int f_g d\rho_0$.

Finally, we have to verify that if $\varphi \in G_0, \varphi(C_0) = C_0$ and we set $g_0 = \varphi | C_0$, then ρ_0 is invariant under left-translation under g_0. Fix a Borel set $B \subseteq G_0$. Then we have that for each $y = g(x_0) \in C_0$,

$$\begin{aligned} f_{g_0 B}(x) &= \mu_{K_0}(g^{-1}(g_0 B \cap gK_0)) \\ &= \mu_{K_0}(g^{-1} g_0 B \cap K_0) \\ &= \mu_{K_0}(g^{-1} g_0 (B \cap g_0^{-1} g K_0)) \\ &= f_B(g_0^{-1}(x)). \end{aligned}$$

So

$$\begin{aligned}\rho_0(g_0B) &= \int_{C_0} f_{g_0B}(x)d\nu_0(x)\\ &= \int_{C_0} f_B(g_0^{-1}(x))d\nu_0(x)\\ &= \int_{C_0} f_B(x)d\nu_0(x)\\ &= \rho_0(B),\end{aligned}$$

by the fact that ν is G-invariant, and so ν_0 is invariant under g_0. $\dashv$

7 Isometric classification of locally compact spaces

We will present here Hjorth's proof of our conjecture that isometry on pseudo-connected locally compact Polish metric spaces is Borel bireducible with the universal countable Borel equivalence relation (similarly for connected or Heine-Borel locally compact spaces.)

Theorem 7.1. (Hjorth) *The equivalence relation of isometry of pseudo-connected locally compact Polish metric spaces is Borel bireducible with the universal countable Borel equivalence relation, E_∞. The same also holds for isometry on connected, Heine-Borel, and connected Heine-Borel locally compact Polish metric spaces.*

Proof. It is not hard to see that E_∞ can be Borel reduced to isometry of connected Heine-Borel locally compact Polish spaces. It is a result of Jackson-Kechris-Louveau, see, e.g., Hjorth-Kechris [1996, p. 241], that E_∞ is Borel bireducible with isomorphism on countable locally finite trees with at least one pending vertex (i.e., countable connected acyclic graphs in which every vertex has finitely many neighbors and there is at least one vertex with only one neighbor). Now every such tree can be viewed as a locally compact Polish metric space in such a way that isomorphisms correspond to isometries: View each edge as replaced by a copy of the interval [0,1] and define the distance in the usual way. Clearly this space is both connected and Heine-Borel.

It remains to show that isometry on pseudo-connected locally compact Polish metric spaces is Borel reducible to E_∞. As usual we will identify the class of pseudo-connected locally compact Polish metric spaces with the set

$$F^{plc}(\mathbb{U}) = \{F \in F(\mathbb{U}) : F \text{ is pseudo-connected locally compact}\}.$$

We also let

$$\begin{aligned}\tilde{F}^{plc}(\mathbb{U}) &= \{F \in F(\mathbb{U}) : F \text{ is almost locally compact} \\ &\quad \text{and } \mathrm{Core}(F) \text{ is pseudo-connected}\}.\end{aligned}$$

It is easy to see that $\tilde{F}^{plc}(\mathbb{U})$ is Borel. (It appears that $F^{plc}(\mathbb{U})$, which is clearly co-analytic, is not Borel, although we have not verified this. It is easy to see that $\{F \in F(\mathbb{U}) : F \text{ is locally compact}\}$ is co-analytic but not Borel.) Also clearly $F^{plc}(\mathbb{U}) \subseteq \tilde{F}^{plc}(\mathbb{U})$ and for $F_1, F_2 \in \tilde{F}^{plc}(\mathbb{U})$.

$$F_1 \cong_i F_2 \Leftrightarrow \mathrm{Core}(F_1) \cong_i \mathrm{Core}(F_2).$$

First we will verify that $\cong_i$ on $\tilde{F}^{plc}(\mathbb{U})$ is Borel. Let

$$\begin{aligned}P(F_1, F_2, x_1, x_2) \quad \Leftrightarrow \quad & F_1, F_2 \in \tilde{F}^{plc}(\mathbb{U}), x_1 \in \mathrm{Core}(F_1), x_2 \in \mathrm{Core}(F_2), \\ & (\mathrm{Core}(F_1), x_1) \cong_i (\mathrm{Core}(F_2), x_2).\end{aligned}$$

Then P is Borel, since, in the notation of 5.7, if $x_1 \in \mathrm{Core}(F_1), x_2 \in \mathrm{Core}(F_2)$

$$(\mathrm{Core}(F_1), x_1) \cong_i (\mathrm{Core}(F_2), x_2) \Leftrightarrow L(\mathrm{Core}(F_1), x_1) = L(\mathrm{Core}(F_2), x_2)$$

and it is easy to see that $L(\mathrm{Core}(F), x)$ is Borel on $\{(F, x) : F \in \tilde{F}^{plc}(\mathbb{U}), x \in \mathrm{Core}(F)\}$. Now $F_1 \cong_i F_2 \Leftrightarrow \exists x_1, x_2 P(F_1, F_2, x_1, x_2)$, and the sections

$$\{(x_1, x_2) \in \mathbb{U}^2 : P(F_1, F_2, x_1, x_2)\}$$

are σ-compact by 5.9, so by the Arsenin-Kunugui Theorem (see Kechris [1995, 35.46]), $\cong_i$ on $\tilde{F}^{plc}(\mathbb{U})$ is Borel.

Next we will find a standard Borel space Ω and a Borel map $J : \tilde{F}^{plc}(\mathbb{U}) \to \Omega$ such that for any $F_1, F_2 \in \tilde{F}^{plc}(\mathbb{U})$:

(i) $J[\{H : H \cong F_1\}]$ is countable,

(ii) $F_1 \not\cong_i F_2 \Rightarrow J[\{H : H \cong F_1\}] \cap J[\{H : H \cong F_2\}] = \emptyset$.

Then by a result of Kechris (see, e.g., Hjorth [∞, 5.2]) $\cong_i$ on $\tilde{F}^{plc}(\mathbb{U})$ can be Borel reduced to a countable Borel equivalence relation, therefore to E_∞, and thus so can $\cong_i$ on $F^{plc}(\mathbb{U})$.

To construct these J, Ω we will use the functions $K_{n,m}(X,x), L(X,x)$ introduced in the proof of 5.7. We will need first a few lemmas establishing some properties of $K_{n,m}$. If there is no danger of confusion we will simply write $K_{n,m}(x)$ instead of $K_{n,m}(X,x)$. We will also give the arguments in case X is not Heine-Borel, the other case being similar.

Lemma 7.2. *Let (X,d) be a pseudo-connected locally compact separable metric space. If $x_i \in X, i \in \mathbb{N}$, and $x_i \to x_\infty \in X$, then*

$$K_{n,m}(x_\infty) \supseteq \overline{T\lim_i} K_{n,m}(x_i),$$

where $\overline{T\lim_i} F_i = \{x \in X :$ every nbhd of x meets F_i for infinitely many $i\}$ is the topological upper limit of (F_i).

Proof. Let $u \in \overline{T\lim_i} K_{n,m}(x_i)$. Then $u = \lim_i u_i$, where $u_i \in K_{n,m}(x_{n_i})$, $n_0 < n_1 < n_2 < \ldots$ Say $u_i = (d(y^i_k, y^i_\ell))^\frown(\rho(y^i_k))$, where $0 \le k, \ell \le m-1, y^i_0 = x_{n_i}, y^i_k \in K_n(x_{n_i})$. Then each $y^i_k, 1 \le k \le m-1$, is the end of a chain $(y^i_k)_0, \ldots, (y^i_k)_n$, where $(y^i_k)_0 = x_{n_i}, d((y^i_k)_j, (y^i_k)_{j+1}) \le \frac{n}{n+1}\rho((y^i_k)_j)$. Then all of the points $(y^i_k)_j, i = 0, 1, \ldots$, are in $K_{n+1}(x_\infty)$ if i is large enough, so there is a subsequence $i_0 < i_1 < \ldots$ such that $(y^{i_p}_k)_j \to (\tilde{y}_k)_j$, so also $y^{i_p}_k \to \tilde{y}_k$. Then $(\tilde{y}_k)_0 = x_\infty$ and thus $\tilde{y}_k \in K_n(x_\infty)$. Also $\tilde{y}_0 = x_\infty$, and $u_{i_p} \to (d(\tilde{y}_k, \tilde{y}_\ell))^\frown(\rho(\tilde{y}_k)) \in K_{n,m}(x_\infty)$, so $u \in K_{n,m}(x_\infty)$. ⊣

Lemma 7.3. *In the notation of 7.2, $x \mapsto K_{n,m}(x)$ is a Baire class 1 function on X and thus so is $x \mapsto L(x)$.*

Proof. It is enough to show that if $F \subseteq \mathbb{R}^{m^2 \times m}$ is closed,

$$\{x \in X : K_{n,m}(x) \cap F \neq \emptyset\}$$

is closed. So let $x_i \in X, x_i \to x_\infty$ be such that $K_{n,m}(x_i) \cap F \neq \emptyset$. We have to show that $K_{n,m}(x_\infty) \cap F \neq \emptyset$. Fix $u_i \in K_{n,m}(x_i) \cap F_i$, say $u_i = (d(y_k, y_\ell))^\frown(\rho(y_k))$, with $0 \le k, \ell \le m-1, x_i = y_0, y_1, \ldots, y_{m-1} \in K_n(x_i)$. Then for large enough i, $x_i, y_1, \ldots, y_{m-1} \in K_{n+1}(x_\infty)$ and so $\{u_i\}$ belongs in some compact set, therefore has a converging subsequence $u_{n_i} \to u$. Clearly $u \in \overline{T\lim_i} K_{n,m}(x_i) \cap F \subseteq K_{n,m}(x_\infty) \cap F$, by Lemma 7.2. ⊣

Lemma 7.4. *Let $(X, d_X), (Y, d_Y)$ be pseudo-connected locally compact separable metric spaces. If $x_i \in X, i \in \mathbb{N}$, and $x_i \to x_\infty$ and if $L(X, x_i) \to L(Y, y)$ for some $y \in Y$, then $L(X, x_\infty) = L(Y, y)$.*

Proof. We have to show that $K_{n,m}(y) = K_{n,m}(x_\infty)$ for all n, m. Now

$$K_{n,m}(x_\infty) \supseteq \overline{T\lim_i} K_{n,m}(x_i)$$

and

$$K_{n,m}(x_i) \to K_{n,m}(y),$$

so

$$K_{n,m}(y) = \overline{T\lim}_i K_{n,m}(x_i) \subseteq K_{n,m}(x_\infty).$$

So it is enough to show that

$$K_{n,m}(x_\infty) \subseteq K_{n,m}(y).$$

Let $u \in K_{n,m}(x_\infty)$, so that

$$u = (d_X(v_k, v_\ell))\hat{\ }(\rho_X(v_k)),$$

$v_0 = x_\infty, v_1, \ldots, v_m \in K_n(x_\infty)$. Let $v_k^1 = x_\infty, v_k^2, \ldots, v_k^{n+1} = v_k$ be such that $d_X(v_k^i, v_{k+1}^i) \leq \frac{n}{n+1}\rho_X(v_k^i)$. Then for i large enough,

$$v_k^0 = x_i, v_k^1, v_k^2, \ldots, v_k^{n+1} = v_k$$

witnesses that $v_k \in K_{n+1}(x_i)$, so that the sequence

$$(1) \quad x_i, (v_k^\ell)_{0 \leq \ell \leq n+1, 1 \leq k \leq m}$$

is in $K_{n+1}(x_i)$ and thus the "$(d(\ ))\hat{\ }(\rho(\ ))$" of that sequence is in $K_{n+1,m(n+1)+1}(x_i)$. Then we can find

$$(2) \quad y, ((v^{(i)})_k^\ell)_{0 \leq \ell \leq n+1, 1 \leq k \leq m}$$

in $K_{n+1}(y)$ so that the "$(d(\ ))\hat{\ }(\rho(\ ))$" of (2) is within

$$\leq d_H(K_{n+1,m(n+1)+1}(x), K_{n+1,m(n+1)+1}(y))$$

from the "$(d(\ ))\hat{\ }(\rho(\ ))$" of (1), where d_H denotes the Hausdorff distance.

As $K_{n+1}(y)$ is compact, we can assume (by going to a subsequence) that $(v^{(i)})_k^\ell \to \tilde{v}_k^\ell$, with $\tilde{v}_k^\ell \in K_{n+1}(y)$. Since

$$d_H(K_{n+1,m(n+1)+1}(x_i), K_{n+1,m(n+1)+1}(y)) \to 0$$

it follows that

$$|d_Y((v^{(i)})_k^0, (v^{(i)})_k^1) - d_X(v_k^0, v_k^1)| \to 0,$$

so since $d_X(v_k^0, v_k^1) = d_X(x_\infty, x_i) \to 0$, it follows that $d_Y((v^{(i)})_k^0, (v^{(i)})_k^1) \to d_Y(\tilde{v}_k^0, \tilde{v}_k^1) = 0$, so $\tilde{v}_k^0 = \tilde{v}_k^1$. By a similar argument, $d_Y(\tilde{v}_k^\ell, \tilde{v}_k^{\ell+1}) = d_X(v_k^\ell, v_k^{\ell+1})$,

$\rho_Y(\tilde{v}_k^\ell) = \rho_X(v_k^\ell)$. This shows that the sequence $(\tilde{v}_k^0 =)\ \tilde{v}_k^1, \tilde{v}_k^2, \ldots, \tilde{v}_k^{n+1}$ witnesses that $\tilde{v}_k = \tilde{v}_k^{n+1} \in K_n(y)$. But also the "$((d(\ ))\hat{}(\rho(\ ))$" of the sequence $y, \tilde{v}_1, \ldots, \tilde{v}_m$ (being the limit of the "$(d(\ ))\hat{}(\rho(\ ))$" of the sequence $y, (v^{(i)})_1^{n+1}, \ldots, (v^{(i)})_m^{n+1})$ is equal to the "$(d(\ ))\hat{}(\rho(\ ))$" of the sequence x_∞, $v_1, \ldots, v_m$, i.e., is equal to u, thus $u \in K_{n,m}(y)$. ⊣

Below we will denote by I the Polish space

$$I = \prod_{n,m} K(\mathbb{R}^{m^2+m})$$

and we will fix a countable open basis $\mathcal{U} = \{U_n\}$ for I.

Lemma 7.5. *Let X be as in 7.2. Fix $x \in X, 0 < \epsilon < \rho(x), \delta > 0$. Then there is $U \in \mathcal{U}$ containing $L(x)$ such that for any $x_1 \in B_\epsilon(x)$ with $L(x_1) \in U$, there is $x_2 \in B_\delta(x)$ with $L(x_1) = L(x_2)$ (i.e., $(X, x_1) \cong_i (X, x_2)$).*

Proof. If this fails, then fix a decreasing sequence $U_n \in \mathcal{U}$ with $\{L(x)\} = \bigcap_n U_n$ and find $x_n \in B_\epsilon(x)$ with $L(x_n) \in U_n$, so that

$$\forall u \in B_\delta(x)(L(u) \neq L(x_n)). \qquad (*)$$

Since $\epsilon < \rho(x)$ we can assume that $x_n \to x_\infty$. Also clearly $L(x_n) \to L(x)$, so by Lemma 7.4, $L(x_\infty) = L(x)$, so $(X, x) \cong_i (X, x_\infty)$, say via the isometry φ. Then for large enough n, $d(x_n, x_\infty) < \delta$, so $d(\varphi(x_n), \varphi(x_\infty)) < \delta$, so $\varphi(x_n) \in B_\delta(x)$ and $L(\varphi(x_n)) = L(x_n)$, contradicting (*). ⊣

For fixed $U \in \mathcal{U}, \epsilon \in \mathbb{Q}^+$, let $X_{U,\epsilon} = \{x \in X : L(x) \in U, \rho(x) > 2\epsilon\}$. Define then the following equivalence relation on $X_{U,\epsilon}$:

$$x R_{U,\epsilon} y \Leftrightarrow \exists x_0 = x, x_1, \ldots, x_n = y \text{ with } x_i \in X_{U,\epsilon} \text{ and } d(x_i, x_{i+1}) < \epsilon.$$

Let $\mathcal{O}_X(x, U, \epsilon)$ be the $R_{U,\epsilon}$-equivalence class of $x \in X_{U,\epsilon}$. Clearly each $\mathcal{O}_X$ is open in $X_{U,\epsilon}$, so there are only countably many $\mathcal{O}_X(x, U, \epsilon)$, as x varies over X, U over $\mathcal{U}$, and ϵ over $\mathbb{Q}^+$.

Put

$$\begin{aligned} \mathcal{T}_X(x, U, \epsilon) \quad \Leftrightarrow \quad & U \in \mathcal{U}, \epsilon \in \mathbb{Q}^+, x \in X_{U,\epsilon}, \forall x_1 \in B_{2\epsilon}(x)\{L(x_1) \in U \\ & \Rightarrow \exists x_2 \in B_\epsilon(x)[L(x_1) = L(x_2)]\}. \end{aligned}$$

Thus, by Lemma 7.5,

$$\forall x \in X \exists U, \epsilon\ \mathcal{T}_X(x, U, \epsilon).$$

Now notice that if $\mathcal{T}_X(x, U, \epsilon)$ and $u \in \mathcal{O}_X(x, U, \epsilon)$, then there is $v \in B_\epsilon(x)$ such that $L(u) = L(v)$. Indeed, using the definition of $\mathcal{T}_X$, we see that if $x_0 = x, x_1, \ldots, x_n = u$ are in $X_{U,\epsilon}$ and $d(x_1, x_{i+1}) < \epsilon$, then, by induction on k, we have that there is $\hat{x}_k$ such that $\hat{x}_k \in B_\epsilon(x)$ and $L(\hat{x}_k) = L(x_k)$. Put $v = \hat{x}_n$.

We use this to show

Lemma 7.6. *Let X, Y be pseudo-connected locally compact Polish metric spaces, $U, V \in \mathcal{U}, \epsilon, \delta \in \mathbb{Q}^+$ and $x \in X, y \in Y$ be such that they satisfy $\mathcal{T}_X(x, U, \epsilon), \mathcal{T}_Y(y, V, \delta)$. Then if $\overline{L(\mathcal{O}_X(x, U, \epsilon))} = \overline{L(\mathcal{O}_Y(y, V, \delta))}$, we have $X \cong_i Y$. (Here $L(\mathcal{O}_X(x, U, \epsilon)) = \{L(X, u) : u \in \mathcal{O}_X(x, U, \epsilon)\}$.).*

Proof. We have $L(y) \in \overline{L(\mathcal{O}_X(x, U, \epsilon))}$, so find $\hat{x}_n \in \mathcal{O}_X(x, U, \epsilon)$ with $L(\hat{x}_n) \to L(y)$. Then, by the preceding remarks, there is $x_n \in B_\epsilon(x)$ so that $L(x_n) = L(\hat{x}_n)$. Since $B_\epsilon(x)$ has compact closure, we can assume that $x_n \to x_\infty$, so by Lemma 7.4, $L(x_\infty) = L(y)$, thus $(X, x_\infty) \cong_i (Y, y)$ and, in particular, $X \cong_i Y$. $\dashv$

Now put

$$\mathcal{T}((F, x), U, \epsilon) \Leftrightarrow F \in \tilde{F}^{plc}(\mathbb{U}), x \in \text{Core}(F), \mathcal{T}_{\text{Core}(F)}(x, U, \epsilon).$$

Then $\mathcal{T}$ is co-analytic (in $\tilde{F}^{plc}(\mathbb{U}) \times \mathbb{U} \times \mathcal{U} \times \mathbb{Q}^+$), since isometry orbits in $\text{Core}(F)$ are closed in $\text{Core}(F)$, thus σ-compact in $\mathbb{U}$. So, by Number Uniformization, we can find Borel functions $U(F, x), \epsilon(F, x)$ such that for $F \in \tilde{F}^{plc}(\mathbb{U}), x \in \text{Core}(F)$,

$$\mathcal{T}((F, x), U(F, x), \epsilon(F, x)).$$

Finally, fix a Borel function $G : \tilde{F}^{plc}(\mathbb{U}) \to \mathbb{U}$ with $G(F) \in \text{Core}(F)$, let $\Omega = F(I) =$ the space of closed subsets of I, and define $J : \tilde{F}^{plc}(\mathbb{U}) \to \Omega$ by

$$J(F) = \overline{L(\mathcal{O}_{\text{Core}(F)}(G(F), U(F, G(F)), \epsilon(F, G(F))).}$$

First, it is clear that

$$J[\{H : H \cong_i F\}]$$

is countable, and, from Lemma 7.6, we see that if $J[\{H : H \cong_i F_1\}] \cap J[\{H : H \cong_i F_2\}] \neq \emptyset$, then $F_1 \cong_i F_2$. Since it is easy to calculate, using 7.3 and the remarks preceding 7.6, that J is Borel, our proof is complete. $\dashv$

Remark. A similar argument also shows that isometry on locally compact Polish metric spaces with only finitely many pseudo-components is Borel bireducible with E_∞.

The preceding result combined with 6.9 has an implication concerning the classification of the isometry equivalence relation on general locally compact Polish metric spaces. We will omit the proof which heavily uses the theory of turbulence as developed in Hjorth [2000], in combination with 6.9.

Theorem 7.7. (Hjorth) *The equivalence relation of isometry on locally compact Polish metric spaces is reducible by a provably $\mathbf{\Delta}^1_2$ function to graph isomorphism (and thus it is bireducible by such functions to graph isomorphism).*

Finally, we conclude with the observation that isometry on the class of almost locally compact Polish metric spaces is Borel bireducible with isometry on the class of locally compact Polish metric spaces. This follows easily using the argument establishing "$\mathcal{G}_4 \subseteq \mathcal{G}_2$" in the proof of 6.3.

8 Some analogies with the model theory of countable structures

In Lemma 2.3, we characterized every closed subgroup of $\mathrm{Iso}(X)$, where X is a Polish metric space, by an infinite sequence of closed subsets of $X^n (n \geq 2)$. This is analogous to the result for closed subgroups of S_∞, see, e.g., Becker-Kechris [1996], 1.5. In fact, if we endow the space $\mathbb{N}$ with the trivial metric (i.e., $d(x, y) = 1$, if $x \neq y$), then S_∞ is exactly $\mathrm{Iso}(\mathbb{N})$, and Lemma 2.3 is indeed a generalization of the results about S_∞.

Similarly, our Lemma 2.4 can be viewed as an analogous result to Becker-Kechris [1996], Theorem 2.7.3, with the Urysohn space $\mathbb{U}$ replacing $\mathbb{N}$.

More explicitly, one can view a structure of the form $\langle X, R_n\rangle_{n\in\mathbb{N}}$, where (R_n) is a sequence of closed relations on X of various arities, as an analog of a structure of the form $\langle \mathbb{N}, S_n\rangle_{n\in\mathbb{N}}$, where (S_n) is a sequence of relations on $\mathbb{N}$ of various arities, and where the notion of an isomorphism between structures of the form $\langle X, R_n\rangle_{n\in\mathbb{N}}$, which are actually *isometries*, replaces the notion of isomorphism between structures of the form $\langle \mathbb{N}, S_n\rangle_{n\in\mathbb{N}}$.

These analogies suggest that actions by groups of isometries might in many ways resemble the logic actions. Nevertheless, there are also noticeable differences between actions of $\mathrm{Iso}(\mathbb{U})$ and those of S_∞. For example, our

Corollary 2.8 established that the action of Iso($\mathbb{U}$) on $F(\mathbb{U})$ induces a universal equivalence relation for all other orbit equivalence relations (in particular, for those induced by Iso($\mathbb{U}$) actions). However, the action of S_∞ on $F(\mathbb{N})$ can be easily seen to induce a concretely classifiable equivalence relation, therefore it is not universal for orbit equivalence relations induced by S_∞ actions.

In the rest of this section we prove a generalization of our Lemma 2.4 and Theorem 2.7.3 of Becker-Kechris [1996]. As a corollary we give a universal action of $U(H)$, the unitary group of an infinite dimensional complex Hilbert space H.

Let G be a Polish group acting continuously on a Polish space X. The action of G on X induces a Borel action of G on the product space $\prod_n F(X^n)$. We are going to give a sufficient condition for the product space to be a universal Borel G-space, i.e., to have the property that for any Borel G-space Y there is a Borel G-embedding f of Y into $\prod_n F(X^n)$, that is, f is a Borel embedding such that, for any $g \in G$ and $y \in Y$,

$$f(g \cdot y) = g \cdot f(y).$$

Note that the orbit equivalence relation for a universal Borel G-space is necessarily universal among all orbit equivalence relations induced by Borel G-actions.

To give the sufficient condition we need some notation first. For $n \in \mathbb{N}, \vec{x} = (x_1, x_2, \ldots, x_n) \in X^n$ and $\vec{O} = (O_1, O_2, \ldots, O_n)$ an n-tuple of open sets in X, we let

$$V_G(\vec{x}, \vec{O}) = \{g \in G : g \cdot x_i \in O_i, \forall 1 \leq i \leq n\}.$$

$V_G(\vec{x}, \vec{O})$ is an open subset of G. Let $D \subseteq X$ be a countable dense subset of X and $\mathcal{A}$ a countable basis for the topology of X.

Let

$$\mathcal{B}_G(D, \mathcal{A}) = \{V_G(\vec{x}, \vec{O}) : \vec{x} \in D^n, \vec{O} \in \mathcal{A}^n, n \in \mathbb{N}\}.$$

Then the topology of G generated by $\mathcal{B}_G(D, \mathcal{A})$ is smaller (coarser) than the original topology on G.

Theorem 8.1. *Let G be a Polish group acting continuously on a Polish space X. If for some countable dense subset D of X and countable open basis $\mathcal{A}$ of X the set $\mathcal{B}_G(D, \mathcal{A})$ is an open basis for G, then $\prod_n F(X^n)$ is a universal Borel G-space.*

Proof. For any $\vec{x} \in X^n$, let $|\vec{x}| = n$. Let

$$Y = \prod_{\vec{x} \in D^{<\mathbb{N}}} F(X^{|\vec{x}|}).$$

Since $D^{<\mathbb{N}}$ is countable, Y is Borel isomorphic to $\prod_{n\in\mathbb{N}}(F(X^n)^{\mathbb{N}})$ as a Borel G-space, and $Y^{\mathbb{N}}$ is Borel isomorphic to Y as a Borel G-space. It is easy to see that Y is Borel isomorphic to a Borel subspace of $\prod_{n\in\mathbb{N}} F(X^n)$ as a Borel G-space. Hence by the proof of Becker-Kechris [1996], Theorem 2.6.1, it suffices to show that there is a Borel G-embedding of $F(G)$ into Y.

We define such an embedding φ as follows. Fix $C \in F(G)$. For each $\vec{x} \in D^{<\mathbb{N}}$, let

$$U^C_{\vec{x}} = \bigcup\{O_1 \times \ldots \times O_n : V_G(\vec{x}, \vec{O}) \cap C = \emptyset\}.$$

Note that $U^C_{\vec{x}}$ is an open subset of $X^{|\vec{x}|}$. Then let

$$\varphi(C) = \left(X^{|\vec{x}|} \setminus U^C_{\vec{x}}\right)_{\vec{x}\in D^{<\mathbb{N}}} \in Y.$$

We claim that φ is as required.

To see that φ is an embedding, let $C_1, C_2 \in F(G)$ and assume $C_1 \neq C_2$. Since $\mathcal{B}_G(D, \mathcal{A})$ is an open basis for G, we may assume that there are $\vec{x} \in D^{<\mathbb{N}}$ and $\vec{O} \in \mathcal{A}^{|\vec{x}|}$ such that

$$V_G(\vec{x}, \vec{O}) \cap C_1 = \emptyset \text{ but } V_G(\vec{x}, \vec{O}) \cap C_2 \neq \emptyset.$$

Then, by the definition in the preceding paragraph, we have that $O_1 \times \ldots \times O_{|\vec{x}|} \subseteq U^{C_1}_{\vec{x}}$. To see that $\varphi(C_1) \neq \varphi(C_2)$, it suffices to verify that $O_1 \times \ldots \times O_{|\vec{x}|} \not\subseteq U^{C_2}_{\vec{x}}$. Assume not. Let $g \in V_G(\vec{x}, \vec{O})$. Then there is $\vec{W}$ with $V_G(\vec{x}, \vec{W}) \cap C_2 = \emptyset$ and $g \cdot x_i \in W_i, \forall i \leq |\vec{x}|$. Thus $g \in V_G(\vec{x}, \vec{W})$ and hence $g \notin C_2$. The argument shows that $V_G(\vec{x}, \vec{O}) \cap C_2 = \emptyset$, contradicting our assumption.

It is straightforward to verify that φ is a G-map. The only point to check here is that for any $g \in G$, $\vec{x}$, and $\vec{O}$,

$$g \cdot V_G(\vec{x}, \vec{O}) = V_G(\vec{x}, g \cdot \vec{O}).$$

It remains to show that φ is Borel. For this it suffices to show that for any $\vec{x} \in D^{<\mathbb{N}}$ and V open in $X^{|\vec{x}|}$, the set $\{C \in F(G) : \varphi(C)_{\vec{x}} \cap V \neq \emptyset\}$ is Borel in $F(G)$. Furthermore, it is enough to check this for $V = O_1 \times \ldots \times O_{|\vec{x}|}$, a basic open set, where $\vec{O} \in \mathcal{A}^{|\vec{x}|}$. The following claim establishes this.

Claim. $\varphi(C)_{\vec{x}} \cap (O_1 \times \ldots \times O_{|\vec{x}|}) \neq \emptyset \Leftrightarrow C \cap V_G(\vec{x}, \vec{O}) \neq \emptyset$.

($\Rightarrow$) Suppose $\varphi(C)_{\vec{x}} \cap (O_1 \times \ldots \times O_{|\vec{x}|}) \neq \emptyset$. We then have that $O_1 \times \ldots \times O_{|\vec{x}|} \not\subseteq U^C_{\vec{x}}$. Therefore, $V_G(\vec{x}, \vec{O}) \cap C \neq \emptyset$, by the definition of $U^C_{\vec{x}}$.

($\Leftarrow$) Suppose $V_G(\vec{x}, \vec{O}) \cap C \neq \emptyset$. Then $(O_1 \times \ldots \times O_{|\vec{x}|}) \not\subseteq U^C_{\vec{x}}$ by the same argument as for the injectivity of φ. Therefore, $\varphi(C)_{\vec{x}} \cap (O_1 \times \ldots \times O_{|\vec{x}|}) \neq \emptyset$.

This completes the proof of the claim, hence the theorem. $\dashv$

Corollary 8.2. *Let X be a Polish metric space and G a closed subgroup of* $\mathrm{Iso}(X)$. *Then the Borel G-space $\prod_n F(X^n)$, with the evaluation action, is a universal Borel G-space.*

Proof. It is enough to verify the condition of Theorem 8.1 for G. First of all, it is obvious that the evaluation action of G on X is continuous. Now let D be an arbitrary countable dense subset of X and $\mathcal{A}$ an arbitrary open basis for X. Let δ be the metric on X. Assume $\delta \leq 1$. Enumerate D as $d_1, d_2, \ldots$. Since the topology on G is the pointwise convergence topology, a compatible metric for G is

$$d_G(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \delta(f(d_i), g(d_i)),$$

for any $f, g \in G$.

To see that $\mathcal{B}_G(D, \mathcal{A})$ is an open basis for G, it suffices to show that for any $f \in G$ and $\epsilon > 0$, there are $\vec{x} \in D^{<\mathbb{N}}$ and $\vec{O} \in \mathcal{A}^{|\vec{x}|}$ such that

$$f \in V_G(\vec{x}, \vec{O}) \subseteq \{g \in G : d_G(f, g) < \epsilon\}.$$

For this let $N \in \mathbb{N}$ be such that $\sum_{i>N} \frac{1}{2^i} < \frac{\epsilon}{2}$. Let $\vec{x} = (d_1, \ldots, d_N)$ and $\vec{O} = (O_1, \ldots, O_N) \in \mathcal{A}^N$ be such that $f(d_i) \in O_i, \forall i \leq N$, and $\mathrm{diam}(O_i) < \frac{\epsilon}{2}$. It is then easy to check that this choice of $\vec{x}$ and $\vec{O}$ works. $\dashv$

This corollary is obviously a generalization of our Lemma 2.4. It is also a generalization of the Becker-Kechris [1996] theorem about universal S_∞-actions since we can again take $X = \mathbb{N}$.

Another example in which Theorem 8.1 applies is to the action on the interval $[0, 1] \subseteq \mathbb{R}$ by its group of homeomorphisms denoted by $H([0, 1])$. This is an example where the action is not by isometries on the underlying space. Furthermore, notice that for the action of $H([0, 1]^2)$ on $[0, 1]^2$ the condition of Theorem 8.1 fails. Thus we do not know if the space $\prod_n F([0,1]^{2n})$ is a universal Borel $H([0,1]^2)$-space.

Finally notice that Corollary 8.2 applies to the unitary group and the orthogonal group.

Corollary 8.3. *Let H be a separable infinite-dimensional complex Hilbert space and $U(H)$ be the group of unitary operators on H. Then the space $\prod_n F(H^n)$, with the evaluation action by $U(H)$, is a universal Borel $U(H)$-space. Similarly, let R be a separable infinite-dimensional real Hilbert space and $O(R)$ be the group of orthogonal operators on R. Then the space $\prod_n F(R^n)$, with the evaluation action by $O(R)$, is a universal Borel $O(R)$-space.*

Proof. Simply note that the strong operator topology on $U(H)$ or $O(R)$ coincides with the pointwise convergence topology. This makes $U(H)$ (resp., $O(R)$) a closed subgroup of $\mathrm{Iso}(H)$ (resp., $\mathrm{Iso}(R)$), as required by Corollary 8.2. ⊣

9 Open Problems

In this final section, we discuss some open problems and directions for further research.

As mentioned in the introduction, there are two open problems about locally compact Polish metric spaces and 0-dimensional Polish metric spaces.

Problem 9.1. Determine the exact complexity (in the hierarchy of Borel reducibility) of the isometry of locally compact Polish metric spaces.

As explained earlier, the conjecture here is that it is Borel bireducible to graph isomorphism.

We also do not know the complexity of isometry in another subclass, namely homogeneous locally compact Polish metric spaces. We do not even know if it is concretely classifiable.

Problem 9.2. Determine the exact complexity of the isometry of 0-dimensional Polish metric spaces.

An analogous problem is the classification of compact metric spaces up to homeomorphism. Hjorth [1999] has shown that this equivalence relation is strictly above graph isomorphism. Also it is known (Kechris-Solecki) that homeomorphism of compact metric spaces is Borel reducible to an equivalence relation induced by a Borel action of a Polish group. In fact, the

Banach-Stone Theorem (see, e.g., Semadeni [1971], 7.8.4) implies that, for compact metric spaces X and Y,

$$\begin{aligned} &X \text{ and } Y \text{ are homeomorphic} \\ \Leftrightarrow\ &C(X) \text{ and } C(Y) \text{ are isometric Banach spaces} \\ \Leftrightarrow\ &C(X) \text{ and } C(Y) \text{ are isometric as Polish spaces,} \end{aligned}$$

where $C(X)$ denotes the space of all continuous functions from X into $\mathbb{R}$ with the supnorm (metric). It is easy to see that the map $X \mapsto C(X)$ is Borel. Thus by 2D, we have an alternative proof of the fact that homeomorphism of compact metric spaces is Borel reducible to an orbit equivalence relation induced by a Borel action of a Polish group. We also see that linear isometry of Banach spacs is above homeomorphism of compact metric spaces. This leads to the following problems.

Problem 9.3. Determine the exact complexity of homeomorphism of compact metric spaces.

Problem 9.4. Determine the exact complexity of isomorphism of separable Banach spaces.

Problem 9.5. Determine the exact complexity of linear isometry of separable Banach spaces.

On this last problem the study of the Gurariǐ space might be relevant.

It might be worth pointing out here that Problem 9.3 bears deeper analogies to the isometry of Polish metric spaces than we have mentioned above. Just as the Urysohn space is an isometrically universal Polish space, the Hilbert cube is a universal Polish space with respect to homeomorphic embeddings (see Kechris [1995]). Compared with the fact that Iso($\mathbb{U}$) is a universal Polish group, the group of homeomorphisms of the Hilbert cube is a universal Polish group as well. Also, by results of Megrelishvili [1996] and Becker-Kechris [1996, 2.6.6], every orbit equivalence relation induced by a Borel action of a Polish group is Borel reducible to the orbit equivalence relation of the evaluation action of some closed subgroup of the homeomorphism group of the Hilbert cube on the Hilbert cube. Analogously, the techniques in our section 2E yield the following result for actions by isometries.

Proposition 9.6. *Let X be a Polish metric space and G a closed subgroup of* Iso(X). *There is a closed subgroup H of* Iso($\mathbb{U}$) *such that $E_G^X \leq_B E_H^{\mathbb{U}}$, where the actions are evaluations.*

Proof. In the notation of 2E, just take $H = G^*$. $\dashv$

It is then natural to ask the following question, which seems to be open.

Problem 9.7. Is there a Polish metric space X and a closed subgroup G of $\mathrm{Iso}(X)$ such that E_G^X is a universal equivalence relation induced by a Borel action of a Polish group?

One can also ask the (seemingly) weaker question of whether it is possible to have a Borel action of a Polish group G by isometries on a Polish metric space X, so that E_G^X is a universal equivalence relation induced by a Borel action of a Polish group.

On classifying isometries, we have the following question about the group $\mathrm{Iso}(\mathbb{U})$.

Problem 9.8. Determine the exact complexity of the equivalence relation induced by the conjugacy action of $\mathrm{Iso}(\mathbb{U})$ on itself. Is it universal for equivalence relations induced by a Borel action of a Polish group?

In connection with this problem, we will compute the complexity of conjugacy in $\mathrm{Iso}(\mathcal{N})$, where $\mathcal{N}$ is the Baire space $\mathbb{N}^{\mathbb{N}}$ endowed with the usual ultrametric

$$d(x,y) = 2^{-n-1} \text{ where } n \text{ is the least number such that } x(n) \neq y(n).$$

Then $\mathcal{N}$ is an ultrametric Polish space. An isometry of $\mathcal{N}$ is usually called a *Lipschitz automorphism* of the Baire space. Here, to be consistent with the other notation employed in this paper, we continue to use $\mathrm{Iso}(\mathcal{N})$ to denote the group of all Lipschitz automorphisms of $\mathbb{N}^{\mathbb{N}}$ and address them as *isometries* of $\mathcal{N}$.

Each isometry of $\mathcal{N}$ is characterized by a coherent sequence of permutations. To be precise, let $\mathbb{N}^n$ denote the set of sequences of natural numbers of length n. Given permutations π, ρ of $\mathbb{N}^n, \mathbb{N}^m$, respectively with $n \leq m$, we write $\pi \leq \rho$ if

$$\rho(s)|n = \pi(s|n),$$

for any $s \in \mathbb{N}^m$. If $\pi_1 \leq \pi_2 \leq \pi_3 \leq \ldots$, where π_n is a permutation of $\mathbb{N}^n$, then $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ given by $f(x) = \bigcup_n \pi_n(x|n)$ is an isometry of $\mathcal{N}$. Conversely, for any isometry f of $\mathcal{N}$, there is a unique sequence $\pi_1 \leq \pi_2 \leq \pi_3 \ldots$, where π_n is a permutation of $\mathbb{N}^n$ such that $f(x) = \bigcup_n \pi_n(x|n)$. In fact, for $s \in \mathbb{N}^n$, let

$$\pi_n(s) = f(s\hat{\ }y)|n,$$

for an arbitrary $y \in \mathbb{N}^{\mathbb{N}}$. Then π_n has the aforementioned property. We should of course make sure that π_n is well defined: let $y_1, y_2 \in \mathbb{N}^{\mathbb{N}}$ be arbitrary. Then $d(s\hat{\ }y_1, s\hat{\ }y_2) \leq 2^{-n}$, from which it follows that $d(f(s\hat{\ }y_1), f(s\hat{\ }y_2)) \leq 2^{-n}$, and hence $f(s\hat{\ }y_1)|n = f(s\hat{\ }y_2)|n$.

Two isometries f, g of $\mathcal{N}$ are *conjugate* to each other, denoted by $f \sim_c g$, if there is $h \in \mathrm{Iso}(\mathcal{N})$ such that

$$f \circ h = h \circ g$$

Being conjugate is an equivalence relation. Moreover, $\sim_c$ is the orbit equivalence relation of the action of $\mathrm{Iso}(\mathcal{N})$ on itself by conjugacy. It is well known (and also follows from the argument in Theorem 4.3 of this paper) that $\mathrm{Iso}(\mathcal{N})$ is isomorphic to a closed subgroup of S_∞. Therefore $\sim_c$ is in particular Borel reducible to graph isomorphism. We will show below that $\sim_c$ is in fact Borel bireducible to graph isomorphism, thus completely characterizing the complexity of this equivalence relation.

Before proceeding to the proof of this result, we recall some relevant background material related to Lipschitz automorphisms. In Dougherty-Jackson-Kechris [1994] Lipschitz automorphisms of $2^{\mathbb{N}}$ were considered. Since $2^{\mathbb{N}}$ is a compact space, the group of Lipschitz automorphisms is a compact Polish group. Thus the conjugacy classification problem is an orbit equivalence relation induced by an action of a compact Polish group and is therefore concretely classifiable. However, in Dougherty-Jackson-Kechris [1994] the authors considered a different classification problem, that is, the classification of Lipschitz automorphisms of $2^{\mathbb{N}}$ up to conjugacy by *Borel* automorphisms. Their results essentially say that this latter equivalence relation is also concretely classifiable. The classification of conjugacy of Borel automorphisms (of some standard Borel space) was also investigated. Clemens (unpublished) showed that it is a $\mathbf{\Sigma}^1_2$-complete equivalence relation and hence is very complicated and beyond the scope of orbit equivalence relations.

Now we are ready to establish our result for Lipschitz automorphisms of $\mathbb{N}^{\mathbb{N}}$.

Theorem 9.9. *The conjugacy equivalence relation on* $\mathrm{Iso}(\mathcal{N})$ *is Borel bireducible with graph isomorphism.*

Proof. As we remarked earlier it is enough to show that graph isomorphism is $\leq_B (\sim_c)$.

In our proof we will use, instead of graphs, the following class of trees on $2\mathbb{N}$, whose isomorphism relation we showed in the proof of Theorem 4.4 to be bireducible to graph isomorphism.

Let $\mathcal{T}$ be the class of all trees on $2\mathbb{N}$ which are nonempty and *splitting*, i.e., for any $T \in \mathcal{T}$ and $s \in T$, there are at least two $t_1, t_2, t_1 \neq t_2$ such that $\text{length}(t_1) = \text{length}(t_2) = \text{length}(s) + 1$ and $s \subseteq t_1, s \subseteq t_2$.

Given $T \in \mathcal{T}$, we will define an isometry f_T of $\mathcal{N}$ by constructing its finite approximations $\pi_T^1 \leq \pi_T^2 \leq \pi_T^3 \leq \ldots$, where each π_T^n is a permutation of $\mathbb{N}^n$.

For this we fix a bijection from $\mathbb{N} \times \{0,1\}$ onto $\mathbb{N}$ and denote it by $(m,i) \mapsto \langle m,i\rangle$. For any $s \in \mathbb{N}^n$, let $p(s) \in \mathbb{N}^n$ be the unique tuple $(m_1, \ldots, m_n)$ such that for some $(i_1, \ldots, i_n) \in 2^n$,

$$s = (\langle m_1, i_1\rangle, \ldots, \langle m_n, i_n\rangle).$$

Now we are ready to define π_T^n. We do this by induction on n. For notational simplicity we start from $n = 0$ and let $\pi_T^0(\emptyset) = \emptyset$. By induction, suppose π_T^n is defined. For $s \in \mathbb{N}^n, k \in \mathbb{N}$, let

$$\pi_T^{n+1}(s\hat{\ }k) = \begin{cases} \pi_T^n(s)\hat{\ }\langle m, 1-i\rangle, \text{ if } k = \langle m,i\rangle \text{ for } p(s)\hat{\ }m \in T, \\ \pi_T^n(s)\hat{\ }k, \text{ otherwise.} \end{cases}$$

It is clear that $\pi_T^n \leq \pi_T^{n+1}$ for all $n \geq 1$. Thus we can let $f_T(x) = \bigcup_n \pi_T^n(x|n)$. Then f_T is an isometry of $\mathcal{N}$ and the assignment $T \mapsto f_T$ is a Borel function.

It is clear from the construction that $p(s) = p(\pi_T^n(s))$, for any $s \in \mathbb{N}^n$. It remains to check that if $T_1, T_2 \in \mathcal{T}$, then

$$T_1 \cong T_2 \Leftrightarrow f_{T_1} \sim_c f_{T_2}.$$

First suppose that φ is an isomorphism from T_1 onto T_2. We construct an isometry g of $\mathcal{N}$ by its finite approximations $\rho_1 \leq \rho_2 \leq \rho_3 \leq \ldots$, where each ρ_n is a permutation of $\mathbb{N}^n$, such that

$$\pi_{T_2}^n = \rho_n \circ \pi_{T_1}^n \cdot \rho_n^{-1},$$

for all $n \geq 1$. This guarantees that $f_{T_2} = g \circ f_{T_1} \circ g^{-1}$.

Let $\rho_0(\emptyset) = \emptyset$. By induction, suppose ρ_n is defined with the inductive assumption that, for any $s \in \mathbb{N}^n$, if $p(s) \in T_1$, then $\varphi(p(s)) = p(\rho_n(s)) \in T_2$, and if $p(\rho_n(s)) \in T_2$, then $p(s) \in T_1$, and $\varphi(p(s)) = p(\rho_n(s))$.

Let $s \in \mathbb{N}^n$ and $k \in \mathbb{N}$. If $p(s) \notin T_1$, then define $\rho_{n+1}(s\hat{\ }k) = \rho_n(s)\hat{\ }k$. Otherwise, $p(s) \in T_1$. If $p(s\hat{\ }k) \in T_1$, then for some $m \in 2\mathbb{N}, p(s\hat{\ }k) =$

$p(s)^\frown\langle m,i\rangle, i\in\{0,1\}$, and since $\varphi(p(s^\frown k))\in T_2$, we have some $m'\in 2\mathbb{N}$ such that

$$\varphi(p(s^\frown k))=\varphi(p(s))^\frown m'.$$

Let $\rho_{n+1}(s^\frown k)=\rho_n(s)^\frown\langle m',i\rangle$. We verify the inductive assumption:

$$\begin{aligned}\varphi(p(s^\frown k))&=\varphi(p(s))^\frown m'=p(\rho_n(s))^\frown m'=p(\rho_n(s)^\frown\langle m',i\rangle)\\&=p(\rho_{n+1}(s^\frown k)).\end{aligned}$$

If $p(s^\frown k)\notin T_1$, note that both sets

$$\{k\in\mathbb{N}:p(s^\frown k)\notin T_1\}\text{ and }\{k\in\mathbb{N}:\varphi(p(s))^\frown k\notin T_2\}$$

are infinite. There is then a bijection γ between the two sets. We define $\rho_{n+1}(s^\frown k)=\rho_n(s)^\frown\gamma(k)$.

It is clear from the definitions that $\rho_n\le\rho_{n+1}$. We need to see that $\pi^{n+1}_{T_2}=\rho_{n+1}\circ\pi^{n+1}_{T_1}\circ\rho^{-1}_{n+1}$.

Let $s\in\mathbb{N}^n$ and $k\in\mathbb{N}$. If $p(s)\notin T_1$, then

$$\pi^{n+1}_{T_2}\circ p_{n+1}(s^\frown k)=\pi^{n+1}_{T_2}(\rho_n(s)^\frown k)=\pi^n_{T_2}\circ\rho_n(s)^\frown k.$$

Since $p\circ\pi^n_{T_2}\circ\rho_n(s)=p\circ\rho_n\circ\pi^n_{T_1}(s)\in T_2$ implies $p(\pi^n_{T_1}(s))=p(s)\in T_1$, and $\rho_{n+1}\cdot\pi^{n+1}_{T_1}(s^\frown k)=\rho_{n+1}(\pi^n_{T_1}(s)^\frown k)=\rho_n\circ\pi^n_{T_1}(s)^\frown k$, we have that $\pi^{n+1}_{T_2}\circ\rho_{n+1}(s^\frown k)=\rho_{n+1}\circ\pi^{n+1}_{T_1}(s^\frown k)$.

If $p(s)\in T_1$ and $p(s^\frown k)\in T_1$, then

$$\pi^{n+1}_{T_2}\cdot\rho_{n+1}(s^\frown k)=\pi^{n+1}_{T_2}(\rho_n(s)^\frown\langle m',i\rangle)$$

(for $\varphi(p(s^\frown k)=\varphi(p(s))^\frown m'$ and $k=\langle m,i\rangle$)

$$\begin{aligned}&=\pi^n_{T_2}(\rho_n(s))^\frown\langle m',1-i\rangle\\&=\rho_n\circ\pi^n_{T_1}(s)^\frown\langle m',1-i\rangle\\&=\rho_{n+1}(\pi^n_{T_1}(s)^\frown k')\end{aligned}$$

(where $k'=\langle m,1-i\rangle$))

$$=\rho_{n+1}\circ\pi^{n+1}_T(s^\frown k).$$

Finally, if $p(s)\in T_1$, but $p(s^\frown k)\notin T_1$, then

$$\begin{aligned}\pi^{n+1}_{T_2}\circ\rho_{n+1}(s^\frown k)&=\pi^{n+1}_{T_2}(\rho_n(s)^\frown\gamma(k))\\&=\pi^n_{T_2}(\rho_n(s))^\frown\gamma(k)\\&=\rho_n\circ\pi^n_{T_1}(s)^\frown\gamma(k)\\&=\rho_{n+1}(\pi^n_{T_1}(s)^\frown k)\\&=\rho_{n+1}\circ\pi^n_{T_1}(s^\frown k).\end{aligned}$$

Now we assume that g is an isometry of $\mathcal{N}$ such that $f_{T_2} = g \circ f_{T_1} \circ g^{-1}$. Let $\rho_1 \leq \rho_2 \leq \rho_3 \leq \ldots$ be the finite approximations of g, where each ρ_n is a permutation of $\mathbb{N}^n$. Then it follows that $\pi^n_{T_2} = \rho_n \circ \pi^n_{T_1} \circ \rho_n^{-1}$. We construct an isomorphism φ from T_1 onto T_2 by induction on the length of $s \in T_1$.

Let $\varphi(\emptyset) = \emptyset$. Suppose φ is already defined from $T_1 \cap (2\mathbb{N})^{\leq n}$ onto $T_2 \cap (2\mathbb{N})^{\leq n}$ in such a way that for any $s \in T_1 \cap (2\mathbb{N})^n$ and $t \in \mathbb{N}^n$ with $p(t) = s, \varphi(x) = p(\rho_n(t))$.

Let $s\hat{\ }k \in T_1 \cap (2\mathbb{N})^{n+1}$ and $t \in \mathbb{N}^n$ with $p(t) = s$. Then $\pi^{n+1}_{T_1}(t\hat{\ }\langle k, 0\rangle) = \pi^n_{T_1}(t)\hat{\ }\langle k, 1\rangle$ and $\pi^{n+1}_{T_1}(t\hat{\ }\langle k, 1\rangle) = \pi^n_{T_1}(t)\hat{\ }\langle k, 0\rangle$. Since $\pi^{n+1}_{T_2} = \rho_{n+1} \circ \pi^{n+1}_{T_1} \circ \rho^{-1}_{n+1}$, we have that, for some m and m', if we let $\rho_{n+1}(t\hat{\ }\langle k, 0\rangle) = \rho_n(t)\hat{\ }m$ and $\rho_{n+1}(t\hat{\ }\langle k, 1\rangle) = \rho_n(t)\hat{\ }m'$, then $\pi^{n+1}_{T_2}(\rho_n(t)\hat{\ }m) = \rho_n(t)\hat{\ }m'$ and $\pi^{n+1}_{T_2}(\rho_n(t)\hat{\ }m') = \rho_n(t)\hat{\ }m$ and $m \neq m'$. It follows from the definition of $\pi^{n+1}_{T_2}$ that $p(\rho_n(t)\hat{\ }m) \in T_2$ and that there is k' such that $m = \langle k', 0\rangle$ and $m' = \langle k', 1\rangle$ and

$$p(\rho_n(t)\hat{\ }m) = p(\rho_n(t))\hat{\ }k' = \varphi(s)\hat{\ }k' \in T_2.$$

We then just let $\varphi(s\hat{\ }k) = \varphi(s)\hat{\ }k' \in T_2$. It is then clear that φ is an isomorphism from T_1 onto T_2. ⊣

Finally, there are some interesting questions about isometry groups.

Problem 9.10. Characterize the isometry groups of connected (or Heine-Borel or pseudo-connected) locally compact Polish metric spaces. Similarly characterize the isometry groups of ultrametric Polish spaces.

The last question is a special case of an old problem of Krasner (see Lemin-Smirnov [1984]) from the 1950's.

References

H. Becker and A.S. Kechris [1996] *The Descriptive Set Theory of Polish Group Actions*, London Math. Soc. Lecture Note Series, **232**, Cambridge Univ. Press, 1996.

R. Dougherty, S. Jackson, and A.S. Kechris [1994] The structure of hyperfinite Borel equivalence relations, *Trans. Amer. Math. Soc.*, **341 (1)** (1994), 193-225.

J. Dugundji [1966] *Topology*, Allyn and Bacon, 1966.

E.G. Effros [1965] Transformation groups and C^*-algebras, *Ann. of Math.*, **81** (1965), 38-55.

H. Friedman and L. Stanley [1989] A Borel reducibility theory for classes of countable structures, *J. Symb. Logic*, **54** (1989), 894-914.

M. Gromov [1999] *Metric Structures for Riemannian and Non-Riemannian Spaces*, Progress in Math., **152**, Birkhäuser, 1999.

P.R. Halmos [1950] *Measure Theory*, Van Nostrand, 1950.

G. Hjorth [2000] *Classification and Orbit Equivalence Relations*, Math. 2000 Surveys and Monographs, **75**, Amer. Math. Soc., 2000.

G. Hjorth [∞], Actions of the classical Banach space, *J. Symb. Logic*, to appear.

G. Hjorth and A.S. Kechris [1996] Borel equivalence relations and classifications of countable models, *Ann. Pure and Appl. Logic*, **82** (1996), 221-272.

W. Hodges [1993] *Model Theory*, Cambridge University Press, 1993.

M. Katětov [1988] On universal metric spaces, *Gen. Topology and its Relations to Modern Analysis and Algebra VI*, Proc. Sixth Prague Topol. Symp. 1986, Z. Frolík (Ed.), Heldermann Verlag, 1988, 323-330.

A.S. Kechris [1995] *Classical Descriptive Set Theory*, Springer-Verlag, 1995.

A.S. Kechris [1999] New directions in descriptive set theory, *Bull. Symb. Logic*, **5(2)** (1999), 161-179.

A.S. Kechris [2000] Actions of Polish groups and Classification Problems, *Analysis and Logic*, London Math. Soc. Lecture Notes Series, Cambridge Univ. Press, 2000 (to appear).

A. Yu. Lemin and Yu. M. Smirnov, Isometry groups of metric and ultrametric spaces and their subgroups, *Russ. Math. Surveys*, **41 (6)** (1986), 213-214.

L.H. Loomis [1949] Haar measure in uniform structures, *Duke Math. J.*, **16** (1949), 193-208.

M.G. Megrelishvili [1996] Free topological G-groups, *New Zealand J. Math.*, **25** (1996), 59-72.

P. Strantzalos [1974] Dynamische Systeme und Topologische Aktionen, *Manuscripta Math.*, **13** (1974), 207-211.

P. Strantzalos [1989] Action by isometries, *Transformation groups (Osaka 1987)*, Lecture Notes in Math., **1375**, Springer, 1989, 319-325.

P. Urysohn [1927] Sur en espace métrique universel, *Bull. Sci. Math.*, **51** (1927), 43-64, 74-90.

V.V. Uspenskiĭ [1986] A universal topological group with a countable basis, *Funct. Anal. and Its Appl.*, **20** (1986), 86-87.

V.V. Uspenskiĭ [1990] On the group of isometries of the Urysohn universal metric space, *Comment. Math. Univ. Carolinae*, **31(1)** (1990), 181-182.

D. van Dantzig and B.L. van der Waerden [1928], Über metrisch homogene Räume, *Abh. Math. Seminar Hamburg*, **6** (1928), 374-376.

A.M. Vershik [1998] The universal Urysohn space, Gromov metric triples and random metrics on the natural numbers, *Russ. Math. Surveys*, **53(5)** (1998), 921-928.

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