UNIVERSITY OF CALIFORNIA

Los Angeles

The Isomorphism Relation Between Countable Models
and
Definable Equivalence Relations

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy
in Mathematics

by

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1998

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University of California, Los Angeles

1998

DEDICATION

To my wife Shuang and our parents, for believing in me.

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ACKNOWLEDGMENTS

First and foremost, I thank my thesis advisor Greg Hjorth for the generous help and guidance he gave me over the years. Thanks are also due to Alekos Kechris, Tony Martin, John Steel and Yiannis Moschovakis for educating me the vast field of mathematical logic. I am grateful to Harvey Friedman for asking the motivating questions and for encouraging me to work on them. Discussions with Howard Becker, Slawek Solecki, Alain Louveau and Boban Veličković are also of great benefit to me. Last but not the least, I thank the Department of Mathematics of UCLA for the continuing financial support over the years, and the Alfred Sloan Foundation for the Dissertation Fellowship that makes it easier for me to write this dissertation.

ABSTRACT OF THE DISSERTATION

The Isomorphism Relation Between Countable Models
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University of California, Los Angeles, 1998

Professor Gregory Hjorth, Chair

In this dissertation we investigate the isomorphism relation between countable models and some other definable equivalence relations. These equivalence relations are either Borel or understood in the context of orbit equivalence relations of Polish group actions. In Chapter 1 we introduce the background and terminology. In Chapter 2, we study the Polish groups which admit complete left-invariant metrics. A characterization of this class of groups is given in model theoretic terms for the category of closed subgroups of S_∞, or the automorphism groups of countable models. In Chapter 3 we prove some results about the homeomorphism groups of compact Polish spaces and their actions. In Chapters 4-6 various dichotomy theorems are proved for the isomorphism relation on invariant Borel classes of linear orderings, simple trees and models in the language of a single unary function. Corollaries of these theorems to answer some questions of Friedman and Stanley are presented in Chapter 7, together with a survey of other positive and negative results about the notion of FS-reducibility and strong FS-reducibility. Chapter 8 deals with a separate topic. In this chapter we study the well-known O notation and its relation with some other Borel equivalence relations arising from analysis

and set theory.

CHAPTER 1

Introduction

1.1 Basic concepts

The topics presented in this dissertation can be unified under the study of equivalence relations from the point of view of descriptive set theory. This is sometimes referred to as invariant descriptive set theory. The underlying spaces for the equivalence relations we consider are all Polish spaces. A *Polish space* is a separable, completely metrizable topological space. The equivalence relations we consider are sometimes Borel, sometimes analytic but nevertheless induced by a Polish group action. These are all definable, and in fact, so simply definable that questions about them are very likely to be decidable within ZFC.

Most of the equivalence relations we are going to discuss in this dissertation are induced by Polish group actions. A *Polish group* is a topological group whose topology is Polish. If G is a Polish group acting continuously on a Polish space X then X is called a *Polish G-space*. The orbit equivalence relation is denoted by E_G^X. Many equivalence relations arise from the situation when X is itself a Polish group, G is a subgroup of X and the action of G on X is just a shift by multiplication (on the right). In this case we also denote the orbit equivalence relation by X/G.

The central concept in invariant descriptive set theory is the reducibility between equivalence relations. Let E and F be two equivalence relations on Polish spaces X and Y, respectively. E is *reducible to F* (or F *reduces* E) if there is a function $\theta : X \to Y$ such that for any $x, y \in X$,

$$xEy \Leftrightarrow \theta(x)F\theta(y).$$

The function θ is called a *reduction (function)* from E to F. If there is no definability restriction to the function θ, then E is reducible to F if and only if the cardinality of the set of equivalence classes of E is smaller than that of F. With the Axiom of Choice this is basically a trivial notion. But things are quite different once we require the function θ to be somewhat definable. The most popular notion is Borel reducibility. E is *Borel reducible to F*, denoted by $E \leq_B F$, if there is a Borel reduction function from E to F. We write $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$, and in this situation E and F are said to be *Borel equivalent* or *Borel bi-reducible*. The structure of Borel reducibility between definable equivalence relations, even between Borel equivalence relations, is an enormously rich and complicated one.

Because of the above remark that reducibility is closely related to the comparison of cardinalities, the study of reducibility can also be viewed as a theory of definable cardinalities. For this reason, the structure (*Borel equivalence relations*, $\leq_B$), or rather the structure of its equivalence classes under $\sim_B$, are informally called Borel cardinalities, or the hierarchy of Borel degrees. The latter terminology is apparently borrowed from recursion theory as an analog of Turing degrees.

Another very useful and indeed powerful point of view to motivate the study of reducibility is to interpret $E \leq_B F$ as classifying the equivalence classes of E by those of F. The reduction function is regarded as providing a way to obtain a complete invariant for the equivalence classes of E. Many classification problems in mathematics can be thought of as trying to find complete invariants for some well defined equivalence relations. The equivalence relation might be among some structures, but usually the structures can be coded by elements of a Polish space. The complete invariant is usually prefered to be some concrete objects such as integers, real or complex numbers, polynomials, measures, etc. Although under heavy notational disguise, the essence of the problem is usually remarkably simple, namely, to try to decide the truth of a statement of the form $E \leq_B F$.

There are many variations of the Borel reducibility notion, depending on the kind of reduction functions allowed in the definition. If the reduction function is continuous the resulting notion is continuous reducibility, denoted by $\leq_c$. If the function is one-to-one the resulting reducibility is denoted by $\sqsubseteq_B$. Clearly, notion like $\sqsubseteq_c$ corresponds to the case when the reduction function is both continuous and one-to-one.

Throughout this dissertation we use ω to denote both the first infinite ordinal and the set of nonnegative natural numbers. ω is a Polish space when given the discrete topology. The Cantor space 2^ω and the Baire space ω^ω are given the product topology of the discrete topology on 2 and ω, respectively. They are examples of Polish spaces. For some of the spaces we consider in the sequel we are more interested in its Borel structure than the concrete Polish topology on it. We call a space (with a Borel structure) a *standard Borel space* if there is a Polish topology for the space generating the same Borel structure. If X is a standard Borel space and G is a Polish group acting on X in a Borel manner, then X is called a *Borel G-space*.

For any Polish space X the identity relation $\mathrm{id}(X)$ on X is the simplest equivalence relation we can consider. It is Borel, in fact closed. It can be induced by a trivial continuous G-action for any Polish group G. The reducibility between identity relations of Polish spaces coincides with the comparison of the cardinality of the spaces. Indeed, there are only countably many such cardinalities.

1.2 Historical development

Let us summarize the previous results we will make use of without proofs in the following chapters. This is definitely not a complete historical account. We merely include some milestones in the development of invariant descriptive set theory. For a comprehensive introduction of the subject we recommend [BK], [Kec1] and [Kec2].

The first results in the subject are probably Silver and Burgess' theorems on the number of equivalence classes for $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_1$ equivalence relations.

Theorem 1.2.1 (Silver) *Let E be a $\mathbf{\Pi}^1_1$ equivalence relation. Then E has either countably many or $2^{\aleph_0}$ many equivalence classes.*

This is the so-called Silver dichotomy for $\mathbf{\Pi}^1_1$ equivalence relations. The second possibility can in fact be strengthened to the statement $\mathrm{id}(2^\omega) \leq_B E$, and in this case we say that E has *perfectly* many equivalence classes. The Silver dichotomy fails for general $\mathbf{\Sigma}^1_1$ equivalence relations. But whether it holds for orbit equivalence relations of a Polish group action is still an outstanding open question. This is called the Topological Vaught Conjecture. Burgess' theorem is a trichotomy for general $\mathbf{\Sigma}^1_1$ equivalence relations.

Theorem 1.2.2 (Burgess) *Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation. Then E has either countably many, $\aleph_1$ many or perfectly many equivalence classes.*

Then comes the isolation of the equivalence relation E_0. It is defined on the Cantor space 2^ω by

$$xE_0y \Leftrightarrow \exists m \forall n \geq m(x(n) = y(n)),\ \forall x, y \in 2^\omega.$$

In [DJK] many interesting properties about E_0 were proved. But the real importance of this equivalence relation lies in the following remarkable theorem.

Theorem 1.2.3 (Harrington-Kechris-Louveau) *Let E be a Borel equivalence relation. Then either $E \leq_B id(2^\omega)$ or else $E_0 \sqsubseteq_c E$.*

This is the Glimm-Effros dichotomy for Borel equivalence relations. The naming of the dichotomy reflects the fact that before the discovery of the theorem more than twenty years of research has been going on in other mathematical fields and with lesser generality. The possibilities in the dichotomy are incompatible. If the first possibility happens, then one can assign, in a Borel fashion, real numbers to the equivalence classes of E as their complete invariants. This gives a nice classification for the equivalence classes of E. In this case E is called *smooth*. If the second possibility happens, then E does not admit such a nice classification, and this is for the concrete reason that E is more complicated than the canonical non-smooth equivalence relation E_0.

If G is a Polish group, then the Glimm-Effros dichotomy for G refers to the statement that for any Borel G-space X the orbit equivalence relation E_G^X has the Glimm-Effros dichotomy. In this definition, one can replace Borel G-space by Polish G-space, by the following theorem in [BK].

Theorem 1.2.4 (Becker-Kechris) *Let X be a Borel G-space. Then there is a Polish G-space Y such that Y is Borel isomorphic to X.*

For various G the Glimm-Effros dichotomy for E_G^X was investigated, e.g., in [BK], [HS], [HK1] and [Be] (also see [HKL] for a good survey of earlier work by other people). Probably the strongest result up to date is Becker's theorem ([Be]).

Theorem 1.2.5 (Becker) *If G admits a complete left-invariant metric then the Glimm-Effros dichotomy for G holds.*

In Chapter 2 we will discuss more about Polish groups admitting complete left-invariant metrics. For now, let us just remark that the Glimm-Effros dichotomy for G implies the Topological Vaught Conjecture for G.

1.3 Logic actions

There is an important application of invariant descriptive set theory to countable model theory. This is through the study of logic actions.

Let

$$S_\infty = \{f \in \omega^\omega \mid f \text{ is one-to-one and onto}\}$$

and equip it with the topology inherited from the Baire space ω^ω. S_∞ is a Polish group.

Let L be a countable language. We consider the space of all countable models of L with domain ω. Under appropriate set-up this will be a Polish space. To see this, assume without loss of generality that L is relational and enumerate the non-logical relation symbols as $(R_i)_{i\in\omega}$. For each i let n_i be the arity of R_i. Let

$$\mathrm{Mod}(L) = \prod_i 2^{\omega^{n_i}}$$

be equipped with the product topology. Then $\mathrm{Mod}(L)$ is a Polish space. Each element of $\mathrm{Mod}(L)$ corresponds to a countable L-structure with domain ω.

Let S_∞ act on $\mathrm{Mod}(L)$ by defining, for each $g \in S_\infty$, $M \in \mathrm{Mod}(L)$ and $i \in \omega$,

$$(g.M)(k_1, \ldots, k_{n_i}) = 1 \Leftrightarrow M(g^{-1}(k_1), \ldots, g^{-1}(k_{n_i})) = 1.$$

This is the logic action of S_∞ on $\mathrm{Mod}(L)$. Note that $\mathrm{Mod}(L)$ is a Polish S_∞-space, i.e., the logic action is continuous. In addition, if $g.M = N$, then g is actually an isomorphism between the models M and N, since the above definition can be rewritten in model theoretic form as

$$R_i^{g.M}(k_1,\dots,k_{n_i}) \Leftrightarrow R_i^M(g^{-1}(k_1),\dots,g^{-1}(k_{n_i})).$$

Therefore the orbit equivalence relation is exactly the isomorphism relation between countable models. For each $M \in \mathrm{Mod}(L)$, let $\mathrm{Aut}(M)$ be the group of all automorphisms of M. Then $\mathrm{Aut}(M)$ is actually a closed subgroup of S_∞, since

$$\mathrm{Aut}(M) = \{g \in S_\infty \mid g.M = M\}.$$

Therefore $\mathrm{Aut}(M)$ is also a Polish group. In this dissertation we will use the following fact freely without mentioning it again.

Theorem 1.3.1 (folklore) *Let L be a countably infinite language. Let G be a Polish group. Then G is a closed subgroup of S_∞ iff $G = Aut(M)$ for some L-structure M.*

The logic action turns out to be typical in all actions of S_∞, in the following sense.

Theorem 1.3.2 (Becker-Kechris) *Let L be a countably infinite language. Then Mod(L) is a universal Borel S_∞-space, in the sense that any Borel S_∞-space is Borel isomorphic to an invariant Borel subspace of Mod(L).*

The infinitary logic $L_{\omega_1\omega}$ comes into play at this point. The formulas in $L_{\omega_1\omega}$ are formed under the combination of Boolean operations, finitely nested quantifications and countable conjunctions and disjunctions. For any sentence φ in $L_{\omega_1\omega}$, we let $\mathrm{Mod}(\varphi)$ denote the space of all models of φ with domain ω. Then it is well-known that $\mathrm{Mod}(\varphi)$, $\varphi \in L_{\omega_1\omega}$, are exactly the invariant Borel subsets of $\mathrm{Mod}(L)$. By Theorem 1.2.4, $\mathrm{Mod}(\varphi)$ can also be viewed as a Polish S_∞-space under an appropriate topology. This action will be called the logic action on $\mathrm{Mod}(\varphi)$.

It follows from the above observations that the Topological Vaught Conjecture for S_∞ is exactly the Vaught Conjecture (for infinitary logic). We make the convention that Vaught's original conjecture shall be called the first-order Vaught Conjecture in this dissertation. Recall its statement: any complete first-order theory has either countably many or $2^{\aleph_0}$ many models up to isomorphism. So the first-order Vaught Conjecture is implied by the general Vaught Conjecture for infinitary logic. Our techniques in the sequel does not make distinctions between the two versions, therefore we will only treat the general version.

Note that Becker's Theorem (Theorem 1.2.5) does not apply to S_∞ actions. It is then desirable to obtain dichotomy theorems for logic actions. In Chapters 4-6 we will prove various dichotomy theorems for logic actions which are at least strengthenings of the Glimm-Effros dichotomy. We should mention that our study still leaves the Vaught Conjecture and the Topological Vaught Conjecture intact.

1.4 Model theory of infinitary logic

The technical details of most of our proofs in the sequel require some model theory for the infinitary logic $L_{\omega_1\omega}$. We summarize the preliminary results in this section. Most of the results presented here can be found in [Ba], [Kei] or [Ho].

Let L be a countable language and F a set of formulas in $L_{\omega_1\omega}$. The set F is called a *fragment* of $L_{\omega_1\omega}$ if it contains all the first-order formulas and is closed under taking subformulas, substitutions of variables and the first-order Boolean operations and quantifications.

Let M and N be L-structures. M and N are *F-elementarily equivalent* if for any sentence $\varphi \in F$, $M \models \varphi$ iff $N \models \varphi$. An embedding f of M into N is an *F-elementary embedding* if for each formula $\varphi(v_0, \ldots, v_n)$ in F and $x_0, \ldots, x_n \in M$,

$$M \models \varphi[x_0, \ldots, x_n] \Leftrightarrow N \models \varphi[f(x_0), \ldots, f(x_n)].$$

If M is a substructure of N and the identity embedding from M into N is an F-elementary embedding, then M is an *F-elementary substructure* of N and is denoted by $M \prec_F N$. By exactly the same proof as the first-order Tarski-Vaught criterion, we have that $M \prec_F N$ iff for any formula $\varphi(v_0, \ldots, v_n)$ in F and $x_0, \ldots, x_{n-1} \in M$, if $N \models (\exists v_n)\varphi[x_0, \ldots, x_{n-1}]$, then there is $x_n \in M$ such that $N \models \varphi[x_0, \ldots, x_n]$. For any formula $\varphi \in F$ a *Skolem function* h_φ for φ in M is a function from M into M such that

$$M \models (\exists v_n)\varphi[x_0, \ldots, x_{n-1}] \Leftrightarrow M \models \varphi[x_0, \ldots, x_{n-1}, h_\varphi(x_0, \ldots, x_{n-1})].$$

Let F be a countable fragment of $L_{\omega_1\omega}$. Let M be an L-structure (not necessarily countable) and A be a subset of M. Fix a system $\mathcal{S}$ of Skolem functions in M for the formulas of F. The *Skolem hull* of A in M with respect to F (and $\mathcal{S}$) is the smallest submodel of M closed under the Skolem functions in $\mathcal{S}$. If A is countable then so is the Skolem hull of A in M with respect to F. By the Tarski-Vaught criterion, the Skolem hull is an F-elementary submodel of M.

By the same proof as in the first-order case, the union of an F-elementary chain is an F-elementary extension of each element of the chain. These facts will be used without quoting in Chapter 2.

Define the *quantifier rank* of a formula φ of $L_{\omega_1\omega}$ inductively as follows:

$$\begin{aligned} \mathrm{qr}(\varphi) &= 0, \quad \text{if } \varphi \text{ is atomic}, \\ \mathrm{qr}(\exists v\varphi) &= \mathrm{qr}(\forall v\varphi) = \mathrm{qr}(\varphi) + 1, \\ \mathrm{qr}(\neg\varphi) &= \mathrm{qr}(\varphi), \\ \mathrm{qr}(\bigwedge \Phi) &= \mathrm{qr}(\bigvee \Phi) = \sup\{\mathrm{qr}(\varphi) \mid \varphi \in \Phi\}. \end{aligned}$$

Two models M and N are *α-equivalent*, denoted by $M \equiv_\alpha N$, if for any sentence φ of quantifier rank α in $L_{\omega_1\omega}$, $M \models \varphi$ iff $N \models \varphi$.

Next we review the Scott analysis for countable models. Fix a countable language L and a countable L-structure M. For any tuple $\vec{a} \in M$, the canonical Scott types are defined by transfinite induction

$$\begin{aligned}
\varphi_0^{\vec{a},M}(\vec{v}) &= \bigwedge\{\theta(\vec{v}) \mid \theta \text{ is atomic or negated atomic and } M \models \theta[\vec{a}]\}, \\
\varphi_{\alpha+1}^{\vec{a},M}(\vec{v}) &= \varphi_\alpha^{\vec{a},M}(\vec{v}) \wedge \bigwedge_{\vec{b}\in M} (\exists \vec{u})\varphi_\alpha^{\vec{a},\vec{b},M}(\vec{v},\vec{u}) \wedge (\forall \vec{u}) \bigvee_{\vec{b}\in M} \varphi_\alpha^{\vec{a},\vec{b},M}(\vec{v},\vec{u}), \\
\varphi_\lambda^{\vec{a},M}(\vec{v}) &= \bigwedge_{\alpha<\lambda} \varphi_\alpha^{\vec{a},M}(\vec{v}) \ (\lambda \text{ a limit})
\end{aligned}$$

The *Scott rank* of M, denoted by $\mathrm{sr}(M)$, is the first ordinal γ such that for any $\vec{a}, \vec{b} \in M$,

$$\varphi_\gamma^{\vec{a},M} = \varphi_\gamma^{\vec{b},M} \Rightarrow \varphi_{\gamma+1}^{\vec{a},M} = \varphi_{\gamma+1}^{\vec{b},M}.$$

If M is countable, then so is $\mathrm{sr}(M)$. The *Scott sentence* of M is defined by

$$\varphi_M = \varphi_{\mathrm{sr}(M)}^{<>,M} \wedge \bigwedge_{\vec{a}\in M} (\exists \vec{v})(\varphi_{\mathrm{sr}(M)}^{\vec{a},M} \rightarrow \varphi_{\mathrm{sr}(M)+1}^{\vec{a},M}).$$

The Scott analysis provides complete invariants for the isomorphism types of countable models.

Theorem 1.4.1 (Scott) *Let L be a countable language and M be an L-structure. Then for any countable L-structure N, $M \cong N$ iff $N \models \varphi_M$ iff $\varphi_N = \varphi_M$.*

In addition, the canonical Scott types have the special property that

$$\varphi_\alpha^{\vec{a},M} = \varphi_\alpha^{\vec{b},M} \Rightarrow (M, \vec{a}) \equiv_\alpha (M, \vec{b}),$$

where $(M, \vec{a})$ and $(M, \vec{b})$ are structures augmented from M by the constants $\vec{a}$ and $\vec{b}$, respectively. By a back and forth argument, it is easy to see that if $\alpha \geq \mathrm{sr}(M)$, then

$$\varphi_\alpha^{\vec{a},M} = \varphi_\alpha^{\vec{b},M} \Rightarrow (M, \vec{a}) \cong (M, \vec{b}),$$

in fact, there is an automorphism $\tau \in \mathrm{Aut}(M)$ such that $\tau(\vec{a}) = \vec{b}$.

If F is a countable fragment of $L_{\omega_1\omega}$ with $\varphi_M \in F$. Then $N \prec_F M$ iff $N \prec_{L_{\omega_1\omega}} M$ iff $N \prec_{L_{\infty\omega}} M$. These facts will be used without quoting in the sequel.

Since all the countable models we consider will have domain ω, each model M is correspondent to a real in the Polish space $\mathrm{Mod}(L)$. Note that the space $\mathrm{Mod}(L)$ is homeomorphic to the Cantor space 2^ω. So fixing L, the models M are coded in a uniform way by reals x_M in 2^ω. For each M, let ${\omega_1}^M$ be the smallest ordinal γ such that γ is not recursive in M, or x_M. ${\omega_1}^M$ is called the *first admissible ordinal* past M. Equivalently, ${\omega_1}^M$ is the smallest ordinal $\gamma > \omega$ such that $\mathsf{L}_\gamma(x_M)$ is admissible, where L is Gödel's hierarchy of constructible sets. We will use the following fact about ${\omega_1}^M$.

Theorem 1.4.2 (Nadel) *For any countable model M, $sr(M) \leq {\omega_1}^M$.*

Let us remark that we will never talk about models of ZF or ZFC in this dissertation, especially we will not denote such by the letter M. Therefore, we will never talk about the first uncountable ordinal in such a model, which is conventionally denoted by ${\omega_1}^M$. Therefore there is no danger of confusion.

For other properties of admissible sets, especially the well-known boundedness principle, we refer the reader to the proofs in Chapter VII of [Ba].

CHAPTER 2

Polish Groups with Complete Left-Invariant Metrics

2.1 Background

A compatible metric d on a Polish group G is *left-invariant* if for all $g, h, k \in G$

$$d(gh, gk) = d(h, k).$$

Similarly we can define *right-invariant* metrics. An *invariant* metric is a metric that is both left-invariant and right-invariant. For Polish groups, any compatible invariant metric d must be complete, i.e., all d-Cauchy sequences are convergent within G (Corollary 1.2.2 of [BK]). G is called CLI if it admits a compatible complete left-invariant metric.

If d_l is a compatible left-invariant metric on a Polish group G, then the metric d_r defined by $d_r(x, y) = d_l(x^{-1}, y^{-1}), \forall x, y \in G$, is a compatible right-invariant metric on G. Moreover, it follows from the continuity of group operations that d_l is complete iff d_r is complete. This means that every statement we prove about left-invariant complete metrics will have a counterpart about right-invariant complete metrics.

Every Polish group admits a compatible left-invariant metric, but in general not every Polish group is CLI. It is not hard to see that S_∞ is not CLI. Nevertheless, the class of CLI groups is a fairly large class. In particular, it contains all locally compact Polish groups (which in turn include all countable discrete groups), all solvable Polish groups, and all Polish groups admitting invariant metrics. In addition, the class of CLI groups is closed under the operation of taking closed subgroups and under countable products. In section 2.2 we show that it is also closed under homomorphisms and group extensions.

In section 2.3 we give criteria for CLI groups for homeomorphism groups of compact Polish spaces. The results seem to indicate that there are not too many CLI groups arising this way.

The study of CLI Polish groups was stimulated by Howard Becker's theorem of the Glimm-Effros dichotomy for Borel actions of CLI groups (see [Be]). [Be] also contains a summary of results related to CLI groups, including the proofs of several facts mentioned in the preceding paragraphs. For proofs of the more elementary facts another good reference is [BK].

To understand the impact of Becker's theorem on the study of Vaught Conjecture, a natural question to ask is: which closed subgroups of S_∞ are CLI? Since closed subgroups of S_∞ are exactly the automorphism groups of countable structures (with domain ω) the question becomes: what kind of countable models have CLI automorphism groups? In section 2.4 we answer this question by giving model theoretic characterizations for the CLI closed subgroups of S_∞.

2.2 Closure under group extensions

In this section we show that the class of CLI groups is closed under homomorphisms and groups extensions.

The following lemma, despite its simplicity, is very useful for the study of CLI Polish groups.

Lemma 2.2.1 (folklore) *Let G be a CLI group. If d is a compatible left-invariant metric on G, then d is complete.*

Proof. Let d_0 be a compatible left-invariant complete metric on G. Let $\{x_n\}$ be a d-Cauchy sequence in G. It suffices to show that $\{x_n\}$ is d_0-Cauchy. Let $\epsilon > 0$. Let $U = \{x \in G \mid d_0(x, 1) < \epsilon\}$. Then there is $\delta > 0$ such that $\{x \in G \mid d(x, 1) < \delta\} \subseteq U$. Fix N with $\forall n, m > N, d(x_n, x_m) < \delta$. Then for any $n, m > N$, by left-invariance of d, we have $d(x_m^{-1}x_n, 1) = d(x_n, x_m) < \delta$. Therefore $d_0(x_m^{-1}x_n, 1) < \epsilon$ since $x_m^{-1}x_n \in U$. By left-invariance of d_0, $d_0(x_n, x_m) = d_0(x_m^{-1}x_n, 1) < \epsilon$. $\dashv$

It follows easily from Lemma 2.2.1 that a subgroup H of a CLI Polish group G with the induced topology is CLI iff H is closed in G iff H is Polish. It is a general fact that if H is a closed subgroup of a Polish group G, then G/H is a Polish space. In case H is a normal subgroup of G, G/H is a Polish group. In addition, any (continuous and open) homomorphic image of G is of the form G/H for some closed normal subgroup H of G. The next lemma studies compatible metrics on G/H.

Lemma 2.2.2 *Let G be a Polish group, d be a compatible left-invariant metric on G and H be a closed normal subgroup of G. Define*

$$d_1(xH, yH) = \inf \{\ d(xh, yk) \mid h, k \in H\}.$$

Then d_1 is a compatible left-invariant metric on G/H. Moreover, we have the following equivalents:

$$\begin{aligned} d_1(xH, yH) &= \inf \{\ d(hx, ky) \mid h, k \in H\} \\ &= \inf \{\ d(x, hy) \mid h \in H\} \\ &= \inf \{\ d(x, yh) \mid h \in H\} \\ &= \inf \{\ d(y^{-1}x, h) \mid h \in H\}. \end{aligned}$$

Proof. By the normality of H, for any $x \in X$, we have $xH = Hx$. The equivalent definitions for d_1 follow immediately from this fact and the left-invariance of d. It is also immediate that d_1 is left-invariant. Among the several facts to check to complete the proof, we show that $d_1(xH, yH) = 0$ implies $xH = yH$. For this, use the fourth definition in the equivalents: $d_1(xH, yH) = 0$ implies that there is a sequence $\{h_n\} \subseteq H$ such that $d(y^{-1}x, h_n) \to 0$, $n \to \infty$; but this means that $h_n \to y^{-1}x$; since H is closed, $y^{-1}x \in H$, hence $xH = yH$. It is now easy to see that d_1 is a metric on G/H and it is compatible with the canonical topology on G/H induced by the one on G. $\dashv$

Theorem 2.2.3 *Let G be a Polish group and H be a closed normal subgroup of G. Then G is CLI iff both H and G/H are CLI.*

Proof. Let d be a compatible left-invariant metric on G. Let $d_0 = d \restriction H$. Then d_0 is a compatible left-invariant metric on H. Let d_1 be the compatible left-invariant metric on G/H as defined in Lemma 2.2.2.

($\Rightarrow$) Suppose in addition that d is complete. Then obviously d_0 is complete. Next we show that d_1 is also complete. Let $\{x_nH\}$ be a d_1-Cauchy sequence in G/H. By choosing a subsequence we may assume that $\forall n < m, d_1(x_nH, x_mH) < 1/2^n$. We define a sequence $\{y_n\} \subset G$ with each $y_n \in x_nH$. Let $y_0 = x_0$. Given $y_i \in x_iH$, since $d_1(x_iH, x_{i+1}H) < 1/2^i$ and $d_1(x_iH, x_{i+1}H) = d_1(y_iH, x_{i+1}H) = \inf\{\, d(y_i, x_{i+1}h) \mid h \in H\}$, there is some $y \in x_{i+1}H$ with $d(y_i, y) < 1/2^i$. Let y_{i+1} be such an y. Now it is easy to check that $\{y_n\}$ is d-Cauchy. So there is $y_\infty \in G$ such that $y_n \to y_\infty, n \to \infty$. Then $x_nH = y_nH \to y_\infty H, n \to \infty$.

($\Leftarrow$) Suppose both H and G/H admit compatible left-invariant complete metrics. Then by Lemma 2.2.1 both d_0 and d_1 are complete. We show that d must be complete. Let $\{x_n\}$ be a d-Cauchy sequence in G. Since $\forall x, y \in G, d_1(xH, yH) \leq d(x, y)$, the sequence $\{x_nH\}$ is d_1-Cauchy in G/H. Then there is $y \in G$ with $x_nH \to yH, n \to \infty$. By left-invariance of d_1, $y^{-1}x_nH \to H, n \to \infty$. So we may assume, without loss of generality that $x_nH \to H, n \to \infty$ and $d_1(x_nH, H) < 1/2^n$. Let $\{h_n\} \subset H$ be such that for each n, $d(x_n, h_n) < 1/2^n$. Then $\{h_n\}$ is d_0-Cauchy in H. Hence there is $h \in H$ with $h_n \to h, n \to \infty$. But then $d(x_n, h) \leq d(x_n, h_n) + d(h_n, h) \leq 1/2^n + d(h_n, h) \to 0, n \to \infty$. This completes the proof. $\dashv$

Corollary 2.2.4 (Hjorth-Solecki) *Any solvable Polish group is CLI.*

Proof. It is well known that abelian groups admit invariant metrics, which are necessarily complete. Now the corollary follows from Theorem 2.2.3 by an easy induction on the length of the derived series of G. $\dashv$

Hjorth-Solecki's proof has not been published, but see [HS] for the kind of argument used in their proof. Our proof is much simpler than theirs.

2.3 Homeomorphism groups of compact Polish spaces

In this section we analyze the homeomorphism groups of various compact Polish spaces and determine when they are CLI.

Let X be a compact Polish space. We denote by $C(X,X)$ the space of all continuous functions from X to X, endowed with the compact-open topology. The topology on $C(X,X)$ is generated by sets of the form $\{f \in C(X,X) \mid f``K \subseteq O\}$ where $K \subseteq X$ is compact and $O \subseteq X$ is open. For any complete metric d_X on X the *supnorm* metric on $C(X,X)$ is defined as

$$d(f,g) = \sup \{\ d_X(f(x),g(x)) \mid x \in X\}.$$

It is a standard fact that d is a compatible complete metric on $C(X,X)$. (See, e.g., [Kec1].) Also we denote by $H(X)$ the group of all homeomorphisms of X onto X, endowed with the compact-open topology. Both $C(X,X)$ and $H(X)$ are Polish, and in fact $H(X)$ is a G_δ subspace of $C(X,X)$. The following theorem gives a sufficient and necessary condition to when $H(X)$ is CLI.

Theorem 2.3.1 *Let X be a compact Polish space. Then $H(X)$ is CLI iff $H(X)$ is closed in $C(X,X)$.*

Proof. Let d be the supnorm metric on $C(X,X)$. d is complete. Let $d_0 = d \restriction H(X)$. Then d_0 is a compatible right-invariant metric on $H(X)$. If $H(X)$ is closed in $C(X,X)$, then d_0 is complete as a metric on $H(X)$. This is because, for any d_0-Cauchy sequence in $H(X)$, it is d-Cauchy in $C(X,X)$, hence it has a limit. Since $H(X)$ is closed, the limit is in $H(X)$. On the other hand, if $H(X)$ admits a compatible left-invariant complete metric, then by lemma 2.2.1, d_0 is complete. This implies that $H(X)$ is closed. $\dashv$

Easy modifications of the above proof give the following corollaries, as pointed out by Howard Becker. In Corollary 2.3.3, note that S_∞ is identified with $H(X)$ where X is the one-point compactification of $\mathbb{Z}$.

Corollary 2.3.2 *Let G be a closed subgroup of $H(X)$ for some compact Polish space X. Then G is CLI iff G is closed in $C(X,X)$.*

Corollary 2.3.3 *Let G be a closed subgroup of S_∞. Then G is CLI iff G is closed in the Baire space ω^ω.*

We omit the proofs of these corollaries. Note that Corollary 2.3.3 implies that S_∞ is not CLI. Moreover, if S_∞ can be topologically embedded as a closed subgroup of a Polish group G, then G is not CLI either. For example, this is true for $H(2^\omega)$. To see this, consider the embedding θ of S_∞ into $H(2^\omega)$ defined by:

$$(\theta(f))(0^n 1x) = 0^{f(n)} 1x,\ \forall n \in \omega, x \in 2^\omega.$$

Then it is easy to check that θ is a topological embedding of S_∞ onto a closed subgroup of $H(2^\omega)$.

In fact we can show that if X is infinite and zero-dimensional then $H(X)$ is never CLI. This is the content of the following theorem.

Theorem 2.3.4 *Let X be an infinite zero-dimensional compact Polish space. Then S_∞ can be topologically embedded as a closed subgroup of $H(X)$. Therefore, $H(X)$ is not CLI.*

Proof. Let D be the Cantor-Bendixson derivative of X, i.e.,

$$D = \{x \in X \mid x \text{ is a limit point of } X\}$$

and let $S = X - D$. We consider two cases.

Case 1. S is finite. In this case D is a closed subset of X. Moreover, D is infinite, zero-dimensional, perfect, compact and Polish. By Brouwer's Theorem (see [Kec1] Theorem 7.4), D is homeomorphic to the Cantor space 2^ω. Hence $H(D)$ is isomorphic to $H(2^\omega)$ as topological groups. Hence S_∞ can be topologically embedded as a closed subgroup of $H(D)$. To finish the proof of this case, it suffices to show that $H(D)$ is a closed subgroup of $H(X)$. But this is easy: just let $\varphi : H(D) \to H(X)$ be defined by

$$\varphi(f)(x) = \begin{cases} f(x) & \text{if } x \in D \\ \\ x & \text{otherwise} \end{cases}$$

It is straightforward to check that φ is an topological embedding and $\varphi``H(D)$ is closed in $H(X)$.

Case 2. S is infinite. In this case we can find an infinite sequence $\{x_n\} \subseteq S$ such that $x_n \to y$ for some $y \in D$. Let Y be the set $\{x_n | n \in \omega\} \cup \{y\}$. Then Y is closed in X. Note that $H(Y)$ is isomorphic to S_∞ as topological groups. Thus to finish the proof of this case, it suffices to show that $H(Y)$ is a closed subgroup of $H(X)$. For this first notice that y is a fixed point of any homeomorphism of Y. We then define $\varphi : H(D) \to H(X)$ by

$$\varphi(f)(x) = \begin{cases} f(x) & \text{if } x \in Y \\ \\ x & \text{otherwise} \end{cases}$$

It is then straightforward to check that φ is a topological embedding and that $\varphi``H(Y)$ is closed in $H(X)$. $\dashv$

Next we consider infinite compact subsets of $\mathbb{R}^n$. Denote the open interval $(-1, 1) \subseteq \mathbb{R}$ by I. We show that it happens quite often that the homeomorphism group is not CLI.

Definition 2.3.5 Let $A \subseteq \mathbb{R}^n$ be a compact subset. A is *fossil* if there is an open set U in $\mathbb{R}^n$ such that $A \cap U$ is homeomorphic to I^k for some $1 \leq k \leq n$.

Theorem 2.3.6 *If $X \subseteq \mathbb{R}^n$ is fossil, then $H(X)$ is not CLI.*

Proof. Suppose U_0 is an open set in $\mathbb{R}^n$ such that $X \cap U_0$ is homeomorphic to I^k, for some $1 \leq k \leq n$, via homeomorphism π. Denote $[-\frac{1}{2}, \frac{1}{2}]^k$ by S. Then there is an open subset U_1 of U_0 such that $\overline{U_1} \subseteq U_0$ and $\pi``(\overline{U_1} \cap X) = S$. Let ∂S denote the boundary of S. Let $G = \{f \in H(S) | f(x) = x, \forall x \in \partial S\}$. Then G is a closed subgroup of $H(S)$. Note that G an be topologically embedded as a closed subgroup of $H(X)$: define $\varphi : G \to H(X)$ by

$$\varphi(f)(x) = \begin{cases} \pi^{-1} f \pi(x) & \text{if } x \in \overline{U_1} \cap X \\ x & \text{otherwise} \end{cases}$$

Then φ is a topological embedding and $\varphi``G$ is closed in $H(X)$.

By Theorem 2.3.1 it suffices to show that G is not closed in $C(S,S)$. We define a sequence $\{f_n\} \subseteq G$ as follows. For every $i \in \omega$, let $S_i = [-\frac{1}{2^{i+1}}, \frac{1}{2^{i+1}}]^k$. (So $S_0 = S$.) Fix $n \in \omega$. For $x \in S_1$, let $f_n(x) = \frac{1}{2^n}x$. For $x \in S_0 - S_1$, let l_x be the straight line with the origin 0 and x on it. Let x_0, x_1, and x_2 be the intersections of l_x with respectively $\partial S_0, \partial S_1$ and ∂S_{n+1} such that $|x - x_0| \leq |x_0|$, $|x - x_1| \leq |x|$ and $|x - x_2| \leq |x|$. Then let $f_n(x)$ be the point $y \in l_x$ such that

$$\frac{|y - x_2|}{|y - x_0|} = \frac{|x - x_1|}{|x - x_0|}.$$

(Let $f_n(x_0) = x_0$.) It is obvious that each f_n is a homeomorphism of S with ∂S unchanged. So $f_n \in G$. Furthermore, $\{f_n\}$ converges to a continuous $f \in C(S,S)$ with $f``S_1 = \{0\}$. So f is not a homeomorphism. $\dashv$

The theorem is also true for $\mathbb{R}^\omega$ if the concept of fossility is extended to $\mathbb{R}^\omega$. Let us remark that one can not hope to prove that for *all* infinite compact Polish spaces the homeomorphism groups are not CLI, since it is possible to construct compact metric spaces which are *rigid*, i.e., have no non-trivial homeomorphism onto itself. However, we are not aware of any example of compact Polish space whose homeomorphism group is uncountable and yet CLI. In fact Theorems 2.3.4 and 2.3.6 seem to suggest that a majority of the compact Polish spaces do not give rise to CLI homeomorphism groups. In some special cases such as the following one, this observation can be put in a more precise form.

Proposition 2.3.7 *Every compact subset of $\mathbb{R}$ is either fossil or zero-dimensional. Therefore, for every compact subset X of $\mathbb{R}$, $H(X)$ is not CLI.*

We omit the proof. The key fact here about $\mathbb{R}$ is that connectedness is the same as path-connectedness. It is not known to us whether the above proposition holds for $n > 1$.

2.4 Automorphism groups of countable structures

In this section we give some model theoretic characterizations for the CLI closed subgroups of S_∞. It turns out that there is an interesting connection between the concepts of CLI groups and *absolutely characterizable* models, the latter concept having existed for about thirty years.

Definition 2.4.1 A countable model M is *absolutely characterizable* if there is no uncountable model of its Scott sentence.

The terminology is following Makkai ([Mak]). These models are so named since their Scott sentences have exactly one model up to isomorphism, hence characterize the models completely. The first result on such models appeared in [Ku] (before the naming).

Theorem 2.4.2 (Kueker) *Let M be a countable model. If* Aut*(M) is countable, then M is absolutely characterizable.*

The theorem we are going to prove in this section is a generalization of this.

Theorem 2.4.3 *The following are equivalent for a countable structure M:*

I) M is absolutely characterizable.

II) Aut*(M) is CLI.*

III) There is no $L_{\omega_1\omega}$-elementary embedding from M into itself which is not onto.

We prove the theorem in two steps. First we show the equivalence between II) and III) using results from section 2.3. Then we establish the equivalence between I) and III). We remark that the condition III) makes it fairly clear that the notion of absolute characterizability is $\mathbf{\Pi}^1_1$, hence absolute between transitive models of ZFC.

Theorem 2.4.4 *Let M be a countable structure of a countable language L. Then* Aut*(M) is CLI iff there is no $L_{\omega_1\omega}$-elementary embedding $j : M \hookrightarrow M$ with $j"M \neq M$.*

Proof. ($\Rightarrow$) Suppose that there is a $j : M \hookrightarrow M$ as above. Then since j is not onto, $j \notin$Aut(M). Fix an enumeration $\{a_i\}_{i\in\omega}$ of M. Let $\gamma =$sr(M). Then for each $n \in \omega$, $\varphi^{a_0,\ldots,a_n}_\gamma = \varphi^{j(a_0),\ldots j(a_n)}_\gamma$. By the property of Scott types, for each $n \in \omega$ there is $f_n \in$Aut(M) such that $f_n(a_i) = j(a_i)$ for $i \leq n$. But this means that

$f_n \to j, n \to \infty$ in the topology of ω^ω. Thus Aut(M) is not closed in the Baire space. By Corollary 2.3.3, Aut(M) is not CLI.

($\Leftarrow$) Suppose that Aut(M) is not CLI. Then by Corollary 2.3.3, there is a sequence $\{f_n\} \subset$Aut(M) and $j : M \to M, j \notin$Aut(M) with $f_n \to j, n \to \infty$ (in the topology of ω^ω). It follows that j is one-one. Let $a_0, ..., a_n \in M$. There is a $k \in \omega$ such that $f_k(a_i) = j(a_i)$ for $i \leq n$. Thus $\varphi_\gamma^{a_0,\dots,a_n} = \varphi_\gamma^{f_k(a_0),\dots,f_k(a_n)} = \varphi_\gamma^{j(a_0),\dots j(a_n)}$. Finally it follows that j is not onto, since otherwise $j \in$Aut(M), contradicting our hypothesis. $\dashv$

It is easy to construct a countable structure M with Aut(M) abelian and M admitting an ($L_{\omega\omega}$-)elementary embedding $j : M \hookrightarrow M$ with $j"M \neq M$. In fact, consider the structure whose underlying set is the linear order with the order type $\mathbb{Z} \cdot \omega$ and the language is $\{S, <\}$, where the interpretation of S in the structure is the canonical successor function and $<$ the natural binary order. The automorphism group is isomorphic to $\mathbb{Z}^\omega$, which is abelian. It is easy to see that this model admits elementary embeddings into itself which are not onto.

Theorem 2.4.5 *Let M be a countable structure of a countable language L. Then* Aut*(M) is CLI iff M is absolutely characterizable.*

Proof. We actually prove the equivalence between III) and I) of Theorem 2.4.3.

($\Rightarrow$) Assume that there is an uncountable model N with $N \models \varphi_M$. Then M and N are $L_{\infty\omega}$- equivalent. Fix an enumeration $\{a_i\}_{i \in \omega}$ of M. Let $\gamma =$sr(M). We can find a sequence $\{b_i\}_{i\in\omega}$ in N such that for any $n \in \omega$, $\varphi_\gamma^{a_0,\dots,a_n,M} = \varphi_\gamma^{b_0,\dots,b_n,N}$. Letting $j(a_i) = b_i$ for $i \in \omega$, j is in fact an embedding from M into N such that for any $\vec{a} \in M$, $(M, \vec{a})$ is $L_{\omega_1\omega}$-equivalent to $(N, j(\vec{a}))$. Let F be a countable fragment of $L_{\omega_1\omega}$ consisting φ_M. By taking the Skolem hull with respect to the formulas in F we can obtain a countable $N_1 \subset N$ such that j"M is a proper subset of N_1 and such that for any $\vec{a} \in M$, $(M, \vec{a})$ is $L_{\omega_1\omega}$-equivalent to $(N_1, j(\vec{a}))$. But this means that $j : M \to N_1$ is an $L_{\omega_1\omega}$-elementary embedding. It follows that $M \cong N_1$ and so there is an $L_{\omega_1\omega}$-elementary embedding of M into itself which is not onto. By Theorem 2.4.4, Aut(M) is not CLI.

($\Leftarrow$) Suppose Aut(M) is not CLI. Then by Theorem 2.4.4 there is an $L_{\omega_1\omega}$-elementary embedding $j : M \to M$ which is not onto. Let $M_0 = j$"M. Then $M_0 \cong M$, $M_0 \prec M$ ($L_{\omega_1\omega}$) and $M_0 \neq M$. It follows that there is a countable structure N_0 such that $M \cong N_0$, $M \prec N_0$ ($L_{\omega_1\omega}$) and $M \neq N_0$. Since Aut(N_0)$\cong$Aut(M) as topological groups, Aut(N_0) is not CLI. Therefore we can iterate this process to get an $L_{\omega_1\omega}$-elementary chain of countable structures

$$M \prec N_0 \prec N_1 \prec \cdots \prec N_\alpha \prec \cdots, \ (\alpha < \omega_1).$$

Let $N = \bigcup_{\alpha<\omega_1} N_\alpha$. Then $|N| = \aleph_1$ and $N_\alpha \prec N$ for every $\alpha < \omega_1$. Therefore $N \models \varphi_M$. $\dashv$

We state the following corollary to emphasize the novelty of this result.

Corollary 2.4.6 *Let M be a countable structure of a countable language L. If* $\mathrm{Aut}(M)$ *is solvable, then M is absolutely characterizable.*

Proof. Follows from Theorem 2.4.5 and Corollary 2.2.4. $\dashv$

Notice that the statement of Corollary 2.4.6 is purely algebraic and model-theoretic, but the only known proof heavily employs topological considerations.

Finally, it is noticeable that Kueker's result has been generalized to models of cardinality $\aleph_1$ ([STV]). Jouko Väänänen raised the question whether our results have similar generalizations; to this we have no answer.

CHAPTER 3

Actions of Homeomorphism Groups of Compact Polish Spaces

3.1 Universal spaces

The existence of universal actions for various kinds of groups has been known for a long time. Often there are different possible realizations of the universal actions, some of which are more *natural* than others. The first result in this subject of study is probably due to Mackey ([Mac]) and Varadarajan ([Va]); they proved the existence of universal G-spaces for locally compact G, in the following sense.

Theorem 3.1.1 (Mackey-Varadarajan) *Let G be a locally compact topological group. Then there is a Borel G-space X_G such that for any separable metrizable space X and any Borel action of G on X, there is a Borel G-embedding from X into X_G. That is, there is a Borel embedding $\varphi : X \to X_G$ such that for any $x \in X$ and $g \in G$, $\varphi(g.x) = g.\varphi(x)$. In fact, the space X_G can be taken to be a compact Polish G-space.*

Such a universal space is usually called a universal Borel G-space. Once the existence of the universal space is established, it makes sense to ask whether there are natural realizations. For separable locally compact groups de Vries ([dV]) showed that the universal space can be taken to be $L^2(G \times G)$, for which the integration is with respect to the Haar measure on G.

Becker and Kechris ([BK]) generalized the Mackey-Varadarajan result to arbitrary Polish groups G.

Theorem 3.1.2 (Becker-Kechris) *For any Polish group G there is a universal Borel G-space.*

Recently Hjorth ([Hj1]) proved a very strong form of such existence theorems.

Theorem 3.1.3 (Hjorth) *For any Polish group G, there is a Polish G-space X_G such that for any Polish G-space X there is a topological G-embedding from X into X_G.*

The universal Borel G-space in the Becker-Kechris result is the infinite power of the *Effros Borel space* $\mathcal{F}(G)$, which is defined to be the space of all closed subsets of G with the standard Borel structure generated by sets of the form

$\{F \in \mathcal{F}(G) | F \cap O \neq \emptyset\}$ for some open subset O of G. The action of G on $\mathcal{F}(G)$ is just the pointwise multiplication (shift).

Another realization of the universal Borel G-space is the infinite power of the space of Lipschitz functions from G to $\mathbb{R}$. Let d be a compatible left-invariant metric on G. Let $\mathcal{L}(G,d)$ be the space of all continuous functions $f : G \to \mathbb{R}$ such that for any $x, y \in G$, $|f(x) - f(y)| \leq d(x,y)$. Let $\mathcal{L}(G,d)$ be equipped with the topology given by subbasic open sets of the form $\{x \in G \mid |f(x) - r| < \epsilon\}$ for $r \in \mathbb{R}$ and $\epsilon > 0$. This is a Polish topology. The action of G on $\mathcal{L}(G,d)$ is defined by $(g.f)(x) = f(g^{-1}x)$, for all $g, x \in G$ and $f \in \mathcal{L}(G,d)$. Thus $\mathcal{L}(G,d)$ becomes a Polish G-space.

The Polish G-space constructed in Hjorth's result is an ingenious, complicated, and rather artificial modification of the space $\mathcal{L}(G,d)^\omega$. There is so far no natural realization of the space with the universal property described in Theorem 3.1.3. The realizations in the preceding paragraphs are considered natural by many people, but their structure could become hopelessly complicated when the group G becomes complicated. For example, in case G is the homeomorphism group $H(X)$ for some compact Polish space X, both the above spaces become remarkably involving.

In this section we prove a result that helps to pin down the complexity of the universal space in some special cases. Here the word "complexity" is used in an intuitive sense. In particular, it is *not* in the sense of the Borel reducibility of the orbit equivalence relations, since it is obvious that all orbit equivalence relations induced by universal actions of a particular Polish group G are Borel equivalent to each other.

Suppose X is a compact Polish space. Suppose that G is a closed subgroup of $H(X)$. For $\vec{x} = (x_1, x_2, \ldots, x_n) \in X^n$ and $\vec{O} = (O_1, O_2, \ldots, O_n)$ a tuple of open sets in X, let

$$\mathcal{U}_G(\vec{x}, \vec{O}) = \{f \in G \mid f(x_i) \in O_i, \forall 1 \leq i \leq n\}.$$

Then $\mathcal{U}_G(\vec{x}, \vec{O})$ is an open set in G. Let $Q \subseteq X$ be a countable dense subset of X. Let $\mathcal{A}$ be a countable basis for the topology of X. Let

$$\mathcal{B}_G = \{\mathcal{U}_G(\vec{x}, \vec{O}) \mid \vec{x} \in Q^n, \vec{O} \in \mathcal{A}^n, n \in \omega\}.$$

Then the topology generated by $\mathcal{B}_G$ is coarser than the original topology on G.

We also recall the following standard definition (see [Kec1]). For a Polish space X, let $K(X)$ be the space of all compact subsets of X. The topology of $K(X)$ is generated by basic open sets of the form $\{K \in K(X) \mid K \cap O \neq \emptyset\}$ and of the form $\{K \in K(X) \mid K \subseteq O\}$. Thus $K(X)$ becomes a Polish space. Moreover, if X is compact, then $K(X)$ is compact and its Borel structure is exactly the same as the Effros Borel structure.

Theorem 3.1.4 *Let G be a Polish subgroup of $H(X)$ for some compact Polish space X. Fix some countable dense subset Q of X and some countable open basis*

$\mathcal{A}$ of X. If $\mathcal{B}_G$ is an open basis for G, then the product space $\prod_{n\in\omega} K(X^n)$ is a universal Borel G-space. Here the action of G on $\prod_{n\in\omega} K(X^n)$ is the product of the application actions of G on $K(X^n)$: for any $g \in G$ and $F \in K(X^n)$,

$$g.F = \{(g(x_1), \dots, g(x_n)) \mid (x_1, \dots, x_n) \in F\}.$$

Proof. For any $\vec{x} \in X^n$, let $|\vec{x}| = n$. Let

$$Y = \prod_{\vec{x}\in Q^{<\omega}} K(X^{|\vec{x}|}).$$

Since $Q^{<\omega}$ is countable, Y is homeomorphic to $\prod_{n\in\omega} K(X^n)^\omega$ as Polish G-spaces. By a diagonalization argument it is easy to see that Y is homeomorphic to a closed subspace of the G-space $\prod_{n\in\omega} K(X^n)$. Hence by the Becker-Kechris proof of Theorem 3.1.2, it suffices to show that there is a Borel G-embedding of $\mathcal{F}(G)$ into Y.

We define such an embedding φ as follows. Fix $F \in \mathcal{F}(G)$. Let U be the complement of F in G. Then U is open in G. For each $\vec{x} \in Q^{<\omega}$ let

$$V_{\vec{x}} = \bigcup\{O_1 \times \cdots \times O_{|\vec{x}|} \mid \mathcal{U}_G(\vec{x}, \vec{O}) \subseteq U\}.$$

Then let $K_{\vec{x}}$ be the complement of $V_{\vec{x}}$ in $X^{|\vec{x}|}$. Since $X^{|\vec{x}|}$ is compact and $V_{\vec{x}}$ is open in $X^{|\vec{x}|}$, $K_{\vec{x}} \in K(X^{|\vec{x}|})$. Finally let $\varphi(F) = (K_{\vec{x}})_{\vec{x}\in Q^{<\omega}}$. We claim that φ is as required.

To see that φ is an embedding, let $F_1, F_2 \in \mathcal{F}(G)$ and $F_1 \neq F_2$. Let U_1 and U_2 be the complements of F_1 and F_2, respectively. Then $U_1 \neq U_2$. Since $\mathcal{B}_G$ is an open basis for G, we may assume there are $\vec{x} \in Q^{<\omega}$ and $\vec{O} \in \mathcal{A}^{|\vec{x}|}$ such that $\mathcal{U}_G(\vec{x}, \vec{O}) \subseteq U_1$ but $\mathcal{U}_G(\vec{x}, \vec{O}) \not\subseteq U_2$. Then following the notation of the the preceding paragraph $O_1 \times \cdots \times O_{|\vec{x}|} \subseteq (V_1)_{\vec{x}}$. It suffices to verify that $O_1 \times \cdots \times O_{|\vec{x}|} \not\subseteq (V_2)_{\vec{x}}$. Suppose not. Let $f \in \mathcal{U}_G(\vec{x}, \vec{O})$. Then there is $\vec{W}$ with $\mathcal{U}_G(\vec{x}, \vec{W}) \subseteq U_2$ and $f(x_i) \in W_i$, $\forall i \leq |\vec{x}|$. Therefore $f \in U_2$. This shows that $\mathcal{U}_G(\vec{x}, \vec{O}) \subseteq U_2$, contradicting our assumption.

It is straightforward to verify that φ is a G-map. Just notice that for any $g \in G$, $\vec{x}$ and $\vec{O}$, $g.\mathcal{U}_G(\vec{x}, \vec{O}) = \mathcal{U}_G(\vec{x}, g\vec{O})$.

It remains to show that φ is Borel measurable. For this it suffices to show that for any $\vec{x} \in Q^{<\omega}$ and V open in $X^{|\vec{x}|}$, $\{F \in \mathcal{F}(G) \mid \varphi(F)_{\vec{x}} \cap V \neq \emptyset\}$ is Borel in $\mathcal{F}(G)$. Furthermore, it is enough to check this for $V = O_1 \times \cdots \times O_{|\vec{x}|}$, a basic open set. The following claim establishes this.

Claim: $\varphi(F)_{\vec{x}} \cap (O_1 \times \cdots \times O_{|\vec{x}|}) \neq \emptyset \Leftrightarrow F \cap \mathcal{U}_G(\vec{x}, \vec{O}) \neq \emptyset$.

($\Rightarrow$) Suppose $\varphi(F)_{\vec{x}} \cap (O_1 \times \cdots \times O_{|\vec{x}|}) \neq \emptyset$. Let U be the complement of F. Using the notation before, we have that $(O_1 \times \cdots \times O_{|\vec{x}|}) \not\subseteq V_{\vec{x}}$. Therefore $\mathcal{U}_G(\vec{x}, \vec{O}) \not\subseteq U$ by the definition of $V_{\vec{x}}$. Thus $\mathcal{U}_G(\vec{x}, \vec{O}) \cap F \neq \emptyset$.

($\Leftarrow$) Suppose $\mathcal{U}_G(\vec{x}, \vec{O}) \not\subseteq U$. Then $(O_1 \times \cdots \times O_{|\vec{x}|}) \not\subseteq V_{\vec{x}}$ by the same argument as for the injectivity of φ. Therefore $\varphi(F)_{\vec{x}} \cap (O_1 \times \cdots \times O_{|\vec{x}|}) \neq \emptyset$.

This finishes the proof of the claim, hence the theorem. $\dashv$

Next we give some examples to rationalize the messy condition of Theorem 3.1.4. For an arbitrary complete metric space (X, d), let $\mathrm{Iso}(X, d)$ be the space of all isometries of (X, d), equipped with the compact-open topology. Then $\mathrm{Iso}(X, d)$ is a Polish space. If X is a compact Polish space and d is a complete metric on X, then $\mathrm{Iso}(X, d)$ is always a compact subgroup of $H(X)$.

Corollary 3.1.5 *For any compact Polish space X and complete metric d on X, $\prod_{n\in\omega} K(X^n)$ is a universal Borel $\mathrm{Iso}(X, d)$-space.*

Proof. Let $G = \mathrm{Iso}(X, d)$, 1_G be the identity and d_G be the supnorm metric on G. Let $V = \{f \in G \mid d_G(f, 1_G) < \epsilon\}$ for some $\epsilon > 0$. By Theorem 3.1.4, it suffices to find a tuple $\vec{x}$ in X and a tuple to open sets $\vec{O}$ in X such that $\mathcal{U}_G(\vec{x}, \vec{O}) \subseteq V$. For this we find an $\frac{\epsilon}{3}$-net N of X and enumerate it as $\{x_1, \ldots, x_n\}$. Let $\vec{x} = (x_1, \ldots, x_n)$. Define the corresponding $\vec{O} = (O_1, \ldots, O_n)$ by

$$O_i = \{y \in X \mid d(y, x_i) < \frac{\epsilon}{3}\}, i = 1, \ldots, n.$$

We check that $\mathcal{U}_G(\vec{x}, \vec{O}) \subseteq V$. Let $f \in \mathcal{U}_G(\vec{x}, \vec{O})$ and $x \in X$. Let $x_i \in N$ with $d(x, x_i) < \frac{\epsilon}{3}$. Then $d(f(x), f(x_i)) = d(x, x_i) < \frac{\epsilon}{3}$ because f is an isometry. Since $f(x_i) \in O_i$, $d(f(x_i), x_i) < \frac{\epsilon}{3}$. Thus

$$d(f(x), x) \leq d(f(x), f(x_i)) + d(f(x_i), x_i) + d(x_i, x) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This shows that $f \in V$. $\dashv$

We should remark that the conclusion of the above corollary is not very interesting since any uncountable Polish space on which $\mathrm{Iso}(X, d)$ trivially acts is already a universal $\mathrm{Iso}(X, d)$-space. This is because $\mathrm{Iso}(X, d)$ is compact, and any orbit equivalence relation of a compact group action is smooth. Nevertheless, the corollary essentially verifies that there is a nonempty class of Polish groups satisfying the condition of Theorem 3.1.4. The next corollary is more useful.

Corollary 3.1.6 *Let X be either the unit interval in $\mathbb{R}$ or the unit circle on the plane. Then $\prod_{n\in\omega} K(X^n)$ is a universal Borel $H(X)$-space.*

Proof. We verify the condition of Theorem 3.1.4 for the unit interval. The other case is similar.

Let $G = H(X)$ and d be the supnorm metric on G. Fix $\epsilon > 0$. It suffices to find a tuple $\vec{x}$ in X and a tuple of open sets $\vec{O}$ in X such that $\mathcal{U}_G(\vec{x}, \vec{O}) \subseteq \{f \in G \mid d(f, 1_G) < \epsilon\}$, where 1_G is the identity of G. For this let $N \subseteq [0, 1]$ be an arbitrary finite $\frac{\epsilon}{2}$-net. Enumerate N as $\{x_1, \ldots, x_n\}$. Then let $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{O} = (O_1, \ldots, O_n)$ where each $O_i =$ the open interval $(x_i - \frac{\epsilon}{2}, x_i + \frac{\epsilon}{2})$. Then $\vec{x}$, $\vec{O}$ are as required. $\dashv$

3.2 Remarks on $H([0,1])$

We saw in Corollary 3.1.6 that $\prod_{n\in\omega} K([0,1]^n)$ is a universal Borel $H([0,1])$-space. The space $\prod_{n\in\omega} K([0,1]^n)$ is in fact a compact Polish $H([0,1])$-space. It is compact because, for any compact space X, the hyperspace $K(X)$ is compact (Theorem 4.26 in [Kec1]), and so by Tychnoff's theorem, the product space $\prod_{n\in\omega} K([0,1]^n)$ is then compact. We remark that the space $\prod_{n\in\omega} K([0,1]^n)$ is not a universal Polish $H([0,1])$-space, that is, it is not the case that for any Polish $H([0,1])$-space Y there is a continuous $H([0,1])$-embedding of Y into it. In fact, there are *no* compact universal Polish $H([0,1])$-spaces.

To show this, we recall a result of Megrelishvili. In [Me] Megrelishvili constructed a Polish G-space which has no compact G-extensions. This implies that there are no compact universal Polish G-spaces. The group G in the construction is G_0^ω, where G_0 is a closed subgroup of $H([0,1])$ defined by

$$G_0 = \{f \in H([0,1]) \mid f(\frac{1}{i}) = \frac{1}{i}, \forall i \geq 1\}.$$

With a standard coding technique we can show that G is homeomorphic to a closed subgroup of $H([0,1])$. For this we construct a topological embedding of G into $H([0,1])$ as follows. First decompose the interval $[0,1]$ into infinitely many closed intervals $I_n = [2^{-n-1}, 2^{-n}]$. Then fix, for each n, a linear transformation θ_n from $[0,1]$ onto I_n. Then for $f = (f_n) \in G$, define

$$\theta(f)(x) = \theta_n f_n \theta_n^{-1}(x), \quad \text{for } x \in I_n,$$

and $\theta(f)(0) = 0$. It is easy to see that θ is a topological embedding.

Next recall a theorem of Mackey-Hjorth (Theorem 2.3.5 of [Kec1]).

Theorem 3.2.1 (Mackey-Hjorth) *Let H be a Polish group and G a closed subgroup of H. Let X be a Polish G-space. Then there is a Polish H-space Y such that*

(i) *X is a closed subset of Y;*

(ii) *Every H-orbit on Y contains exactly one G-orbit on X.*

Corollary 3.2.2 *There is no compact universal Polish $H([0,1])$-space.*

Proof. Let $H = H([0,1])$ and G be the group in Megrelishvili's theorem. We verified that G is a closed subgroup of H. Now assume there were a compact universal Polish H-space X. We show that X would be a compact universal Polish G-space, contradictory to Megrelishvili's theorem.

Let Y be an arbitrary Polish G-space. By Theorem 3.2.1 there is a Polish H-space Z such that Y is a closed subset of Z. By our assumption there is a continuous H-embedding j from Z into X. Let i be the identity embedding from

Y into Z, then i is a continuous G-embedding. Combining i and j together, the map $j \circ i$ is a continuous G-embedding from Y into X. This shows that X is a universal Polish G-space. $\dashv$

Actions of $H([0,1])$ appear to have close relationship with those of S_∞. However, it is still open whether the universal $H([0,1])$ action is reducible to S_∞ actions. Various $H([0,1])$ actions have been verified to be reducible to S_∞ actions. For example, the application actions on $[0,1]^\omega$ and $K([0,1])$, and the action by conjugacy of $H([0,1])$ on itself (see [Hj2]). In contrast to these results, $H([0,1]^2)$ is known to have turbulent actions, hence by Hjorth's turbulence theory ([Hj2]) the universal action of $H([0,1]^2)$ is not reducible to any actions of S_∞.

3.3 Action by conjugacy on $C(X,X)$

Any Polish group is isomorphic to a closed subgroup of $H(X)$ for some compact Polish space X. This is a theorem of Uspenskii (see [Kec1] for a proof). In fact the X can be taken as the Hilbert cube $[0,1]^\omega$. Thus groups of the form $H(X)$ can be viewed as forming a kind of "filter" in the class of all Polish groups. Correspondingly the actions of $H(X)$ also play a typical role in all Polish group actions.

In this section we focus on the conjugacy action of $H(X)$ on $C(X,X)$ for compact Polish spaces X. More explicitly, the action of $H(X)$ on $C(X,X)$ is defined by

$$g.f = g \circ f \circ g^{-1}, \forall g \in H(X), f \in C(X,X).$$

Throughout this section we denote the orbit equivalence relation to be E^C_H, if there is no danger of confusion what X is, instead of the clumsy $E^{C(X,X)}_{H(X)}$.

Some special cases of the conjugacy action of $H(X)$ on itself have been previously examined. The action of S_∞ on itself by conjugacy is smooth; the orbit equivalence relation of conjugacy action of $H([0,1])$ on itself is Borel equivalent to that of the universal S_∞-action (see [Hj2]). Our study of the complexity of E^C_H can be viewed as one step further along this line of research, since the space $H(X)$ is an invariant G_δ subspace of $C(X,X)$.

Proposition 3.3.1 *Let X be the one point compactification of $\mathbb{Z}$. ($H(X)$ is then homeomorphic to S_∞.) Then E^C_H is Borel.*

Proof. We first assign some countable models as complete invariants for the orbits. Specifically, we define a Borel reduction φ such that for each $f \in C(X,X)$ $\varphi(f)$ is a countable model, and such that $f E^C_H g$ iff $\varphi(f) \cong \varphi(g)$. Then we show that the isomorphism relation on the kind of countable models in question is Borel. It will be clear from the construction that the reduction φ is Borel.

Enumerate X as $\{1,2,\ldots,\infty\}$. Then for each $f \in C(X,X)$, $\varphi(f)$ is just the countable model $M_f = (X, \{\infty\}, f)$, whose signature consists of a unary relation symbol and a unary function symbol. It is easy to see that φ is a Borel reduction.

Each M_f can be regarded a directed graph, with a directed edge present from x to y if and only if $f(x) = y$. Note that $f(x) = x$ corresponds to a loop at node x. In general, if $f(x_1) = x_2, f(x_2) = x_3, \ldots, f(x_{n-1}) = x_n, f(x_n) = x_1$ and all x_i's are distinct, then there is a cycle connecting $x_1, x_2, \ldots, x_n$ and back to x_1.

We say that two nodes x and y are *connected* if there are $n, m \in \omega$ such that $f^n(x) = f^m(y)$. The directed graph M_f can be decomposed into at most countably many connected components. It is easy to check the following facts, using the fact that $f \in C(X, X)$:

(1) Each connected component C of M_f contains at most one cycle Z. In addition, for any $x \in C - Z$ and $y \in Z$, there is $n \in \omega$ such that $f^n(x) = y$. Intuitively, this means that if there is a cycle in a connected component, then the cycle lies on an end of the component.

(2) There is at most one infinitely branching node in M_f. If there is such a node b, then we always have $f(\infty) = b$. In other words, the infinitely branching node can only occur in the connected component of ∞.

(3) There is always a cycle in the the component of ∞. In fact, if $f(\infty) = \infty$, then there is a loop at ∞; otherwise, $f(\infty) = n$ for some finite $n \in \omega$, so $f^{-1}(\{n\})$ is co-finite, hence there are only finitely many values in the sequence $\{f^k(n)\}_{k \in \omega}$.

(4) If a component does not contain any cycles, then it is an infinite finitely branching directed tree (Here the term "tree" is understood in the graph theoretical sense, in particular, a tree is always connected).

Now to see that the isomorphism relation between M_f's is Borel, it is enough to verify that the following two kinds of isomorphism relations are Borel. Note that the position of ∞ is given by the unary relation in the structure, so we can focus on finitely branching part of M_f.

Claim 1: The isomorphism between finitely branching trees is Borel. For any finitely branching tree T and $x \in T$, let (T, x) be the augmented structure with constant x. Then for two such trees T_1 and T_2, $T_1 \cong T_2$ iff there are $x_1 \in T_1$ and $x_2 \in T_2$ such that $(T_1, x_1) \cong (T_2, x_2)$. Therefore it suffices to show that the isomorphism between finitely branching trees with a distinguished element is Borel. For this we treat the "upper" half U_T of (T, x) and "lower" half L_T of (T, x) separately. Rigorously, $U_T = \{y \in T \mid \exists n \in \omega(f^n(y) = x)\}$. So U_T is a finitely branching tree with an end element. Define a level function l on the nodes of U_T as follows. For $y \in U_T$, $l(y) = n$ if $f^n(y) = x$. Then for (T_1, x_1) and (T_2, x_2), define U^* to be the tree of partial isomorphisms between the levels of U_{T_1} and U_{T_2}. Then U^* is finitely branching. We make the convention that, when the two corresponding levels of U_{T_1} and U_{T_2} are both empty, the U^* at this level is constructed by trivially extending the previous level without branching. Following the similar idea, we define the concept of levels on L_T. To begin with, let the level of x in T be $l(x) = 0$. Let $D_0 = \{x\}$. Let $D_1 = \{y \in T \mid f(x) = y\}$. For $y \in D_1$, define $l(y) = 1$. In general, let $D_{i+1} = \{y \in L_T \mid f(y) \in D_i \text{ or } \exists x \in D_i(f(x) = y)\} - \bigcup\{D_j \mid j \leq i\}$. For any $y \in D_{i+1}$, let $l(y) = i + 1$. It is straightforward to see that the level function is well defined on all nodes of T, and it is invariant under isomorphisms of (T, x). Now given (T_1, x_1) and (T_2, x_2), let L^* be the tree of partial isomorphisms between the levels of L_{T_1} and L_{T_2} preserving the direction of the arrows between nodes. L^* is again a finitely branching tree. Now $(T_1, x_1) \cong (T_2, x_2)$ iff both U^* and L^* has infinite branches iff (by König's lemma) both U^* and L^* are infinite. Therefore this relation is Borel.

Inspired by the fact (1) above, let us call the directed graphs with exactly one cycle on the bottom *pseudo-trees.*

Claim 2: The isomorphism between finitely branching pseudo-trees is Borel. Here we can define a level function similarly as above, except that we start with the cycle and define the level of any node on the cycle to be 0. Again consider the partial isomorphisms between the corresponding levels of two finitely branching pseudo-trees. These partial isomorphisms constitute a finitely branching tree. The two pseudo-trees are isomorphic exactly when the comparison tree has an infinite branch. This is if and only if the comparison tree is infinite, by König's lemma. Therefore the relation is Borel. $\dashv$

Note that the proof of this fact is considerably more involving than that of the fact that the conjugacy action of S_∞ on itself is smooth. Let us also remark that if we let S_∞ act on the Baire space ω^ω by conjugacy, then the orbit equivalence relation is exactly the isomorphism relation between models with a unary function, which (by [FS]) reduces any orbit equivlance relation induced by S_∞ action. In particular, it is not Borel any more.

The proof of Proposition 3.3.1 actually shows that E_H^C is Borel equivalent to the identity of countable sets of reals. Note that the isomorphism relation between finitely branching trees is Borel equivalent to a countable equivalence relation ([HK3]), that is, an equivalence relation each of whose equivalence classes is countable. Therefore, a countable set of finitely branching trees is then correspondent to a countable set of reals by a Borel coding. In particular, this implies that E_H^C is not smooth.

With a little bit more notation and argument similar to the above proof, one can actually show that for any countable compact Polish space X, the equivalence relation E_H^C is Borel. This requires a transfinite induction on the Cantor-Bendixson rank of X. The above proof is correspondent to the simplest situation in the case when the Cantor-Bendixson rank is 2. In the general case, if the Cantor-Bendixson rank of X is $1+\alpha$, then E_H^C is Borel equivalent to the α-th iterate of the "countable sets of" operation on the set of reals. We will explain more on these equivalence relations in Section 5.3.

CHAPTER 4

Dichotomy Theorems for Linear Orderings

4.1 An overview

In the next several chapters we deal with special cases of logic actions. First we review the concepts. Let L be a countable language and let σ be a sentence in $L_{\omega_1\omega}$. Let $F \subset L_{\omega_1\omega}$ be a countable fragment containing σ. It is well known (see [Sa]) that the space of all countable models of σ, denoted by $\mathrm{Mod}(\sigma)$, can be endowed with a Polish topology τ_F, so that the isomorphism relation on the models can be naturally viewed as the orbit equivalence relation of a continuous S_∞-action on $(\mathrm{Mod}(\sigma), \tau_F)$. Let us denote this isomorphism relation by $\cong\restriction \mathrm{Mod}(\sigma)$.

The infinitary version of Vaught Conjecture is just the Silver dichotomy for $\cong\restriction \mathrm{Mod}(\sigma)$, which says that either there are countably many or there are perfectly many models up to isomorphism in $\mathrm{Mod}(\sigma)$. This conjecture is still open. However, there are instances where this dichotomy is established. For example, one can consider a sentence σ in $L_{\omega_1\omega}$, where $L = \{<\}$ contains a single binary relation symbol, all of whose models are trees. Vaught Conjecture for trees was proved by Steel ([St]). Indeed, prior to Steel's result, Vaught Conjecture for linear orderings was proved by Rubin ([Ru]), and the first-order version for models with a unary function by Miller and Marcus ([Mar]).

One can ask whether $\cong\restriction \mathrm{Mod}(\sigma)$ has the Glimm-Effros dichotomy. That is, either $\cong\restriction \mathrm{Mod}(\sigma)$ is smooth or else $E_0 \sqsubseteq_c \cong\restriction \mathrm{Mod}(\sigma)$. We will abbreviate the question as whether σ has the Glimm-Effros dichotomy. It is known that for some σ in the language of groups the Glimm-Effros dichotomy fails. (see [HK1].) Also notice that the Glimm-Effros dichotomy for σ is a statement stronger than the Vaught Conjecture for σ. Therefore, there are serious limitations on the statements we can hope to be true. As a first step, we focus on classes of σ for which the Vaught Conjecture is known (to be true). In this chapter we will actually focus on linear orderings. We establish the Glimm-Effros dichotomy theorem and make some remarks about further dichotomies.

Theorem 4.1.1 (The Glimm-Effros dichotomy for linear orderings)
Let $L = \{<\}$, where $<$ is a binary relation symbol. Let $\sigma \in L_{\omega_1\omega}$ be a sentence all of whose models are linear orderings. Then the Glimm-Effros dichotomy holds for σ.

Section 4.2 will be devoted to the proof of this theorem. Comparing to the statement of the Vaught Conjecture, the Glimm-Effros dichotomy theorems provide extra structural information for the isomorphism relations on models in these classes. This structural information can help us answer some questions asked by Friedman and Stanley ([FS]). In Chapter 6 we will formulate the questions and give a survey of the results around them.

Recall that an invariant Borel class is called *Borel-complete* if its isomorphism relation reduces arbitrary isomorphism relation on countable models. The isomorphism relation on a Borel-complete class is necessarily $\mathbf{\Sigma}^1_1$-complete as a two-dimensional set (see [FS]). A very strong dichotomy one can consider is of the form "any invariant Borel class of certain models is either Borel-complete or its isomorphism relation is Borel". In view of the Harrington-Kechris-Louveau result ([HKL]), the Glimm-Effros dichotomy follows from this dichotomy as an immediate corollary. We do not know whether this dichotomy holds for linear orderings. But let us remark that this is the best one can hope for in some sense. In Chapter 6 we shall prove that any Borel equivalence relation induced by S_∞-action is bi-reducible to the isomorphism relation on an invariant Borel class of linear orderings.

If we restrict out attention to first-order theories, then it is conceivable to obtain sharper dichotomies. Indeed, Schirmann ([Sc]) was able to prove the following dichotomy by making use of ideas in [Ru].

Theorem 4.1.2 (Schirmann) *Let T be a complete first-order theory of linear orders. Then either T is $\aleph_0$-categorical (i.e., there is exactly one countable model of T up to isomorphism) or else* $\mathrm{Mod}(T)$ *is Borel-complete.*

4.2 The Glimm-Effros dichotomy

In this section we prove the Glimm-Effros dichotomy theorem for linear orderings. The structure of the proof resembles that of Steel's proof of Vaught Conjecture for trees ([St]). The new element here is the consideration of automorphisms of the linear orderings.

First let us fix some notation. For any structure M we use $\mathrm{sr}(M)$ to denote the Scott rank of M. For ordinal α and $\vec{a} \in M$, let $\varphi_\alpha^{\vec{a},M}$ denote the canonical Scott α-type of $\vec{a}$ in M. Sometimes the superscript M is omitted if there is no confusion. Also we let φ^M denote the canonical Scott sentence of M. The basic definitions and properties of Scott ranks, canonical Scott types and Scott sentences can be found in [Ba] and [Na], in which other unexplained terminology and notation regarding Scott analysis can also be found.

Throughout the section we use L to denote the language with a single binary relation symbol $<$, unless stated otherwise for some of the more general results. If $M = (|M|, <^M)$ is a linear ordering and $a, b \in M$, we will write $a < b$ for $a <^M b$ or

$M \models a < b$, when there is no confusion. We will also freely use the abbreviations $\leq$, $>$ and $\geq$ with their usual meanings.

Let $M = (|M|, <^M)$ be a countable linear ordering. For any $a, b \in M$ with $a \leq b$, let $[a,b]^M$ denote the sub-ordering $(\{c \in M | a \leq c \leq b\}, <^M)$ and call it the *closed interval* in M with endpoints a and b. In a similar manner we can also define the *open interval* $(a,b)^M$, the *half-open half-closed intervals* $(a,b]^M$ and $[a,b)^M$, and various infinite intervals $(-\infty, a]^M$, $(a, +\infty)^M$, etc. For closed intervals it is easy to see that

$$\mathrm{sr}([a,b]^M) \leq \mathrm{sr}(M,a,b) \leq \mathrm{sr}(M) \leq \mathrm{sr}(M,a,b) + 2,$$

where (M,a,b) is the expansion of M by constants a, b. Similar inequalities hold also for other kinds of intervals.

A sub-ordering $S \subseteq M$ is called a *segment* if $\forall a,b,c \in M, (a,b \in S \wedge a \leq c \leq b \Rightarrow c \in S)$. Obviously any interval in M is a segment. We do not know whether $\mathrm{sr}(S) \leq \mathrm{sr}(M)$ for every segment S in M. We conjecture that it is true.

For each countable linear ordering M, we define the *interval rank* of M, denoted by $\mathrm{ir}(M)$, to be $\sup\{\mathrm{sr}([a,b]^M) \mid a \leq^M b\}$. It is obvious that $\mathrm{ir}(M) \leq \mathrm{sr}(M)$ by the above chain inequalities. Moreover, if M has two endpoints, then $\mathrm{ir}(M) = \mathrm{sr}(M)$.

We shall use the following theorem of Becker-Kechris ([BK]), built on earlier results of Suzuki and Miller.

Theorem 4.2.1 (Becker-Kechris) *Let L be a countable language, $F \subset L_{\omega_1\omega}$ a countable fragment and T a theory in F. Then either $E_0 \sqsubseteq_c \cong\restriction Mod(T)$ or T has a countable F-atomic model.*

The following lemma can be seen as a technical converse of the inequality $\mathrm{ir}(M) \leq \mathrm{sr}(M)$. It was originally stated in [St] under a slightly stronger hypothesis for trees. The proof is essentially the same as that appeared in [St]. For the convenience of the reader we include a complete proof here. In the statement, $\mathrm{Mod}(\equiv_\lambda M)$ stands for $\mathrm{Mod}(\varphi_\lambda^{\emptyset,M})$.

Lemma 4.2.2 (Steel) *Let M be a countable linear ordering and λ be a limit ordinal. Suppose $E_0 \not\sqsubseteq_c \cong\restriction Mod(\equiv_\lambda M)$. If $ir(M) < \lambda$, then $sr(M) \leq \lambda \cdot 3$.*

Proof. Assume $\mathrm{sr}(M) > \lambda \cdot 3$. Let F be the smallest fragment of $L_{\omega_1\omega}$ so that for all $\vec{a} \in M$ and $\alpha < \lambda \cdot 2$, $\varphi_\alpha^{\vec{a},M} \in F$. Let $T = \mathrm{Th}(M) \cap F$. By Theorem 4.2.1, T has a countable model N prime over F. By the construction of F, we have that $\mathrm{sr}(N) \geq \lambda \cdot 2$. On the other hand, by atomicity of N, $\mathrm{sr}(N) \leq \sup\{\mathrm{qr}(\psi) | \psi \in F\} = \lambda \cdot 2$. Therefore, $\mathrm{sr}(N) = \lambda \cdot 2$.

By the atomicity of N again, we can construct an F-elementary embedding j from N into M. Without loss of generality, we may assume that $\forall a \in M - j\text{"}N \exists \vec{b} \in j\text{"}N$ ($(a, \vec{b})$ satisfies no complete formula of T). (To see this, run a proof of the fact that atomic models are prime, e.g., cf. [Kei], and in the back-and-forth argument try to make the image of the elementary embedding maximal.)

Let $R = \{c \in M | \exists a, b \in j"N(c \in [a,b]^M)\}$. Then R is a segment of M. Let $\{x_i | i \in \mathbb{Z}\}$ be a cofinal sequence of N such that $x_i \leq x_{i+1}$ for all $i \in \mathbb{Z}$. Then for each $i \in \mathbb{Z}$, $[x_i, x_{i+1}]^N \cong [j(x_i), j(x_{i+1})]^R$ since their Scott ranks are $< \lambda$ and j is F-elementary. It follows that $N \cong R$. If $M = R$, then $N \cong M$, which is a contradiction since they have different Scott ranks. Therefore $M - R \neq \emptyset$. Let $S = (M - R) \cup j"N$. Then $S \prec_F M$.

Let F' be the smallest fragment of $L_{\omega_1\omega}$ such that for all $\vec{a} \in M$ and $\alpha < \lambda$, $\varphi_\alpha^{\vec{a},M} \in F'$. Let $T' = \mathrm{Th}(N) \cap F' = \mathrm{Th}(M) \cap F'$. By the same argument as before, we can find a countable model $N' \prec_{F'} N$ with $\mathrm{sr}(N') = \lambda$. Let $S' = (M-R) \cup j"N'$. Then $S' \prec_{F'} S$.

Without loss of generality we may assume that $\exists a \in j"N'(\mathrm{sr}([a, +\infty)^M) \geq \lambda \cdot 3)$ and $\exists b \in M \forall c \in j"N(c < b)$. Fix such an a and a b. Since $\mathrm{ir}(M) < \lambda$, $S \prec_F M$ and $a, b \in S$, we have $\mathrm{sr}([a,b]^S) < \lambda$. On the other hand, $(S, a, b) \equiv_\lambda (S', a, b)$. Therefore, there is an isomorphism $\pi : [a,b]^S \cong [a,b]^{S'}$. Note that π cannot map $j"N$ onto $j"N'$, since otherwise there would be an isomorphism of $\{x \in N | j(x) \geq a\}$ with $\{y \in N' | j(y) \geq a\}$, but they have different Scott ranks, a contradiction. We consider two cases.

Case 1: $\exists c \in j"N \cap [a,b]^S(\pi(c) \notin j"N')$.

Now $([a,b]^{S'}, \pi(c)) \equiv_\lambda ([a,b]^S, \pi(c))$. It follows that there is an isomorphism τ between these structures. Now $\tau \circ \pi$ is an automorphism of $[a,b]^S$ mapping c to $\pi(c)$, where $c \in j"N$ and $\pi(c) \notin j"N'$. Note that in fact $\pi(c) \notin j"N$.

Now by the construction of j we can find $\vec{d} \in j"N$ so that $(\pi(c), \vec{d})$ satisfies, in M, hence in S, no complete formula of T. Let ρ be an automorphism of S trivially extending $\tau \circ \pi$. Let $\vec{e} = \rho^{-1}(\vec{d})$. Then $\vec{e} \in j"N$. But then $(c, \vec{e}) = \rho^{-1}(\pi(c), \vec{d})$, so $(c, \vec{e})$ satisfies in S, hence in M, no complete formula of T. This is a contradiction to $c \in j"N$.

Case 2: $\exists c \in [a,b]^S - j"N(\pi(c) \in j"N')$.

The proof is similar to the proof for Case 1. ⊣

Steel conjectured that the lemma might still be true without the hypothesis about $\cong \upharpoonright \mathrm{Mod}(\equiv_\lambda M)$. This is still open. We should mention that if this conjecture is verified, then our proof of the main theorem could be put much shorter, if not simpler.

It turns out that the E_0-nonembeddability condition is a nuisance in many parts of our proof. To deal with this let us recall the following dichotomy theorem of Becker and Hjorth-Kechris ([HK1]), which is applicable to all the equivalence relations we consider in this paper.

Theorem 4.2.3 (Becker, Hjorth-Kechris) *Let G be a Polish group acting in a Borel manner on a Polish space X, with the orbit equivalence relation E. Then exactly one of the following holds:*

(I) *There is a map $U : X \to 2^{<\omega_1}$ which is C-measurable in the codes such that for any $x, y \in X$, $xEy \Leftrightarrow U(x) = U(y)$.*

(II) $E_0 \sqsubseteq_c E$.

Moreover, (II) *is equivalent to:*

(II)' *There exists an E-ergodic, non-atomic probability Borel measure on X, where E-ergodic means that every E-invariant Borel set has measure 0 or 1 and E-non-atomic means that every E-equivalence class has measure 0.*

It follows from the theorem that if E_0 is reducible to E via a C-measurable function then in fact $E_0 \sqsubseteq_c E$. We will use this fact implicitly in the sequel.

The next lemma is a preparation to Lemma 4.2.5. First, let us recall some notation. For linear orderings M and N, we define the *concatenation* of M and N by

$$M + N = (|M| \dot{\cup} |N|, <^{M+N})$$

where $\dot{\cup}$ means disjoint union and, assuming the domains are disjoint for simplicity, for $a, b \in |M| \cup |N|$,

$$\begin{aligned} a <^{M+N} b \quad \Leftrightarrow \quad & ((a \in M \wedge b \in N) \\ & \vee (a \in M \wedge b \in M \wedge a <^M b) \\ & \vee (a \in N \wedge b \in N \wedge a <^N b)). \end{aligned}$$

The operation of concatenation is associative, up to isomorphism.

Lemma 4.2.4 *Let $M = A + N + B$ be the concatenation of linear orderings A, N and B. Let α be an ordinal. Suppose for any N_0, N_1 with $N_0 \equiv_\alpha N_1 \equiv_\alpha N$ and for any isomorphism $\pi : A + N_0 + B \cong A + N_1 + B$, $\pi``N_0 \subseteq N_1$. Then $\cong \restriction Mod(\equiv_\alpha N) \leq_B \cong \restriction Mod(\equiv_\alpha M)$. In particular, if $E_0 \sqsubseteq_c \cong \restriction Mod(\equiv_\alpha N)$, then $E_0 \sqsubseteq_c \cong \restriction Mod(\equiv_\alpha M)$.*

Proof. The hypothesis implies that if $N_0 \equiv_\alpha N_1 \equiv_\alpha N$ and π is an isomorphism between $A + N_0 + B$ and $A + N_1 + B$, then $\pi \restriction N_0$ is an isomorphism between N_0 and N_1. Now define a Borel mapping Φ:Mod($\equiv_\alpha N$) $\rightarrow$ Mod($\equiv_\alpha M$) by putting $\Phi(N_0) = A + N_0 + B$, for any $N_0 \equiv_\alpha N$. Then Φ witnesses that $\cong \restriction \mathrm{Mod}(\equiv_\alpha N) \leq_B \cong \restriction \mathrm{Mod}(\equiv_\alpha M)$. The second conclusion follows from the first and Theorem 4.2.3. $\dashv$

The following technical lemma is one of the two main ingredients in our main proof. Note that the hypothesis of this lemma is weaker than that of Lemma 4.2.2.

Lemma 4.2.5 *Let M be a countable linear ordering and λ be a limit ordinal. Suppose $E_0 \not\sqsubseteq_c \cong \restriction Mod(\equiv_\lambda M)$. Suppose*

$$\forall a, b \in M (\varphi_\lambda^{a,M} = \varphi_\lambda^{b,M} \Longrightarrow sr([a, b]^M) < \lambda).$$

Then $sr(M) \leq \lambda \cdot 7$.

Proof. For any $a \in M$, let

$$S^0_a = \bigcup\{[b,c]^M | a \in [b,c]^M \wedge \varphi_\lambda^{b,M} = \varphi_\lambda^{c,M}\}.$$

For $i \in \omega$, let $S^{i+1}_a = \bigcup\{S^0_b | b \in S^i_a\}$ and $S_a = \bigcup_{i\in\omega} S^i_a$. Then we have the following facts.

(i) $a \in S^0_a \subseteq \ldots \subseteq S^i_a \subseteq S^{i+1}_a \subseteq \ldots \subseteq S_a$. Moreover, if $S^i_a = S^{i+1}_a$ for some i, then $S_a = S^i_a$.

(ii) All S^i_a and S_a are segments in M. For S^i_a this follows from an easy induction on i, for each a. Then by (i), S_a is a segment.

(iii) For all $b \in S_a$ and $c \in M$, if $\varphi_\lambda^{b,M} = \varphi_\lambda^{c,M}$, then $c \in S_a$. This is because if $b \in S^i_a$ then in fact $c \in S^{i+1}_a$.

(iv) If $b \in S_a$, then $S_b \subseteq S_a$. Suppose $b \in S^i_a$. An easy induction on $j \in \omega$ shows that $S^j_b \subseteq S^{i+j+1}_a$.

(v) If $b \notin S_a$, then $S_b \cap S_a = \emptyset$. We prove by induction on i that $S^i_b \cap S_a = \emptyset$. For notational simplicity assume $b < a$. First consider the base step $i = 0$. Assume there are $d_0 \leq c \leq d_1$ so that $b \in [d_0, d_1]^M$, $\varphi_\lambda^{d_0,M} = \varphi_\lambda^{d_1,M}$ and $c \in S_a$. Note that $a \notin [d_0, d_1]$ since otherwise $b \in S^0_a$, a contradiction. It follows that $c \leq d_1 < a$. By (ii), $d_1 \in S_a$. Then by (iii), $d_0 \in S_a$, whence $b \in S_a$ by (ii) again, a contradiction. This finishes the proof for the base step. For inductive steps the proof is similar.

(vi) $\{S_a | a \in M\}$ is a partition of M into disjoint segments. It is enough to show that if $b \in S_a$ then $S_b = S_a$. On direction is provided by (iv). Assume $S_a \not\subseteq S_b$. By (iv) again, $a \notin S_b$. By (v), $S_a \cap S_b = \emptyset$, a contradiction.

(vii) If τ is an automorphism of M so that $\tau(a) = b$, then $S_a = S_b$. This follows easily from the hypothesis and our construction of S_a.

(viii) For any $b, c \in S_a$ with $b \leq c$, $\mathrm{sr}([b,c]^M) < \lambda$. It is enough to prove by induction on i that $\mathrm{sr}([b,c]^M) < \lambda$ for $b, c \in S^i_a$ with $b \leq c$. Consider the base step $i = 0$. Suppose $a, b \in [b_0, b_1]^M$, $a, c \in [c_0, c_1]^M$ with $\varphi_\lambda^{b_0,M} = \varphi_\lambda^{b_1,M}$ and $\varphi_\lambda^{c_0,M} = \varphi_\lambda^{c_1,M}$. A typical case is $b_0 \leq b \leq a \leq c \leq c_1$. By the hypothesis, $\mathrm{sr}([b,a]^M) < \lambda$ and $\mathrm{sr}([a,c]^M) < \lambda$. Since λ is a limit ordinal, it is easy to see that $\mathrm{sr}([b,c]^M) < \lambda$. For inductive steps the proof is similar.

(ix) $\mathrm{sr}(S_a) \leq \lambda \cdot 6$. First we verify that the hypothesis of Lemma 4.2.4 holds with S_a in place of N and $\lambda \cdot 2$ in place of α. Let $N_0 \equiv_{\lambda\cdot 2} N_1 \equiv_{\lambda\cdot 2} S_a$ and let $\pi : M(N_0) \cong M(N_1)$ where $M(N_i)$ is the linear ordering obtained from M by replacing S_a with N_i, for $i = 0, 1$. Note that $M(N_i) \equiv_{\lambda\cdot 2} M$, for $i = 0, 1$. Let $x \in N_0 \subseteq M(N_0)$. If $\pi(x) \notin N_1$, then $\pi(x) \in M(N_1) - N_1 = M - S_a$. Since π is an isomorphism, $\varphi_\lambda^{x,M(N_0)} = \varphi_\lambda^{\pi(x),M(N_1)} = \varphi_\lambda^{\pi(x),M}$. Let $y \in S_a$ be such that $\varphi_\lambda^{y,S_a} = \varphi_\lambda^{x,N_0}$. Then $\varphi_\lambda^{y,M} = \varphi_\lambda^{x,M(N_0)} = \varphi_\lambda^{x,M}$. By (iii), this is a contradiction to the construction of S_a. Now it follows from our assumption that $E_0 \not\leq_c \cong\restriction \mathrm{Mod}(\equiv_{\lambda\cdot 2} M)$. Applying Lemma 4.2.4 we get that $E_0 \not\leq_c \cong\restriction \mathrm{Mod}(\equiv_{\lambda\cdot 2} S_a)$. By (viii) and Lemma 4.2.2, $\mathrm{sr}(S_a) \leq \lambda \cdot 6$.

Now suppose $\mathrm{sr}(M) > \lambda \cdot 7$. Then there are $\vec{a}, \vec{b} \in M$ such that $\varphi_{\lambda\cdot 7}^{\vec{a},M} = \varphi_{\lambda\cdot 7}^{\vec{b},M}$ but there is no automorphism of M mapping $\vec{a}$ to $\vec{b}$. Without loss of generality we may assume $S_{a_i} = S_{b_j}$, for all $1 \leq i, j \leq n$. (Say $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$). This is because any automorphism of M respects the partition $\{S_x | x \in M\}$ by (vii), and conversely, any automorphism of S_x trivially extends to an automorphism of M, by (ii). Let $S = S_{a_i}$. We will arrive at a contradiction if we can show that $\varphi^{\vec{a},S}_{\lambda\cdot 6} = \varphi^{\vec{b},S}_{\lambda\cdot 6}$, since by (ix) there would be an automorphism of S mapping $\vec{a}$ to $\vec{b}$. For this we show a more general claim.

Claim: For any α and $\vec{a}, \vec{b} \in S$, if $(M, \vec{a}) \equiv_{\lambda+\alpha} (M, \vec{b})$, then $(S, \vec{a}) \equiv_\alpha (S, \vec{b})$.

We prove the claim by induction on α. The proof is obvious when $\alpha = 0$ or α is a limit. Suppose $(M, \vec{a}) \equiv_{\lambda+\alpha+1} (M, \vec{b})$. Let $c \in S$. There is $d \in M$ such that $(M, \vec{a}, c) \equiv_{\lambda+\alpha} (M, \vec{b}, d)$. In particular, $\varphi^{c,M}_\lambda = \varphi^{d,M}_\lambda$. Therefore by (iii), $d \in S$. We have just shown that

$$\forall c \in S \exists d \in S\ (M, \vec{a}, c) \equiv_{\lambda+\alpha} (M, \vec{b}, d)$$

and by symmetry

$$\forall d \in S \exists c \in S\ (M, \vec{a}, c) \equiv_{\lambda+\alpha} (M, \vec{b}, d).$$

By inductive hypothesis, we have

$$\forall c \in S \exists d \in S\ (S, \vec{a}, c) \equiv_\alpha (S, \vec{b}, d)$$

and

$$\forall d \in S \exists c \in S\ (S, \vec{a}, c) \equiv_\alpha (S, \vec{b}, d).$$

This is just $(S, \vec{a}) \equiv_{\alpha+1} (S, \vec{b})$. ⊣

The computation of the Scott rank in this lemma is obviously not optimal, but it is good enough for our purpose.

Next we develop a technique to obtain elementary submodels of an arbitrary segment in a linear ordering. The elementary submodel will have (approximately) desired interval ranks. First we have the following very general lemma.

Lemma 4.2.6 *Let L be an arbitrary countable language and M a countable model in L. Let λ be an ordinal and α be a limit ordinal with $\lambda \leq \alpha \leq sr(M)$. Suppose $E_0 \not\leq_c \cong\restriction Mod(\equiv_\lambda M)$. Then there is a countable model N such that $N \prec_\alpha M$ and $sr(N) = \alpha$.*

Proof. Let F be the smallest fragment of $L_{\omega_1\omega}$ such that for all $\vec{a} \in M$ and $\beta < \alpha$, $\varphi^{\vec{a},M}_\beta \in F$. Let $T = \mathrm{Th}(M) \cap F$. Since $E_0 \not\leq_c \cong\restriction \mathrm{Mod}(\equiv_\lambda M)$, we have $E_0 \not\leq_c \cong\restriction \mathrm{Mod}(\equiv_\alpha M)$. By Theorem 4.2.1 we get a countable F-atomic model N with $N \prec_F M$. Therefore $N \prec_\alpha M$.

Since $\alpha \leq \mathrm{sr}(M)$, $\mathrm{sr}(N) \geq \alpha$ by the choice of F. On the other hand, by atomicity of N, every tuple $\vec{b} \in N$ satisfies a complete formula in F. Since $\sup\{\mathrm{qr}(\theta) | \theta \in F\} = \alpha$, $\mathrm{sr}(N) \leq \alpha$. Therefore $\mathrm{sr}(N) = \alpha$. ⊣

However, the use of Lemma 4.2.6 is restricted, since we do not know how to derive the E_0-nonembeddability condition for arbitrary segments. There is only one specific case where we do know the answer positively, that is, for the intervals of a linear ordering, the E_0-nonembeddability condition passes along. This is the content of the following lemma.

Lemma 4.2.7 *Let M be a countable linear ordering and $a, b \in M$ with $a \leq b$. If $E_0 \sqsubseteq_c \cong\restriction \mathit{Mod}(\equiv_\lambda [a,b]^M)$, then $E_0 \sqsubseteq_c \cong\restriction \mathit{Mod}(\equiv_\lambda M)$.*

Proof. Denote $X = \mathrm{Mod}(\equiv_\lambda [a,b]^M)$, $Y = \mathrm{Mod}(\equiv_\lambda M)$ and $E_X = \cong\restriction X$, $E_Y = \cong\restriction Y$. By Theorem 4.2.3 there is an E_X-ergodic, non-atomic probability Borel measure μ on X. Let $\Phi : X \to Y$ be defined as follows: for any $N \equiv_\lambda [a,b]^M$, let $\Phi(N)$ be obtained from M by replacing $[a,b]^M$ by N. Then Φ is Borel measurable. Let $\nu = \Phi\mu$ be the image measure on Y. Then ν is a probability Borel measure. We check that ν is E_Y-ergodic and non-atomic. By Theorem 4.2.3 again, this implies $E_0 \sqsubseteq_c E_Y$.

ν is E_Y-ergodic because $\Phi(N) \cong \Phi(N')$ whenever $N \cong N'$. To see ν is E_Y-non-atomic, just notice that, for any E_Y equivalence class $\mathcal{C}$, $\Phi^{-1}\mathcal{C}$ consists of at most countably many E_X equivalence classes. This is because, for any $M \in \mathcal{C}$, if $N \in \Phi^{-1}(M)$ then N is isomorphic to a closed interval in M. But since M is countable, there are at most countably many closed intervals in M. $\dashv$

The following lemma is another main ingredient in our proof.

Lemma 4.2.8 *Let M be a countable linear ordering and S be a segment in M with $ir(S) \geq \lambda$, where λ is a limit ordinal. Suppose $E_0 \not\sqsubseteq_c \cong\restriction \mathit{Mod}(\equiv_\lambda M)$. Then for any limit ordinal η with $\lambda \leq \eta \leq ir(S)$, there is a countable model N such that $N \prec_\lambda S$ and $\eta \leq ir(N) \leq \eta + \omega$.*

Proof. Fix a limit η with $\lambda \leq \eta \leq \mathrm{ir}(S)$. Let $\{a_i\}_{i \in \mathbb{Z}}$ be a cofinal, increasing sequence in S such that for some $i \in \mathbb{Z}$, $\mathrm{sr}([a_i, a_{i+1}]^M) \geq \eta$. By Lemma 4.2.7, $E_0 \not\sqsubseteq_c \cong\restriction \mathrm{Mod}(\equiv_\lambda [a_i, a_{i+1}]^M)$ for every $i \in \mathbb{Z}$. Applying Lemma 4.2.6 to each $[a_i, a_{i+1}]^M$ whose Scott rank is $\geq \eta$ we may obtain a λ-elementary submodel $N_i \prec_\lambda [a_i, a_{i+1}]^M$ with $\mathrm{sr}(N_i) = \eta$. Let N be obtained from S by substituting N_i for each applicable $[a_i, a_{i+1}]^M$. Then $N \prec_\lambda S$. Obviously $\mathrm{ir}(N) \geq \eta$. To compute the upper bound for $\mathrm{ir}(N)$, let $x, y \in N$ with $x \leq y$. The interval $[x,y]^N$ consists of at most finitely many $[a_i, a_{i+1}]^M$ or N_i. A direct computation shows that $\mathrm{sr}([x,y]^N) \leq \eta + \omega$. Hence $\mathrm{ir}(N) \leq \eta + \omega$. $\dashv$

We will need the following theorem which is essentially due to Sacks. The reader can find a quick proof of it in [St]; a formulation of the theorem closer to ours can be found in [Mak]; the equivalence of these formulations follows from a theorem of Becker-Kechris ([BK]). The notations are explained in [St].

Theorem 4.2.9 (Sacks) *Let L be an arbitrary countable language and $\sigma \in L_{\omega_1 \omega}$ be a sentence. Suppose there is $\alpha < \omega_1$ such that for any countable model M of σ, whenever $sr(M) \geq \alpha$, $sr(M) < \omega_1{}^M$. Then $\cong\restriction \mathit{Mod}(\sigma)$ is Borel.*

Thus by the result of Harrington-Kechris-Louveau in [HKL] $\cong\restriction \mathrm{Mod}(\sigma)$ has the Glimm-Effros property.

Recall that E_0 is bi-reducible to $E_0(\mathbb{Z})$, where $E_0(\mathbb{Z})$ is the equivalence relation on $2^{\mathbb{Z}}$ defined by

$$xE_0(\mathbb{Z})y \ \Leftrightarrow\ \exists s \in \mathbb{Z}\forall t \in \mathbb{Z}(x(t) = y(s+t)).$$

For notational simplicity we will also write E_0 for $E_0(\mathbb{Z})$.

Now we are ready to prove the main theorem of this chapter. *Proof of Theorem 4.1.1.* Assume the theorem fails; let σ be a counterexample.

Claim 1: For any $\alpha < \omega_1$, there is a countable model M of σ and there are $a, b \in M$ such that $\varphi^{a,M}_{{\omega_1}^M} = \varphi^{b,M}_{{\omega_1}^M}$ and $\mathrm{sr}([a,b]^M) = \mathrm{sr}(M) = {\omega_1}^M > \alpha$.

Otherwise, let $\alpha_0 < {\omega_1}^M$ be such that for any countable model M of σ with $\mathrm{sr}(M) = {\omega_1}^M > \alpha_0$, we have

$$\forall a, b \in M(\ \mathrm{sr}([a,b]^M) < {\omega_1}^M \ \vee\ \exists\beta < {\omega_1}^M(\varphi^{a,M}_{\beta} \neq \varphi^{b,M}_{\beta})).$$

By boundedness, the occurrences of ${\omega_1}^M$ in the above formula can be replaced by some $\beta_0 < {\omega_1}^M$ for each M. For each M with $\mathrm{sr}(M) > \alpha_0$, let β_M be the smallest ordinal $< {\omega_1}^M$ such that

$$\forall a, b \in M(\ \mathrm{sr}([a,b]^M) < \beta_M \ \vee\ \varphi^{a,M}_{\beta_M} \neq \varphi^{b,M}_{\beta_M}).$$

Regarding each M as a real $x \in 2^\omega$, we then have

$$(\forall x \in 2^\omega)(x \models \sigma \Longrightarrow (\mathrm{sr}(x) \leq \alpha_0 \ \vee\ (\exists e \in \omega)(\ \{e\}^M \in WO \ \wedge\ |\{e\}^M| = \beta_M)).$$

This is of the form $\forall x \exists e P(x,e)$ where P is $\mathbf{\Pi}^1_1$. By the Easy Uniformization Theorem (see [Mo]) there is a Borel $f : 2^\omega \to \omega$ such that $\forall x P(x, f(x))$. Then $\{\ \{f(x)\}^x \mid x \in 2^\omega\}$ is a $\mathbf{\Sigma}^1_1$ set of wellorders. Hence there is a countable limit ordinal $\gamma > \alpha_0$ such that $\forall x(|\{f(x)\}^x| < \gamma)$. This is to say that for all countable models M of σ, either $\mathrm{sr}(M) < {\omega_1}^M$, or

$$\forall a, b \in M(\varphi^{a,M}_{\gamma} = \varphi^{b,M}_{\gamma} \Longrightarrow \mathrm{sr}([a,b]^M) < \gamma).$$

Now applying Lemma 4.2.5 we get that, for any countable model M of σ, either $\mathrm{sr}(M) < {\omega_1}^M$ or $\mathrm{sr}(M) \leq \gamma \cdot 7$. By Theorem 4.2.9, $\cong\restriction \mathrm{Mod}(\sigma)$ is Borel, hence σ has the Glimm-Effros property. $\dashv$

For the rest of the proof we fix $\lambda = \mathrm{qr}(\sigma) + \omega$ and a countable model M of σ given by Claim 1 such that $\mathrm{sr}(M) > \lambda$. Fix $a_0, a_1 \in M$ with $\varphi^{a_0,M}_{{\omega_1}^M} = \varphi^{a_1,M}_{{\omega_1}^M}$ and $\mathrm{sr}([a_0,a_1]^M) = {\omega_1}^M$. By the property of Scott types there is an automorphism τ of M such that $\tau(a_0) = a_1$.

For any $a, b \in M$, let $a \sim b$ iff $(a = b \vee (a < b \wedge \mathrm{sr}([a,b]^M) < {\omega_1}^M) \vee (a > b \wedge \mathrm{sr}([b,a]^M) < {\omega_1}^M))$. Then $\sim$ is an equivalence relation whose equivalence classes are segments in M. For each $a \in M$, let $C_a = [a]_\sim$ and call it the *component* of a. Note that for $a \leq b$, $C_a \cap C_b = \emptyset$ iff $\mathrm{sr}([a,b]^M) = {\omega_1}^M$. Let $\mathcal{C} = (\{C_a | a \in M\}, <^{\mathcal{C}})$ where $C_a <^{\mathcal{C}} C_b \Leftrightarrow (a < b \wedge C_a \neq C_b)$. Then $\mathcal{C}$ is the linear ordering of the nonempty components of M. When there is no confusion we omit the superscript $\mathcal{C}$ in $<^{\mathcal{C}}$. By boundedness $\mathrm{ir}(C) < {\omega_1}^M$ for every component $C \in \mathcal{C}$.

Claim 2: $\mathcal{C}$ is dense.

Otherwise assume $C_a, C_b \in \mathcal{C}$ such that $C_a < C_b$ and there is no $C \in \mathcal{C}$ with $C_a < C < C_b$. Then for any $c \in [a,b]^M$, $\mathrm{sr}([a,c]^M) < {\omega_1}^M$ or $\mathrm{sr}([c,b]^M) < {\omega_1}^M$. By boundedness, we have $\gamma < {\omega_1}^M$ such that $\mathrm{sr}([a,c]^M) < \gamma$ or $\mathrm{sr}([c,b]^M) < \gamma$. Let $I = \{c \in [a,b]^M \,|\, \mathrm{sr}([a,c]^M) < \gamma\}$. Then I is definable over $[a,b]^M$ from a formula of Scott rank $< \gamma + \omega$. Let $J = [a,b]^M - I$. By Lemma 4.2.7, $E_0 \not\sqsubseteq_c \cong\restriction \mathrm{Mod}(\equiv_\lambda [a,b]^M)$. It follows that $E_0 \not\sqsubseteq_c \cong\restriction \mathrm{Mod}(\equiv_\lambda I)$. By Lemma 4.2.2, $\mathrm{sr}(I) < {\omega_1}^M$. A similar argument shows that $\mathrm{sr}(J) < {\omega_1}^M$. Hence $\mathrm{sr}([a,b]^M) < {\omega_1}^M$, which is a contradiction to $C_a \neq C_b$. $\dashv$

Let $M_l = \{a \in M \,|\, \forall s \in \mathbb{Z}(a < \tau^s(a_0))\}$. We may assume that either $M_l = \emptyset$ or M_l is a single component with $\mathrm{ir}(M_l) \leq \lambda \cdot 2$. This is because, in case $M_l \neq \emptyset$, we can apply Lemma 4.2.8 to obtain M_l' with $M_l' \prec_\lambda M_l$ and $\mathrm{ir}(M_l') \leq \lambda \cdot 2$. Let M' be the model obtained from M by replacing M_l with M_l'. Then $M' \equiv_\lambda M$ and $\mathrm{Mod}(\equiv_\lambda M') = \mathrm{Mod}(\equiv_\lambda M)$. We then consider M' instead of M from now on. Let $M_r = \{a \in M \,|\, \forall s \in \mathbb{Z}(a > \tau^s(a_0))\}$. For similar reasons we may assume that either $M_r = \emptyset$ or M_r is a single component with $\mathrm{ir}(M_r) \leq \lambda \cdot 2$.

The rest of the proof is devoted to showing that $E_0 \sqsubseteq_c \cong\restriction \mathrm{Mod}(\equiv_\lambda M)$, which is a contradiction to the hypothesis. For this we consider two cases.

Case 1: For any $\alpha < {\omega_1}^M$, $\{C_d \,|\, \mathrm{ir}(C_d) > \alpha\}$ is dense in $[C_a, C_b]^{\mathcal{C}}$.

First note that if $\tau(a) = b$ then $\tau\text{''}C_a = C_b$. By the case hypothesis, there is a component $D_0 \in [C_a, C_b]^{\mathcal{C}}$ with $\mathrm{ir}(D_0) \geq \lambda \cdot 7$. For $s \in \mathbb{Z}$ let $D_s = \tau^{s}\text{''}D_0$. Then $< D_s | s \in \mathbb{Z} >$ is a sequence of components such that $\forall s \in \mathbb{Z}$

(i)$D_s = \emptyset$ and $D_s \cap D_{s+1} = \emptyset$,
(ii)$\tau\text{''}D_s = D_{s+1}$, $\tau^{-1}\text{''}D_s = D_{s-1}$, and
(iii)$\mathrm{ir}(D_s) \geq \lambda \cdot 7$.

Let $\{K_i | i \in \omega\}$ be an enumeration of all components of M for which $\lambda \cdot 3 \leq \mathrm{sr}(K_i) \leq \lambda \cdot 6, \forall i \in \omega$ and $K_i \ncong K_j, \forall i,j \in \omega, i \neq j$. By Lemma 4.2.8 we obtain K_i' such that $\mathrm{ir}(K_i') \leq \lambda \cdot 2$ and $K_i' \prec_\lambda K_i$. By Lemma 4.2.8 again, let $N_0, N_1 \prec_\lambda D_0$ with $\lambda \cdot 3 \leq \mathrm{ir}(N_0) \leq \lambda \cdot 4$ and $\lambda \cdot 5 \leq \mathrm{ir}(N_1) \leq \lambda \cdot 6$.

Given $x \in 2^{\mathbb{Z}}$. Let M_x be the model obtained from M by replacing each component C with C', where C' is defined by

. if $C \cong K_i$ for some $i \in \omega$, let $C' \cong K_i'$;

. if $C = D_s$ for some $s \in \mathbb{Z}$, then let

$$C' \cong \begin{cases} N_0 & \text{if } x(s) = 0, \\ N_1 & \text{if } x(s) = 1. \end{cases}$$

. in all other cases $C' = C$.

It is easy to see that for each $x \in 2^{\mathbb{Z}}$, $M_x \equiv_\lambda M$. In order to see $\forall x, y \in \mathbb{Z}$, $xE_0y \Leftrightarrow M_x \cong M_y$, it is enough to prove the following claim.

Claim 3: For any $a, b \in M_x$, $\mathrm{sr}([a,b]^{M_x}) < {\omega_1}^M$ iff there is a C' with $a, b \in C'$.

By our case hypothesis, if a and b are not in the same C' then there are densely many $C \in \mathcal{C}$ in between a and b which are not changed by the above definition. In particular for any $\alpha < {\omega_1}^M$ there is a subinterval of $[a,b]^{M_x}$ with Scott rank $\geq \alpha$. This implies that $\mathrm{sr}([a,b]^{M_x}) \geq {\omega_1}^M$. $\dashv$

Finally the assignment $x \mapsto M_x$ is easily seen to be Borel. So we obtain a reduction from E_0 to $\cong \restriction \mathrm{Mod}(\equiv_\lambda M)$. This finishes the proof of Case 1.

Case 2: There is $\alpha < {\omega_1}^M$, $\{C_d \mid \mathrm{ir}(C_d) > \alpha\}$ is not dense in $[C_a, C_b]^{\mathcal{C}}$.

Fix such an $\alpha_0 > \lambda \cdot 2$. We may assume that there is no $C \in \mathcal{C}$ with $\mathrm{ir}(C) > \alpha_0$. This is because we can consider another model M' instead of M which is defined as follows. Let $d, e \in [a_0, a_1]^M$ with $C_d < C_e$ such that there is no $C \in (C_d, C_e)^{\mathcal{C}}$ with $\mathrm{ir}(C) > \alpha_0$. Let S be the smallest segment in M with $C_e, C_{\tau(d)} \subseteq S$. Let $S' \prec_\lambda S$ be given by Lemma 4.2.8 with $\mathrm{ir}(S') \leq \lambda \cdot 2$. Let M' be obtained from M by replacing each segment $\tau^s(S)$ by an isomorphic copy of S', for all $s \in \mathbb{Z}$. Then $M' \equiv_\lambda M$. Moreover, there is no component in M' with interval rank $> \alpha_0$, since each component in M' is isomorphic to one of the components in the set $(C_d, C_e)^{\mathcal{C}} \cup \{S', M_l, M_r\}$, by an argument similar to the proof of Claim 3.

Now let D_0 be the smallest segment in M containing C_{a_0} and C_{a_1}. Then $\mathrm{ir}(D_0) = {\omega_1}^M$. Now for $s \in \mathbb{Z}$ let $D_s = \tau^{s}{}'' D_0$. This defines a sequence $< D_s \mid s \in \mathbb{Z} >$ of segments satisfying the properties (i) and (ii) as in Case 1, and satisfying

(iv) $\mathrm{ir}(D_s) \geq \alpha_0 \cdot 6$.

By Lemma 4.2.8 get $N_0, N_1 \prec_\lambda D_0$ with $\alpha_0 \cdot 2 \leq \mathrm{ir}(N_0) \leq \alpha_0 \cdot 3$ and $\alpha_0 \cdot 4 \leq \mathrm{ir}(N_1) \leq \alpha_0 \cdot 5$. Given $x \in 2^{\mathbb{Z}}$, let M_x be the model obtained from M by replacing each D_s by D'_s, where

$$D'_s \cong \begin{cases} N_0 & \text{if } x(s) = 0, \\ N_1 & \text{if } x(s) = 1. \end{cases}$$

The rest of the proof is similar to Case 1. $\dashv$

CHAPTER 5

Dichotomy Theorems for Simple Trees

5.1 Scott ranks and automorphisms

Because there are many concepts of trees, it is necessary to give a formal definition of the kind of trees we are considering here. Following Steel [St], a structure $\mathcal{T} = (T, <^{\mathcal{T}})$ is called a *tree* if $<^{\mathcal{T}}$ is a partial order on T such that for any $t \in T$, $<^{\mathcal{T}}$ linearly orders the set $\{s \in T \mid s <^{\mathcal{T}} t\}$. In this chapter we focus on a subclass. A countable tree $\mathcal{T}$ is called *simple* if it has a least element and for any $t \in T$, the set $\{s \in T \mid s <^{\mathcal{T}} t\}$ is finite.

Note that our tree concept is quite different from that in graph theory, where a tree is defined to be an acyclic graph. The major difference is that our trees contain all linear orderings, which is not the case for graph theoretic trees. The simple trees are closer to the graph theoretical concept. One can view a simple tree as just a graph theoretic tree with a distinguished element, namely the root. In fact simple trees were called countable rooted trees of height $\leq \omega$ in [FS].

Throughout this chapter we fix a language $L = \{<\}$ with a binary relation symbol $<$. The class of all simple trees, which we denote by $\mathbb{T}_\omega$, is indeed an invariant Borel class. In this chapter we investigate the isomorphism relation on invariant Borel subclasses of $\mathbb{T}_\omega$ and prove some dichotomy results.

In the rest of this section we prove some lemmas on the Scott ranks and automorphisms of simple trees. The object is to prepare enough terminology and facts for the proof of the key lemma in the next section. However, some of the lemmas proved here might also be interesting in their own rights. First let us introduce some ad hoc notation for simple trees.

Let $\mathcal{T}$ be a simple tree and t be an element of $\mathcal{T}$. We let $\mathcal{T}_t$ denote the structure $(\{s \in \mathcal{T} \mid s <^{\mathcal{T}} t\}, <^{\mathcal{T}})$, that is, the subtree of $\mathcal{T}$ rooted by t. $\mathcal{T}_t$ is also a simple tree. We define the *level* of t, $l(t)$, to be the cardinality of the set $\{s \in \mathcal{T} \mid s <^{\mathcal{T}} t\}$. The *parent* of t, $p(t)$, is the immediate predecessor of t in $\mathcal{T}$. An element $s \in \mathcal{T}$ is a *child* of t if $p(s) = t$. For any $t_1, t_2 \in \mathcal{T}$, the *meet* of t_1 and t_2, $m(t_1, t_2)$, is defined to be the largest element $s \in \mathcal{T}$ with $s \leq^{\mathcal{T}} t_1$ and $s \leq^{\mathcal{T}} t_2$. If $m(t_1, t_2)$ is neither t_1 nor t_2, then t_1 and t_2 are *incomparable*. For $n \geq 1$, we define an equivalence relation

$\sim_{\mathcal{T},n}$ on all n-tuples of $\mathcal{T}$ as follows. For $\vec{s} = (s_0, \ldots, s_{n-1})$ and $\vec{t} = (t_0, \ldots, t_{n-1})$,

$$\vec{s} \sim_{\mathcal{T},n} \vec{t} \iff \text{there is an automorphism } \tau : \mathcal{T} \to \mathcal{T} \text{ such that } \tau(\vec{s}) = \vec{t},$$
$$\text{i.e., } \tau(s_i) = t_i \text{ for all } i < n.$$

The automorphism τ in the definition is called a *witness* for $\vec{s} \sim_{\mathcal{T},n} \vec{t}$. If τ is a witness for $s \sim_{\mathcal{T},1} t$, where $s, t \in \mathcal{T}$, then it is called a *minimal witness* for $s \sim_{\mathcal{T},1} t$ if $\tau(t) = s$ and $\tau(u) = u$ for any $u \in \mathcal{T}$ with $m(u,s), m(u,t) \leq^{\mathcal{T}} m(s,t)$. When there is no danger of confusion we drop some of the superscripts and subscripts for notational simplicity.

We first establish some facts about the equivalence relation $\sim$.

Lemma 5.1.1 *If $s, t \in \mathcal{T}$ and $s \sim t$, then there is a minimal witness for $s \sim t$.*

Proof. Let τ be an arbitrary witness for $s \sim t$. We construct a minimal witness τ_0. First note that $l(s) = l(t)$ and $\tau(p(s)) = p(t)$ since τ is an automorphism. It follows from an induction that $\tau(m(s,t)) = m(s,t)$, and therefore $\mathcal{T}_{m(s,t)}$ is invariant under τ. For $u \notin \mathcal{T}_{m(s,t)}$, let $\tau_0(u) = u$. Let s' be the child of $m(s,t)$ with $s' \leq s$ and let $t' = \tau(s')$. Then τ sends $\mathcal{T}_{s'}$ to $\mathcal{T}_{t'}$. For $u \in \mathcal{T}_{m(s,t)} - \mathcal{T}_{s'} - \mathcal{T}_{t'}$, let $\tau_0(u) = u$. For $u \in \mathcal{T}_{s'}$, let $\tau_0(u) = \tau(u)$. For $u \in \mathcal{T}_{t'}$, let $\tau_0(u) = \tau^{-1}(u)$. It is then easy to check that τ_0 is a minimal witness for $s \sim t$. $\dashv$

Lemma 5.1.2 *Let $\mathcal{T}$ be a simple tree and $\vec{s}, \vec{t}$ are n-tuples of elements of $\mathcal{T}$, where $n > 1$. Then $\vec{s} \sim \vec{t}$ if and only if both the following conditions hold:*

(i) for any $\vec{s'} = (s_{i_0}, \ldots, s_{i_k})$, a k-tuple (for some $k < n$) which is a proper subsequence of $\vec{s}$, and its corresponding tuple $\vec{t'} = (t_{i_0}, \ldots, t_{i_k})$, $\vec{s'} \sim \vec{t'}$, and

(ii) for any $i, j < n$, $l(m(s_i, s_j)) = l(m(t_i, t_j))$.

Proof. The "only if" direction is obvious. For the "if" direction, consider the set $M = \{m(s_i, t_i) \,|\, i < n\}$. Let w_0 be a maximal element of M. We consider two cases.

CASE 1. $M \neq \{w_0\}$. Without loss of generality we can assume that $w_0 = m(s_i, t_i)$ exactly when $i < p$ for some $p < n$. Then for any $i \geq p$, $s_i, t_i \notin \mathcal{T}_{w_0}$. This is because, assuming $s_i \in \mathcal{T}_{w_0}$, we have $m(s_i, s_0) \geq w_0$, hence by (ii), $m(t_i, t_0) \geq w_0$ since $l(m(t_i, t_0)) = l(m(s_i, s_0)) \geq l(w_0)$; this shows that $t_i \in \mathcal{T}_{w_0}$, hence $m(s_i, t_i) \geq w_0$; by the maximality of w_0, $m(s_i, t_i) = w_0$, hence by our assumption $i < p$. Now by (i) we have $(s_0, \ldots, s_{p-1}) \sim (t_0, \ldots, t_{p-1})$ and $(s_p, \ldots, s_{n-1}) \sim (t_p, \ldots, t_{n-1})$. Moreover, by a similar argument as in the proof of the preceding lemma, there is an automorphism τ_0 witnessing the first equivalence and such that $\mathcal{T}_{w_0}$ is invariant under τ_0 and $\tau_0(u) = u$ for all $u \notin \mathcal{T}_{w_0}$. To finish the proof of this case, it is enough to show that the second equivalence has a witness τ_1 with $\tau_1(w_0) = w_0$. Let τ be an arbitrary witness for $(s_p \ldots, s_{n-1}) \sim (t_p, \ldots, t_{n-1})$. Focusing on the nontrivial case, suppose $\tau(w_0) \neq w_0$. Let $w_1 = m(w_0, \tau(w_0))$ and w_2 be the child of w_1 with $w_2 \leq w_0$. Then $\tau(w_1) = w_1$ and $\tau(w_2) \neq w_2$. Let $w_2' = \tau^{-1}(w_2)$. Then $\mathcal{T}_{w_2} \cup \mathcal{T}_{w_2'}$ does not contain any $m(s_j, t_j)$ for $j \geq p$, since any such $m(s_j, t_j)$ is fixed by τ. Now define τ_1: for $u \notin \mathcal{T}_{w_2} \cup \mathcal{T}_{w_2'}$, let $\tau_1(u) = \tau(u)$; for $u \in \mathcal{T}_{w_2}$, let $\tau_1(u) = u$; for $u \in \mathcal{T}_{w_2'}$, let $\tau_1(u) = \tau^2(u)$. Then τ_1 is as required.

CASE 2. $M = \{w_0\}$. By (1) there is an automorphism τ witnessing that $(s_1, \ldots, s_{n-1}) \sim (t_1, \ldots, t_{n-1})$. We construct a witness τ_0 for $\vec{s} \sim \vec{t}$. If $\tau(s_0) = t_0$, there is nothing to prove and we let $\tau_0 = \tau$. Suppose $\tau(s_0) = s_0' \neq t_0$. Let w_0 be the child of $m(s_0', t_0)$ with $w_0 \leq t_0$, let u_0' be the child of $m(s_0', t_0)$ with $u_0' \leq s_0'$, and let $u_0 = \tau^{-1}(u_0')$. Since $s_0 \sim t_0$, we have $s_0' \sim t_0$ and hence by the preceding lemma there is a minimal witness τ_1. Let $\tau_0 = \tau_1 \circ \tau$. We verify that τ_0 is as required. Obviously, $\tau_0(s_0) = t_0$. We only have to check that $\tau_0(s_i) = t_i$ for all $i \geq 1$. For this first notice that $\mathcal{T}_{u_0}$ does not contain any s_i, $i \geq 1$. This is because, if $s_i \in \mathcal{T}_{u_0}$, then $m(s_0, s_i) \in \mathcal{T}_{u_0}$ and $l(m(s_0, s_i)) \leq l(u_0)$; by (ii), $l(m(t_0, t_i)) \leq l(u_0) = l(w_0) = l(u_0')$, hence $t_i \in \mathcal{T}_{w_0}$; but $\tau(s_i) = t_i$ and $\tau(u_0) = u_0'$, therefore $t_i \in \mathcal{T}_{u_0'}$, this is a contradiction. For a similar reason $\mathcal{T}_{w_0}$ does not contain any t_i, $i \geq 1$. Now fix $i \geq 1$. We have $\tau_0(s_i) = \tau_1(\tau(s_i)) = \tau_1(t_i)$. But $t_i \notin \mathcal{T}_{w_0}$ and $t_i \notin \mathcal{T}_{u_0'}$ since $s_i \notin \mathcal{T}_{u_0}$. Therefore $\tau_1(t_i) = t_i$ by the minimality of τ_1. This finishes the proof of the lemma. $\dashv$

Lemma 5.1.3 *Let $\mathcal{T}$ be a simple tree and $\vec{s}, \vec{t}$ are n-tuples of elements of $\mathcal{T}$. Then $\vec{s} \sim \vec{t}$ if and only if both the following conditions hold:*

(i) for any $i < n$, $s_i \sim t_i$, and

(ii) for any $i, j < n$, $l(m(s_i, s_j)) = l(m(t_i, t_j))$.

Proof. By an induction on n and use the preceding lemma for the inductive step. $\dashv$

The following lemma relates the above study of automorphisms to the Scott ranks of simple trees. For a comprehensive treatise of Scott analysis see [Ba]. For a simple tree $\mathcal{T}$, we denote the Scott rank of $\mathcal{T}$ by $\mathrm{sr}(\mathcal{T})$. If α is an ordinal and $\vec{t} \in \mathcal{T}$, we denote the canonical Scott α-type of $\vec{t}$ by $\varphi_\alpha^{\vec{t}, \mathcal{T}}$. The canonical Scott sentence is denoted by $\varphi^{\mathcal{T}}$. Sometimes the superscript $\mathcal{T}$ is omitted if there is no confusion. We let ${\omega_1}^M$ denote the ordinal height of the least admissible set A containing $\mathcal{T}$, that is, $A = L_{{\omega_1}^M}[\mathcal{T}]$. The countable fragment of $L_{\omega_1 \omega}$ within A is then denoted by L_A.

Lemma 5.1.4 *Let $\mathcal{T}$ be a simple tree and A be the least admissible set containing $\mathcal{T}$. The followings are equivalent:*

(i) $sr(\mathcal{T}) < {\omega_1}^M$,

(ii) for any $\vec{t} \in \mathcal{T}$, the set $\{\vec{s} \in \mathcal{T} \mid \vec{s} \sim \vec{t}\}$ is in A,

(iii) for any $t \in \mathcal{T}$, the set $\{s \in \mathcal{T} \mid s \sim t\}$ is in A.

Proof. We show that (i)⇒(iii)⇒(ii).

For (i)⇒(iii), note that $s \sim t \iff \varphi^s_\alpha = \varphi^t_\alpha$, where $\alpha = \mathrm{sr}(\mathcal{T}) < {\omega_1}^M$.

(iii)⇒(ii) follows from the preceding lemma.

For (ii)⇒(i), let $\alpha = \mathrm{sr}(\mathcal{T})$ and let $P_{\vec{t}} = \{\vec{s} \in \mathcal{T} \,|\, \vec{s} \sim \vec{t}\} \in A$. We then have that the structure $(\mathcal{T}, \vec{t}, P_{\vec{t}}) \in A$. Since $\vec{s} \sim \vec{t} \iff \varphi^{\vec{s}}_\alpha = \varphi^{\vec{t}}_\alpha$, $P_{\vec{t}}$ is definable over the structure $(\mathcal{T}, \vec{t})$ by some formula of $L_{\omega_1\omega}$ without any other parameters. By Theorem VII.7.5 of [Ba], $P_{\vec{t}}$ is definable over $(\mathcal{T}, \vec{t})$ by a formula $\psi^{\vec{t}}$ of L_A without any other parameters. Let $\gamma(\vec{t})$ be the rank of $\psi^{\vec{t}}$. Then $\gamma(\vec{t}) < {\omega_1}^M$. Moreover, $\vec{s} \sim \vec{t} \iff \varphi^{\vec{s}}_{\gamma(\vec{t})} = \varphi^{\vec{t}}_{\gamma(\vec{t})}$. Let $\gamma_0(\vec{t})$ be the least ordinal γ such that for any $\vec{s}$, $\vec{s} \sim \vec{t} \iff \varphi^{\vec{s}}_\gamma = \varphi^{\vec{t}}_\gamma$. Let $\beta = \sup\{\gamma_0(\vec{t}) \,|\, \vec{t} \in \mathcal{T}\}$. By boundedness, $\beta < {\omega_1}^M$. But $\mathrm{sr}(\mathcal{T}) \leq \beta$, since if $(\mathcal{T}, \vec{s}) \equiv_\beta (\mathcal{T}, \vec{t})$, then $\vec{s} \sim \vec{t}$. This shows that $\mathrm{sr}(\mathcal{T}) < {\omega_1}^M$. ⊣

Lemma 5.1.5 *Let $\mathcal{T}$ be a simple tree with sr$(\mathcal{T}) = {\omega_1}^M$. Then there is a $t \in \mathcal{T}$ with sr$(\mathcal{T}_t) = {\omega_1}^M$ and infinitely many $s \in \mathcal{T}$ with $s \sim t$.*

Proof. Assume not. We then show that $\mathrm{sr}(\mathcal{T}) < {\omega_1}^M$. By the preceding lemma it is enough to check that for any $t \in \mathcal{T}$, the set $P_t = \{s \in \mathcal{T} \,|\, s \sim t\}$ is in the least admissible set A containing $\mathcal{T}$. We show this by induction on $l(t)$. If $l(t) = 0$ then t is the root, therefore $P_t = \{t\} \in A$. Suppose $l(t) > 0$. If P_t is finite, then $P_t \in A$. If P_t is infinite, by our assumption $\mathrm{sr}(\mathcal{T}_t) < {\omega_1}^M$. Let $\gamma < {\omega_1}^M$ be a limit ordinal bigger than $\mathrm{sr}(\mathcal{T}_t)$. We claim that $s \sim t \iff p(s) \sim p(t)$ and $\varphi^s_\gamma = \varphi^t_\gamma$. To see this, note that the second condition on the right hand side implies that $\mathcal{T}_s$ and $\mathcal{T}_t$ are isomorphic to each other. Thus if τ is a witness for $p(s) \sim p(t)$, we can find τ_1 a minimal witness for $\tau(s) \sim t$. Letting $\tau_0 = \tau_1 \circ \tau$, τ_0 is then a witness for $s \sim t$. This proves the claim. The claim together with the inductive hypothesis shows that $P_t \in A$. ⊣

5.2 A strong dichotomy theorem

In this section we prove the following dichotomy theorem: For any invariant Borel class of simple trees, either its isomorphism relation is Borel or else it is Borel complete. We give a complete proof of the theorem here and leave the implications and remarks to the next section.

The plan of proof is as follows. We start with an $L_{\omega_1\omega}$ sentence σ describing an invariant Borel class of simple trees. Suppose the quantifier rank of this sentence is λ. Suppose $\cong\restriction \mathrm{Mod}(\sigma)$ is non-Borel. First we obtain a tree $\mathcal{T}$ in $\mathrm{Mod}(\sigma)$ which is complicated in the sense that it has a lot of automorphisms moving its high rank subtrees around. Then we manipulate these high rank subtrees of $\mathcal{T}$ to obtain a lot of different trees, in fact the number of which equals the number of all simple trees. The effect is that the class of trees λ-equivalent to $\mathcal{T}$, denoted $\mathrm{Mod}(\equiv_\lambda \mathcal{T})$, can code all simple trees in a faithful manner. This coding will provide a Borel reduction of $\mathbb{T}_\omega$ into Mod($\equiv_\lambda \mathcal{T}$), thus proving that the class Mod(σ) is Borel complete. We carry out this plan and gradually clarify the vague terms in the rest of the section.

The following lemma is the key lemma we promised to give near the beginning of section 5.1. The lemma guarantees the existence of a complicated tree we need in our proof of the main theorem. In the statement of the lemma a subtree of a simple tree $\mathcal{T}$ means a substructure of $\mathcal{T}$ that is closed under $<^{\mathcal{T}}$.

Lemma 5.2.1 *Let $\mathcal{T}$ be a simple tree with $sr(\mathcal{T}) = {\omega_1}^M$. Then $\mathcal{T}$ contains a subtree $\mathcal{T}^0$ with the following properties:*

(i) For any $t \in \mathcal{T}^0$, $sr(\mathcal{T}_t) = {\omega_1}^M$;

(ii) Any $t \in \mathcal{T}^0$ has either one child or infinitely many children in $\mathcal{T}^0$;

(iii) For any $t \in \mathcal{T}^0$, there is $u \in \mathcal{T}^0$ such that $u \geq t$ and u has infinitely many children in $\mathcal{T}^0$;

(iv) For any $s, t \in \mathcal{T}^0$, if $l(s) = l(t)$, then $s \sim_{\mathcal{T}} t$;

Proof. We construct $\mathcal{T}^0$ in ω many stages. At the end of each stage n we obtain a subtree $\mathcal{T}^0_n$ of finite height satisfying (i) and (iv). By extending $\mathcal{T}^0_n$ to $\mathcal{T}^0_{n+1}$ we will meet the requirements (ii) and (iii). Let T^0_0 be the empty subtree of $\mathcal{T}$.

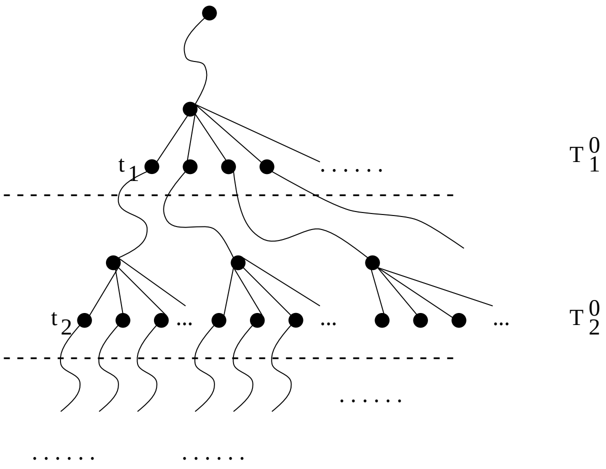

Figure 5.1: The construction of $\mathcal{T}^0$.

By Lemma 5.1.5, there is a $t_1 \in \mathcal{T}$ with $\mathrm{sr}(\mathcal{T}_{t_1}) = {\omega_1}^M$ and there are infinitely many $s \in \mathcal{T}$ with $s \sim t_1$. Without loss of generality we may assume that there are infinitely many such s with $p(s) = p(t_1)$. This is because, if the assumption fails for t_1 we can instead consider $p(t_1)$, which satisfies also that $\mathrm{sr}(\mathcal{T}_{p(t_1)}) = {\omega_1}^M$ and there are infinitely many s with $s \sim p(t_1)$. Note that $l(p(t_1)) = l(t_1) - 1$. By an easy induction the procedure stops before we reach the root of $\mathcal{T}$.

Now let $\mathcal{T}_1^0 = \{s \in \mathcal{T} \mid s < t_1 \text{ or } (s \sim t_1 \text{ and } p(s) = p(t_1)\}$. Then (i) and (iv) hold. For any $t \in \mathcal{T}_1^0$ that is not a terminal node (ii) and (iii) also follow from the definition. Note that $\mathcal{T}_{t_1}$ is definable from $(\mathcal{T}, t_1)$ by a quantifier free formula, so $\omega_1^{\mathcal{T}_{t_1}} \leq \omega_1{}^M = \mathrm{sr}(\mathcal{T}_{t_1})$. Hence by Nadel, $\omega_1^{\mathcal{T}_{t_1}} = \omega_1{}^M$.

Now suppose $\mathcal{T}_n^0$ has been defined. Fix t_n an arbitrary terminal node in $\mathcal{T}_n^0$. Consider $\mathcal{T}_{t_n}$. By Lemma 5.1.5 and the argument above, we can find $t_{n+1} < t_n$ with $\mathrm{sr}(\mathcal{T}_{t_{n+1}}) = \omega_1{}^M$ and such that there are infinitely many $s \in \mathcal{T}$ with $s \sim t$ and $p(s) = p(t_{n+1})$. Let $\{t_n^i\}_{i \in \omega}$ be an enumeration of all terminal nodes of $\mathcal{T}_n^0$, with $t_n^0 = t_n$. For each $i \in \omega$, let τ_i be a witness for $t_n \sim t_n^i$. Let $S = \{s \in \mathcal{T} \mid s < t_{n+1} \text{ or } (s \sim t_{n+1} \text{ and } p(s) = p(t_{n+1})\}$. Let $\mathcal{T}_{n+1}^0 = \mathcal{T}_n^0 \cup \bigcup\{\tau_i(S) \mid i \in \omega\}$. Then (i) and (iv) hold for elements in $\mathcal{T}_{n+1}^0$ and (ii) and (iii) hold for elements in $\mathcal{T}_n^0$.

Eventually let $\mathcal{T}^0$ be the increasing union of all $\mathcal{T}_n^0$. $\mathcal{T}^0$ is as required. $\dashv$

If $t \in \mathcal{T}^0$ has infinitely many children in $\mathcal{T}^0$, we say that t is *(infinitely) splitting.* Otherwise, t has only one child in $\mathcal{T}^0$, and we say it is *non-splitting.* For any $t \in \mathcal{T}^0$, we let $l^0(t)$ denote the cardinality of the set $\{s \in \mathcal{T}^0 \mid s < t \text{ and } s \text{ is splitting}\}$ and call it the *relative level* of t in $\mathcal{T}^0$.

We now proceed to the proof of our main theorem. At this point let me warn the reader that some parts of the proof are going to be written in vague terms rather than rigorous notions. For some reason we feel that this is a better way to give the proof. Due to the nature of the subject, any attempt to make the argument more formal is bound to increase the complexity of the notation. Yet we feel that it is better to give a clearer intuition than to bury the reader into messy and less motivated arguments.

Let σ be an $L_{\omega_1\omega}$ sentence describing an invariant Borel class of simple trees. Let $\lambda > \omega$ be a countable limit ordinal bigger than the quantifier rank of σ. Suppose that $\cong\restriction \mathrm{Mod}(\sigma)$ is non-Borel. By Sack's theorem (Theorem 4.2.9) there is a model $\mathcal{T}$ of σ with $\mathrm{sr}(\mathcal{T}) = \omega_1{}^M > \lambda$. Let $\mathcal{T}^0$ be the subtree of $\mathcal{T}$ given by the preceding lemma.

We make use of the following lemma of Steel.

Lemma 5.2.2 (Steel) *Let $\mathcal{T}$ be a simple tree with $sr(\mathcal{T}) = \omega_1{}^M$ and let $\lambda < \omega_1{}^M$ be a limit ordinal. Then $id(2^\omega) \leq_B \cong\restriction Mod(\equiv_\lambda \mathcal{T})$.*

In the statement $\mathrm{Mod}(\equiv_\lambda \mathcal{T})$ is an abbreviation of $\mathrm{Mod}(\varphi_\lambda^{\emptyset,\mathcal{T}})$. For a proof of the lemma, see [St]. We define an infinite sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_i, \ldots$ of simple trees by induction on i. For each $i \in \omega$, we fix an arbitrary $t_i \in \mathcal{T}^0$ with $l^0(t_i) = i+1$ and $p(t_i)$ splitting. $\mathcal{U}_i$ shall be chosen from $\mathrm{Mod}(\equiv_\lambda \mathcal{T}_{t_i})$ so that it is not isomorphic to any $\mathcal{T}_t$, for $t \in \mathcal{T}$. In addition, $\mathcal{U}_i$ shall not be isomorphic to any $(\mathcal{U}_j)_t$, for $j < i$ and $t \in \mathcal{U}_j$. Since there are at most countably many trees for $\mathcal{U}_i$ to avoid, such a $\mathcal{U}_i$ exists by the preceding lemma.

We are now ready to code $\mathbb{T}_\omega$ into $\mathrm{Mod}(\equiv_\lambda \mathcal{T})$. Given an arbitrary simple tree $\mathcal{S}$, we construct a simple tree $\tilde{\mathcal{S}}$ by replacing some subtrees of $\mathcal{T}$ by some $\mathcal{U}_i$. The construction will ensure that for any simple trees $\mathcal{S}_1$ and $\mathcal{S}_2$,

$$\mathcal{S}_1 \cong \mathcal{S}_2 \;\Leftrightarrow\; \tilde{\mathcal{S}}_1 \cong \tilde{\mathcal{S}}_2.$$

Here is the construction of $\tilde{\mathcal{S}}$.

First we associate a $t(s) \in \mathcal{T}^0$ with each $s \in \mathcal{S}$ so that the following properties hold:

(1) $p(t(s))$ is splitting,
(2) $l^0(p(t(s))) = l(s)$,
(3) if $s' \neq s$, then $t(s')$ and $t(s)$ are incomparable,
(4) if $s' < s$, then $p(t(s')) < p(t(s))$.

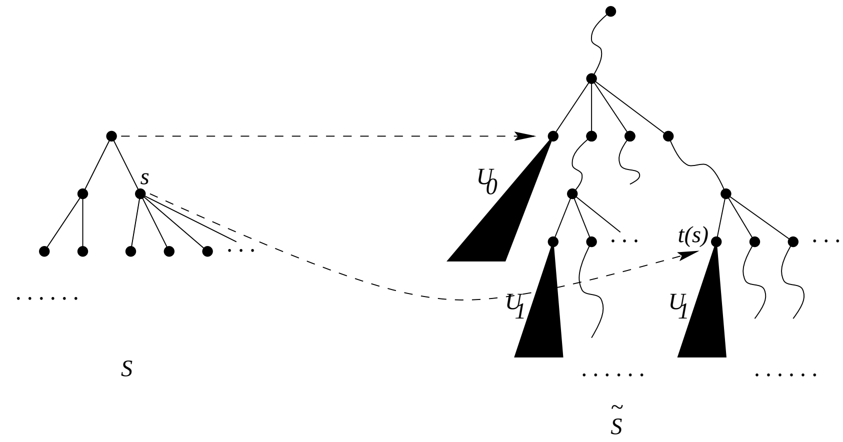

Figure 5.2: The construction of $\tilde{\mathcal{S}}$.

It is quite easy to see that the assignment can be carried out with a lot of free room. Intuitively, the assignment is constructed in a top down manner for elements of $\mathcal{S}$. For each $s \in \mathcal{S}$, we first find a corresponding splitting element in $\mathcal{T}^0$ whose relative level is the same as $l(s)$. This element should be greater than $p(t(p(s)))$ and incomparable with any other $t(s')$. Then we assign $t(s)$ to be a child of this element.

$\tilde{\mathcal{S}}$ is then obtained from $\mathcal{T}$ by replacing all $\mathcal{T}_{t(s)}$ by $\mathcal{U}_{l(s)}$. This finishes the construction.

We claim that $\mathcal{S} \mapsto \tilde{\mathcal{S}}$ is in fact a Borel reduction function. It is apparently Borel because all the models involved in the construction has Scott rank $\leq {\omega_1}^M < \omega_1$ (see [BK] for the theorem relating boundedness of Scott ranks with Borelness). Let $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{T}_\omega$. It is also easy to see that if $\mathcal{S}_1 \cong \mathcal{S}_2$, then $\tilde{\mathcal{S}}_1 \cong \tilde{\mathcal{S}}_2$. In fact any isomorphism between $\mathcal{S}_1$ and $\mathcal{S}_2$ naturally induces an isomorphism between $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}_2$. This was made possible by Lemma 5.2.1 (iv). To see that if $\tilde{\mathcal{S}}_1 \cong \tilde{\mathcal{S}}_2$ then

$\mathcal{S}_1 \cong \mathcal{S}_2$, it is enough to show that $\mathcal{S}$ can in fact be recovered from $\tilde{\mathcal{S}}$, as done by the following procedure.

First we search for $t \in \tilde{\mathcal{S}}$ of smallest level with $\tilde{\mathcal{S}}_t \cong \mathcal{U}_0$. If no such t exists then $\mathcal{S}$ is empty. If such a t exists then it follows from the definition of the sequence $\{\mathcal{U}_i\}_{i\in\omega}$ that it is unique. This t corresponds to the root of $\mathcal{S}$. Next we remove $\tilde{\mathcal{S}}_t$ from $\tilde{\mathcal{S}}$ and in the intersection of the remaining part with $\tilde{\mathcal{S}}_{p(t)}$ search for occurrences of $\mathcal{U}_1$ in a similar manner as above. The occurrences of $\mathcal{U}_1$ correspond to the first level element of $\mathcal{S}$, again by the definition of $\{\mathcal{U}_i\}_{i\in\omega}$. In general, elements of level n in $\mathcal{S}$ are recovered in the n-th step of this procedure. And eventually all of $\mathcal{S}$ can be recovered. Therefore, if $\tilde{\mathcal{S}}_1 \cong \tilde{\mathcal{S}}_2$, then the above procedures recover isomorphic trees, thus $\mathcal{S}_1 \cong \mathcal{S}_2$.

Thus we have proven the theorem:

Theorem 5.2.3 *Let σ be an $L_{\omega_1\omega}$ sentence all of whose models are simple trees. Then either $\cong\restriction$ Mod(σ) is Borel or Mod(σ) is Borel complete.*

5.3 Corollaries and remarks

Recall that the Glimm-Effros dichotomy holds for Borel equivalence relations ([HKL]). Hence it follows immediately from Theorem 5.2.3 that the Glimm-Effros dichotomy holds for simple trees.

Corollary 5.3.1 (Glimm-Effros dichotomy for simple trees)
Let σ be an $L_{\omega_1\omega}$ sentence all of whose models are simple trees. Then the Glimm-Effros dichotomy holds for σ.

It is also possible to prove the Glimm-Effros dichotomy theorem directly by a modification of the proof in the preceding sections. Let us point out the main idea. By Lemma 5.1.5 we obtain a tree with infinitely many nodes leading identical subtrees. Moreover, we may assume that these infinitely many nodes have a common parent. Then by Lemma 5.2.2 each of the subtree is capable of coding a real. Thus putting these together, the original tree is then capable of coding a countable set of reals. This shows that the identity of countable sets of reals is Borel reducible to $\cong\restriction$ Mod(σ), which implies that $E_0 \leq_B \cong\restriction \mathrm{Mod}(\sigma)$. So the direct proof of the Glimm-Effros dichotomy avoids the complicated Lemma 5.2.1 and the embedding constructed in the main proof of section 5.2.

Another corollary of Theorem 5.2.3 is about a question of Friedman and Stanley. In [FS] the authors defined an ω_1 sequence $\{\mathbb{S}_\alpha\}_{\alpha\in\omega_1}$ of invariant Borel classes of simple trees. Each $\mathbb{S}_\alpha$ is just the class of wellfounded trees of rank α. They proved that

$$\mathbb{S}_0 <_B \mathbb{S}_1 <_B \mathbb{S}_2 <_B \ldots <_B \mathbb{S}_\alpha <_B \ldots$$

Hence the isomorphism relations are strictly increasing in terms of Borel reducibility. They also noticed another way to view the tower, that is, $\mathbb{S}_2$ can be identified with ω, $\mathbb{S}_3$ with the reals (in the Cantor space), $\mathbb{S}_4$ with the space of countable sets of reals, $\mathbb{S}_5$ with the space of countable sets of countable sets of reals, etc. The isomorphism relations between trees are then viewed as the identity relations of the corresponding objects. Moreover, each of the spaces can be given a topology which is Polish.

The Scott sentence of any countable model can be viewed as an element of $\mathbb{S}_\alpha$, where α is the Scott rank of the model. Thus the Scott analysis assigns to each countable model an element of some classes in the above tower. When we consider an invariant Borel class whose isomorphism relation is Borel, this assignment is then a Borel reduction by boundedness. This shows that the Friedman-Stanley tower dominates all Borel isomorphism relations on invariant Borel classes of countable models.

Recall the following definition of *α-completeness*: an invariant Borel class $\mathbb{A}$ is α-complete if $\mathbb{S}_\alpha \leq_B \mathbb{A}$. A basic question of the subject is whether an invariant Borel class is Borel complete if it is α-complete for all $\alpha < \omega_1$. Our theorem answers the question for invariant Borel classes of simple trees.

Corollary 5.3.2 *Let $\mathbb{A}$ be an invariant Borel class of simple trees. If $\mathbb{A}$ is α-complete for all $\alpha < \omega_1$, then $\mathbb{A}$ is Borel complete.*

Proof. If $\cong\restriction \mathbb{A}$ is Borel then $\mathbb{A} \leq_B \mathbb{S}_\alpha$ for some $\alpha < \omega_1$, by the above observation about Scott analysis. The hypothesis then implies that $\cong\restriction \mathbb{A}$ is non-Borel. Therefore $\mathbb{A}$ is Borel complete by Theorem 5.2.3. $\dashv$

Note that Theorem 5.2.3 cannot be generalized to arbitrary countable models. In fact, let $\mathbb{A}$ be the class of countable abelian torsion groups. The $\cong\restriction \mathbb{A}$ is not Borel (Theorem 6 of [FS]), yet $\mathbb{S}_4 \not\leq_B \mathbb{A}$ (Theorem 5 of [FS]). (This latter fact is strengthened in [HK1] to $E_0 \not\leq_B \cong\restriction \mathbb{A}$.) Hence $\mathbb{A}$ is not Borel complete.

CHAPTER 6

Dichotomy Theorems for Models with One Unary Function

6.1 The strong dichotomy

In this chapter we fix a language $L = \{R\}$, where R is a binary relation symbol. We consider models of language L with the property

$$\forall x \forall y \forall z (\, R(x,y) \wedge R(x,z) \Rightarrow y = z).$$

This is an invariant Borel class, which we denote by $\mathbb{U}$. The relation can be viewed as a partial unary function on the model. Let us denote this partial function by F, i.e., $F(x) = y \Leftrightarrow R(x,y)$. We will use the notation F^n, $n \in \omega$, in the intuitive sense

$$F^0(x) = x,$$

$$F^n(x) = y \quad \Leftrightarrow \quad \exists x_1 \dots \exists x_n (\, R(x,x_1) \wedge \cdots \wedge R(x_{n-1},x_n) \wedge x_n = y), \;\; n > 0$$

Then one can define a partial order $<_F$ as follows

$$x <_F y \Leftrightarrow \exists n > 0 (F^n(y) = x) \wedge \forall m (F^m(x) \neq y).$$

For any simple tree, the parent function is a partial unary function on it. Therefore we are considering models more general than simple trees. With some help from the proof of Theorem 5.2.3, we prove in the rest of this section the following strong dichotomy theorem for invariant Borel subclasses of $\mathbb{U}$.

Theorem 6.1.1 *Let σ be an $L_{\omega_1\omega}$ sentence such that Mod(σ) $\subseteq \mathbb{U}$. Then either $\cong\restriction$ Mod(σ) is Borel or else Mod(σ) is Borel complete.*

First we need some notation. Let $M \in \mathbb{U}$. For any $x, y \in M$, x and y are said *connected* if there are $n, m \in \omega$ such that $F^n(x) = F^m(y)$. For each $x \in M$, the *(connected) component* of x is $C_x = \{y \in M \mid x \text{ and } y \text{ are connected}\}$. It is easy to check that being connected is an equivalence relation and the components are the equivalence classes, therefore giving a partition of M. M is *connected* if every pair of elements in M are connected. There are only three kinds of components C, as follows:

Type I: There is no $<_F$-minimal element in C. With respect to $<_F$, C is an infinite tree without root. This happens when $F(x)$ is always defined for any $x \in C$ and $F^n(x) \neq x$, for any $n > 0$ and $x \in C$.

Type II: There are more than one $<_F$-minimal elements in C. In this case the $<_F$-minimal elements constitute a finite directed cycle with respect to F. This happens when $F(x)$ is always defined for any $x \in C$ and $F^n(x) = x$ for some $x \in C$ and $n > 1$.

Type III: There is a $<_F$-least element in C. With respect to $<_F$, C is just a simple tree. This happens either when $F(x) = x$ for some $x \in C$ or when $F(x)$ is not defined for some $x \in C$. In either cases the special element is the root.

For each $x \in M$, let T_x be the structure with domain $\{y \in M \mid x <_F y \vee x = y\}$ and the partial order $<_F$. Then T_x is always a simple tree.

Lemma 6.1.2 *Let $M \in \mathbb{U}$ be such that $sr(M) = {\omega_1}^M$. Then there is $x \in M$ such that $sr(T_x) = {\omega_1}^M$.*

Proof. We first claim that there must be a component C with $\mathrm{sr}(C) = {\omega_1}^M$. Assume not. Then for any $x \in M$, $\mathrm{sr}(C_x) < {\omega_1}^M$. By boundedness, there is a limit ordinal $\gamma < {\omega_1}^M$ such that $\mathrm{sr}(C_x) < \gamma$, $\forall x \in M$. We demonstrate that $\mathrm{sr}(M) < \gamma + \omega$, hence a contradiction. For this it suffices to show that for any $\vec{a}, \vec{b} \in M$, if $(M, \vec{a}) \equiv_{\gamma+\omega} (M, \vec{b})$ then there is an automorphism τ of M with $\tau(\vec{a}) = \vec{b}$. Without loss of generality we may assume all elements in $\vec{a}$ are in a single component, say C_0. Then it follows that all elements in $\vec{b}$ are in a single component as well, say C_1. Moreover $(C_0, \vec{a}) \equiv_\gamma (C_1, \vec{b})$. Since $\mathrm{sr}(C_0) < \gamma$, there is an isomorphism $\tau : C_0 \to C_1$ with $\tau(\vec{a}) = \vec{b}$. It is then obvious that τ can be extended to an automorphism of M.

Therefore we may assume that M is connected. To see that there is $x \in M$ with $\mathrm{sr}(T_x) = {\omega_1}^M$, we consider the three types of M. If M is of type III, there is nothing to prove. If M is of type II, then M is the union of finitely many simple trees with their roots tied together by a cycle. One of these simple trees is as required. The only non-trivial case is when M is of type I.

Suppose M is of type I and suppose that for any $x \in M$, $\mathrm{sr}(T_x) < {\omega_1}^M$. Then by boundedness there is a limit ordinal $\gamma < {\omega_1}^M$ such that $\mathrm{sr}(T_x) < \gamma$, $\forall x \in M$. We demonstrate that $\mathrm{sr}(M) < \gamma + \omega$, hence a contradiction. For this it suffices to show that for any $\vec{a}, \vec{b} \in M$, if $(M, \vec{a}) \equiv_{\gamma+\omega} (M, \vec{b})$ then there is an automorphism τ of M with $\tau(\vec{a}) = \vec{b}$. Let $x \in M$ be such that $x <_F$ all elements in $\vec{a}$ and $\vec{b}$. Such x exists since M is connected and of type I. Then it follows that $(T_x, \vec{a}) \equiv_\gamma (T_x, \vec{b})$. Since $\mathrm{sr}(T_x) < \gamma$, there is an automorphism τ of T_x such that $\tau(\vec{a}) = \vec{b}$. It is easy to see that τ can be extended to an automorphism of M. $\dashv$

Now the proof of Theorem 6.1.1 follows the same line of proof as Theorem 5.2.3. *Proof of Theorem 6.1.1.* Let σ be an $L_{\omega_1\omega}$ sentence describing an invariant Borel subclass of $\mathbb{U}$. Let $\lambda > \omega$ be a countable limit ordinal bigger than the quantifier rank of σ. Suppose that $\cong\restriction \mathrm{Mod}(\sigma)$ is non-Borel. By Theorem 4.2.9 there is a model M of σ with $\mathrm{sr}(M) = {\omega_1}^M > \lambda$. By Lemma 6.1.2 there is $x \in M$ such that $\mathrm{sr}(T_x) = {\omega_1}^M = {\omega_1}^{T_x} > \lambda$. Therefore by the proof of Theorem 5.2.3 there is a Borel reduction of $\mathbb{T}_\omega$ into $\mathrm{Mod}(\equiv_\lambda T_x)$. This induces an embedding of $\mathbb{T}_\omega$ into $\mathrm{Mod}(\equiv_\lambda M)$, provided that the previous embedding was chosen so that the isomorphic types of all T_y, $y \in M$ are avoided by the coding blocks. This shows that $\mathrm{Mod}(\equiv_\lambda M)$, hence $\cong\restriction \mathrm{Mod}(\sigma)$, is Borel complete. $\dashv$

Let us state the by far usual corollaries.

Corollary 6.1.3 (Vaught Conjecture, Marcus-Miller-Steel) *Let σ be an $L_{\omega_1\omega}$ sentence such that Mod(σ) $\subseteq \mathbb{U}$. Then σ has either countably many or perfectly many models up to isomorphism.*

Corollary 6.1.4 (Glimm-Effros dichotomy) *Let σ be an $L_{\omega_1\omega}$ sentence such that Mod(σ) $\subseteq \mathbb{U}$. Then either $\cong\restriction$ Mod(σ) is smooth or else $E_0 \leq_B \cong\restriction$ Mod(σ).*

Corollary 6.1.5 *Let σ be an $L_{\omega_1\omega}$ sentence such that Mod(σ) $\subseteq \mathbb{U}$. If Mod(σ) is α-complete for all $\alpha < \omega_1$, then Mod(σ) is Borel complete.*

6.2 Remarks on first-order theories

The proofs of Theorems 5.2.3 and 6.1.1 say a little more about first-order theories in the language of one unary function symbol.

Theorem 6.2.1 *Let T be a first-order theory in the language of one unary function symbol. Then the followings are equivalent:*
(i) There is a model M of T with sr(M) $= {\omega_1}^M$.
(ii) For any ordinal $\alpha < \omega_1$, there is a model M of T with sr(M) $= {\omega_1}^M > \alpha$.
(iii) $\cong\restriction$ Mod(T) is non-Borel.
(iv) Mod(T) is Borel complete.

Proof. (i)⇒(iv) by the proof of Theorem 6.1.1 or 5.2.3. (iv)⇒(iii) by a theorem of Friedman and Stanley. ($\cong\restriction \mathrm{Mod}(T)$ is indeed $\mathbf{\Sigma}^1_1$-complete.) (iii)⇒(ii) by Theorem 4.2.9. (ii)⇒(i) is obvious. $\dashv$

As to the possibility of further dichotomy results on first-order theories, we consider the analogue of Schirmann's Theorem (Theorem 4.1.2). Recall that Theorem 4.1.2 says that any invariant Borel class of linear orderings with more than one element described by a first-order theory must be Borel complete. One could ask the question whether this is true for theories in the language of one unary function symbol. We remark that it is false.

Recall from Section 5.3 that $\mathbb{S}_\alpha$ is the invariant Borel class of wellfounded trees of rank α, for $\alpha < \omega_1$. In particular, there are $\aleph_0$ many models in $\mathbb{S}_2$ and perfectly many models in $\mathbb{S}_\alpha$ for any $\alpha > 2$. Let $\mathbb{S}_{<\omega} = \bigcup_{n\in\omega} \mathbb{S}_n$. Then $\mathbb{S}_{<\omega}$ is also an invariant Borel class.

Proposition 6.2.2 *Let L be the language with one unary function symbol and consider the invariant Borel classes $\mathbb{S}_\alpha$, $\alpha < \omega_1$, in the language L. Then the followings hold:*

(i) For any $n \in \omega$, $\mathbb{S}_n$ is finitely axiomatizable.

(ii) $\mathbb{S}_{<\omega}$ is not first-order axiomatizable.

(iii) For any $\omega \le \alpha < \omega_1$, $\mathbb{S}_\alpha$ is not first-order axiomatizable.

Proof. $\mathbb{S}_\alpha$ was defined in the language of simple trees, i.e., in the language of a binary relation symbol. But it is not hard to see that it could be reformulated so as to be defined in the language of a unary function symbol. For the root r of the simple tree, just let $F(r) = r$.

(i) We only deal with the non-trivial cases when $n \ge 2$. For $n \ge 2$, let T_n be the theory of a single sentence

$$\exists x \forall y (F^{n-1}(y) = x).$$

Then T_n describes all simple trees with at most n levels, and these are exactly the wellfounded trees of rank n.

(ii) Assume $\mathbb{S}_{<\omega}$ were first-order axiomatizable. Let Γ be a first-order theory with $\mathrm{Mod}(\Gamma) = \mathbb{S}_{<\omega}$. Now consider the language $L' = L \cup \{c_i \mid i \in \omega\}$ where each c_i is a constant symbol. Consider the L' theory $\Gamma' = \Gamma \cup \{c_{i+1} < c_i \mid i \in \omega\}$. Γ' is consistent by the Compactness Theorem. But any model of Γ' can not be a wellfounded tree, a contradiction.

(iii) Same as (ii). $\dashv$

It follows that there are first-order theories of one unary function such that the isomorphism relation on their models realize the Borel equivalence relations $\mathrm{id}(\omega)$, $\mathrm{id}(2^\omega)$, etc. This is quite different from the linear ordering case.

In Proposition 6.2.2 (i), the theory being finitely axiomatizable is not essential for its realization of the Borel equivalence relations. For example, there is a theory T which is not finitely axiomatizable and has exactly $\aleph_0$ many countable models. The following is such a theory:

$$\begin{aligned} T = \{ \quad & \forall x \forall y (F(x) = F(y) \Rightarrow x = y) \ (F \text{ is one-one}), \\ & \forall x (F(x) \ne x), \\ & \forall x (F^2(x) \ne x), \\ & \ldots \ (\text{there are no finite cycles})\} \end{aligned}$$

A complete invariant for the models of T is the number of components (finite or infinite). Each of the components looks like a $\mathbb{Z}$-chain. T is not finitely axiomatizable. If it were, then it could be axiomatized by a finite subset, whereas any finite subset of T can only rule out finite cycles of a bounded length.

It is not clear what happens with equivalence relations beyond that of $\mathbb{S}_{<\omega}$. Proposition 6.2.2 (ii) and (iii) show that the canonical examples stop to work; but we do not know whether there are other invariant Borel classes bi-reducible to any of $\mathbb{S}_{<\omega}$ or $\mathbb{S}_{\alpha}$, $\alpha \geq \omega$ and can be first-order axiomatized. In general, the same question remains unanswered for Borel equivalence relations not in the Friedman-Stanley tower.

CHAPTER 7

FS-Reducibility Theory

7.1 Interpretations between $L_{\omega_1\omega}$ theories

Interpretation is an important topic in model theory. By interpreting one structure in another, we are usually able to obtain interesting properties of the interpreted structure from those of the interpreting one, or vice versa. If the interpretation is general enough, one can usually obtain interpretations of a class of models in another class. Formally, letting A and B be two classes of models, an interpretation Γ of A in B can be viewed as just a map of A into B such that for any property ϕ there is a property ϕ_Γ, for any model $M \in A$, $M \models \phi \Leftrightarrow \Gamma(M) \models \phi_\Gamma$. To make the notion workable we may require that Γ be reasonably definable, ϕ be a sentence in some language K, ϕ_Γ be some sentence in some language L, and the map $\phi \mapsto \phi_\Gamma$ be reasonably definable as well. It is also reasonable to require that Γ preserves the isomorphism types, that is, for any $M, N \in A$, $M \cong N \Leftrightarrow \Gamma(M) \cong \Gamma(N)$.

We are not trying to give a formal definition for arbitrary interpretations. In fact, there are already many different definitions for interpretability notions (see [Ho]). Most definitions in [Ho] make extra restrictions than the formulation above in order for the interpretations to preserve stronger properties of classes of models. Here we just intend to remark that interpretations are closely related to reducibility notions. From the above formulation it is clear that interpretations are strengthenings of reducibility. Also notice that when ϕ and ϕ_Γ are sentences in infinitary logic we obtain an $L_{\omega_1\omega}$ interpretability notion. This is indeed what the authors of [FS] proposed as the minimal requirement for a decent interpretability notion. Their proposal amounts to the definitions below.

Consider two countable languages L and K. Let A and B be invariant Borel subsets of $\mathrm{Mod}(L)$ and $\mathrm{Mod}(K)$, respectively. Then in fact there are an $L_{\omega_1\omega}$ sentence ϕ and a $K_{\omega_1\omega}$ sentence ψ such that $A = \mathrm{Mod}(\phi)$ and $B = \mathrm{Mod}(\psi)$. Recall that A is *Borel reducible* to B, denoted by $A \leq_B B$, if there is a Borel mapping $\Phi : A \to B$ such that for any $x, y \in A$, $x \cong y \iff \Phi(x) \cong \Phi(y)$. Such a Φ is called a *Borel reduction* from A to B. Note that this is just the usual definition for $\cong\restriction A \leq_B \cong\restriction B$. We write $A \sim_B B$ for $A \leq_B B$ and $B \leq_B A$.

We say that A is *FS-reducible* to B, and denote $A \leq_{FS} B$, if there is a Borel reduction Φ from A to B such that for any invariant Borel $A_0 \subseteq A$, the invariant closure of $\Phi"A_0$ in B, in symbols $[\Phi"A_0]_{S_\infty}$, is Borel. The intuitive meaning of

$A \leq_{FS} B$ is that there is an interpretability notion which associates a model of ψ to each model of ϕ, so that $L_{\omega_1\omega}$ sentences logically stronger than ϕ correspond to $K_{\omega_1\omega}$ sentences logically stronger than ψ.

It is easy to see that $\leq_{FS}$ is a partial order. Note also that if A is an invariant Borel subclass of B then $A \leq_{FS} B$ via the identity embedding. By reformulating the interpretation results from model theory we get the first non-trivial results about FS-reducibility. The following theorem is well-known in model theory under a different formulation. The reader may find a proof in [Ho] (e.g., Section 5.5 of [Ho]).

Theorem 7.1.1 *Let L be a countable language. Let A be one of the following invariant Borel classes:*

(i) *the class of all countable graphs;*
(ii) *the class of all countable directed graphs;*
(iii) *Mod(K) where K is the language with one binary relation symbol;*
(iv) *the class of all countable lattices;*
(v) *the class of all countable lattices of height ≤ 4;*
(vi) *the class of all countable groups;*
(vii) *the class of all countable solvable groups;*
(viii) *the class of all countable nilpotent groups;*
(ix) *the class of all countable nilpotent groups of rank 2.*

Then Mod(L) $\leq_{FS}$ A. In fact, for each L there are an $K_{\omega_1\omega}$ sentence σ_L in the appropriate language K and a Borel isomorphism Φ such that Mod(L) $\leq_{FS}$ Mod(σ_L) via Φ.

As to the proofs, (i) is Theorem 5.5.1 of [Ho]; (ii) and (iii) follow from (i); (v) is Theorem 5.5.2 of [Ho]; (iv) follows from (v); (ix) can be found in [Ho] Appendix A.3; (vi), (vii) and (viii) follow from (ix) and the fact that they are invariant Borel classes. All the proofs in [Ho] are for finite language L, but they work equally well for infinite languages. An immediate corollary of the above theorem is that Vaught Conjecture for any of these classes is equivalent to the full Vaught Conjecture. [BK] gave more historical remarks on this matter.

As remarked in [FS] and shown in [Ho], a true interpretability notion usually defines the interpretation of atomic formulas and then extends by logical complexity. Various countable fragments of $L_{\omega_1\omega}$ are usually preserved in this situation, in the sense that a formula in the fragment is interpreted by another formula in the same fragment. Of particular interest is the fragment of first-order formulas, and this leads to the following definition. For invariant Borel classes A and B, we say that A is *strongly FS-reducible* to B, denoted $A \leq_{sFS} B$, if $A \leq_{FS} B$ via θ and for any first-order theory T, there is a first-order theory T' such that

$$[\theta``\{M \in A \mid M \models T\}]_{S_\infty} = \{N \in B \mid N \models T'\}.$$

In Theorem 7.1.1, all $\leq_{FS}$ can in fact be replaced by $\leq_{sFS}$. The virtue of this is that it follows that the first-order Vaught Conjecture for any of those classes is equivalent to the full first-order Vaught Conjecture. (It does not make any sense to talk about the first-order Vaught Conjecture for the classes of all solvable groups or nilpotent groups, since these classes are not first-order axiomatizable. But we could replace these classes by, for example, solvable groups of rank n for some fixed n.)

In fact the notion of FS-reducibility makes sense for general equivalence relations (but not for sFS-reducibility). In the sequel we also talk about FS-reducibility between arbitrary equivalence relations. We also write $E \sim_B F$ for $E \leq_B F$ and $F \leq_B E$.

We show in the following theorem that for a variety of Borel equivalence relations FS-reducibility coincides with the usual Borel reducibility. The author learned the theorem from Hjorth, who attributed the ideas to earlier arguments of Burgess and Kechris.

Theorem 7.1.2 *Let G be a Polish group acting in a Borel manner on a Polish space X. Let F be a Borel equivalence relation on a Polish space Y. Suppose $E_G^X \leq_B F$ via θ. Then*

(i) *$E_G^X \leq_{FS} F$. In fact, $E_G^X \leq_{FS} F$ via θ.*

(ii) *Suppose $[\theta``X]_F = Y$. Then $E_G^X \sim_B F$.*

Proof. (i) Let B be an invariant Borel subset of X and $A = [\theta``B]_F$. Then A is $\mathbf{\Sigma}_1^1$. Let α be the action of G on X. Let z be a real such that $\alpha, \theta, B, F \in \Delta_1^1(z)$. We claim that $\forall y \in A \exists x \in \Delta_1^1(y,z)(x \in B \wedge \theta(x) F y)$. The claim would imply that $A = \{y \in Y \mid \exists x \in \Delta_1^1(y,z)(x \in B \wedge \theta(x) F y)\}$, which shows that A is $\mathbf{\Pi}_1^1$, hence Borel.

Now fix $y \in A$. Let $x_0 \in B$ be such that $\theta(x_0) F y$. It is easy to see that there is a Borel code w for $[x_0]_G$ such that $w \in \Delta_1^1(y,z)$. We show that $\exists x \in \Delta_1^1(w,z) \cap [x_0]_G$, which is enough for the claim.

Let $\{U_n\}_{n \in \omega}$ be an enumeration of the basic open sets that effectively presents the topology of X. Assume that $U_0 = X$ and $U_1 = \emptyset$. Let $C_n = X - U_n$ for each $n \in \omega$. Suppose $[x_0]_G$ is $\mathbf{\Pi}_\gamma^0$. Then

$$x \in [x_0]_G \Leftrightarrow \forall i \exists j \cdots \exists k \; x \in C_{n(i,j,\cdots,k)}.$$

The group of quantifiers on the right hand side can be viewed as a game quantifier for a game played on some wellfounded tree $T \subseteq \omega^{<\omega}$ recursive in w. Also recursive in w is the function $n(\cdot, \ldots)$ from the set of terminal nodes of T to ω.

We consider the following game $G(l,c)$ for $l \in \omega$ and c a Borel code for a closed set of X:

$$\begin{array}{llllll}
\mathrm{I} & i_0 & & i_1 & i_2 & \\
 & & & & & \cdots \\
\mathrm{II} & & m, j_0 & j_1 & j_2 &
\end{array}$$

All elements in the play are in ω. The part $< i_0, j_0, i_1, j_1, \cdots >$ of the play is on the tree T. If II plays outside of T for the first time then she loses. Otherwise the game terminates when $< i_0, j_0, \cdots >$ is a terminal node of T. Then II wins iff

(i) $\overline{U_m} \subseteq U_l$, where $\overline{U_m}$ is the closure of U_m, and

(ii) $\mathrm{diam}(U_m) \leq \frac{1}{2}\mathrm{diam}(U_l)$, and

(iii) if $\exists x \in [x_0]_G \exists^* g \in G(g.x \in U_l \cap B_c)$, then $\exists x \in [x_0]_G \exists^* g \in G(g.x \in U_m \cap B_c \cap C_{n(i_0, j_0, \cdots)})$,

where B_c denotes the closed set coded by c, and $\exists^* g \in G$ denotes "there are nonmeager many $g \in G$".

We show that II has a winning strategy for $G(l, c)$ for any l, c. To see this, suppose $\exists x \in [x_0]_G \exists^* g \in G(g.x \in U_l \cap B_c)$. Write $[x_0]_G = \bigcap_{i_0} \bigcup_{j_0} D_{i_0, j_0}$, where each D_{i_0, j_0} is $\mathbf{\Pi}^0_\nu$ for some $\nu < \gamma$. Then since

$$U_l = \bigcup \{U_m \mid \overline{U_m} \subseteq U_l \wedge \mathrm{diam}(U_m) \leq \frac{1}{2}\mathrm{diam}(U_l)\},$$

we have that for any i_0,

$$\begin{aligned} [x_0]_G \cap U_l \cap B_c \ \subseteq \ & \bigcup \{[x_0]_G \cap D_{i_0, j_0} \cap U_m \cap B_c \mid j_0 \in \omega \\ & \wedge \overline{U_m} \subseteq U_l \wedge \mathrm{diam}(U_m) \leq \tfrac{1}{2}\mathrm{diam}(U_l)\}. \end{aligned}$$

Note that the union on the right hand side is countable. So there is m and j_0 such that $\exists x \in [x_0]_G \exists^* g \in G(g.x \in D_{i_0, j_0} \cap U_m \cap B_c)$. Similar argument works also for subsequent moves. Since the game terminates after a finite number of moves, we have that II wins the game.

In fact II has a winning strategy $\Gamma(l, c)$ which is $\Delta^1_1(w, z, l, c)$. First note that $\exists x \in [x_0]_G \exists^* g \in G(g.x \in \ldots)$ iff $\forall x \in [x_0]_G \exists^* g \in G(g.x \in \ldots)$, so it is a Borel condition. It follows that the winning condition of the game is $\Delta^1_1(w, z, l, c)$. In order to define a winning play for II, we can choose $m, j_0, j_1, \cdots$ to be lexicographically least so as to satisfy the conditions as in the preceding paragraph.

Now let $P = \{u \in \omega^{<\omega} \mid \exists v \in \omega^{<\omega} < u, v > \text{ is a terminal node of } T\}$, where $< u, v >$ is the shuffle of u and v. Then P is the set of all legitimate plays of I. Fix $<_P$ to be a $\Delta^1_1(w, z)$ linear ordering which induces an enumeration of P. For $i \in \omega$, let u_i be the i-th element of P under $<_P$.

For the construction of an $x \in \Delta^1_1(w, z) \cap [x_0]_G$, we define a sequence $\{m(i)\}_{i \in \omega}$ in a $\Delta^1_1(w, z)$ manner such that

(1) $\forall i \in \omega$, $\overline{U_{m(i+1)}} \subseteq U_{m(i)}$, and

(2) $\forall i \in \omega$, $\mathrm{diam}(U_{m(i)}) \leq \frac{1}{2^i}$, and

(3) $\forall u \in P \exists v \exists i \in \omega \forall i' \geq i \exists x \in [x_0]_G \exists^* g \in G(g.x \in C_{n(<u,v>)} \cap U_{m(i')})$.

Granting such a sequence, let x be the unique point in $\bigcap_{i \in \omega} U_{m(i)}$. Then $\forall u \in P \exists v \exists i \in \omega \forall i' \geq i(C_{n(<u,v>)} \cap U_{m(i')} \neq \emptyset)$. It follows that $\forall u \in P \exists v$ (x is a limit point of $C_{n(<u,v>)}$). Hence $\forall u \in P \exists v(x \in C_{n(<u,v>)})$. This implies that $x \in [x_0]_G$.

To define the sequence $\{m(i)\}_{i\in\omega}$, we use the following induction, in which another sequence $\{c(i)\}_{i\in\omega}$ is simultaneously defined. At stage -1 let $m(-1)=0$ ($U_{m(-1)}=X$) and $B_{c(-1)}=X$. At stage i play the game $G(m(i-1),c(i-1))$. Let I play u_i and II respond according to $\Gamma(m(i-1),c(i-1))$. Let $m(i)$ be the m played by II. Let v be the other part played by II. Then let $B_{c(i+1)}=B_c\cap C_{n(<u_i,v>)}$.

It is now easy to see that (1)-(3) are satisfied. For (3), let $u\in P$. Suppose $u=u_i$. Then by the above construction, there is v such that for $i'\geq i$, $B_{c(i')}\subseteq C_{n(<u,v>)}$. This finishes the proof of (i).

Note that we have described a Borel function φ from $[\theta``X]_F$ into $X$ such that $\theta(\varphi(y))Fy$, $\forall y\in[\theta``X]_F$. Since θ is a reduction, so is φ. This shows that, if $[\theta``X]_F=Y$, we have $F\leq_B E^X_G$, hence $E^X_G\sim_B F$. ⊣

Note that Theorem 7.1.2 (ii) actually bypasses the concept of FS-reducibility. We do not know whether similar statements as Theorem 7.1.2 (i) and (ii) are true for arbitrary Borel equivalence relations.

7.2 An application of the Glimm-Effros dichotomies

The Glimm-Effros dichotomy can be viewed as a structural theorem strengthening the Vaught Conjecture. Therefore, it should have consequences about the structures of classes of countable models which the Vaught Conjecture does not entail. In this section we present some applications of our Glimm-Effros dichotomy theorems in the previous chapters to the study of FS-reducibility notion. In particular, we prove the non-existence of such interpretability notion between certain Borel invariant classes.

Recall that the theory of graphs can interpret any other theory in a faithful manner, in the sense of Theorem 7.1.1 (i). In an attempt to push this further to theories of simpler classes of models, Friedman and Stanley considered in particular the class of countable linear orderings $\mathbb{L}$, the class of simple trees $\mathbb{T}_\omega$ and the class of models in the language of a unary function $\mathbb{U}$. They proved the following theorem in [FS].

Theorem 7.2.1 (Friedman-Stanley) *Let L be an arbitrary countable language. Then*

(i) Mod(L) $\leq_B \mathbb{T}_\omega$.
(ii) Mod(L) $\leq_B \mathbb{U}$.
(iii) Mod(L) $\leq_B \mathbb{L}$.

Friedman and Stanley asked whether Theorem 7.2.1 could be improved to statements such as $\mathrm{Mod}(L)\leq_{FS}\mathbb{L}$ or $\mathrm{Mod}(L)\leq_{FS}\mathbb{T}_\omega$. They observed that if one of these were true, then the Vaught Conjecture would follow. As Hjorth points out, our Glimm-Effros dichotomy theorems answer the questions negatively.

Theorem 7.2.2 *If* $A \le_{FS} B$ *via* Φ *and* $\cong\restriction [\Phi\text{''}A]_{S_\infty}$ *has the Glimm-Effros property, then so does* $\cong\restriction A$.

Proof. Without loss of generality assume $B = [\Phi\text{''}A]_{S_\infty}$. It is enough to show that if $E_0 \sqsubseteq_c \cong\restriction B$, then $E_0 \sqsubseteq_c \cong\restriction A$. Define $P = \{(y, x) \in B \times A \,|\, \Phi(x) \cong y\}$. Then $P \subseteq B \times A$ is $\mathbf{\Sigma}^1_1$. By our assumption $P_B = B$. By the Jankov-von Neumann Uniformization Theorem ([Kec1], Theorem 29.9), P has a C-measurable uniformization function Ψ. Since Φ is a reduction from $\cong\restriction A$ to $\cong\restriction B$, Ψ is a reduction of $\cong\restriction B$ to $\cong\restriction A$. It follows that E_0 is reducible to $\cong\restriction A$ via a C-measurable function. By the remark after Theorem 4.2.3, $E_0 \sqsubseteq_c \cong\restriction A$. $\dashv$

Theorem 7.2.3 *Let* L *be a countable language with at least one relation symbol of arity* ≥ 2. *Then*

(i) Mod(L*)* $\not\le_{FS} \mathbb{L}$.

(ii) Mod(L*)* $\not\le_{FS} \mathbb{T}_\omega$.

(iii) Mod(L*)* $\not\le_{FS} \mathbb{U}$.

Proof. Let K be the language of groups and ψ be the $K_{\omega_1\omega}$ sentence for which the Glimm-Effros property fails. By Theorem 7.1.1 (iii) there exists an $L_{\omega_1\omega}$ sentence ϕ and an FS-reduction ψ from Mod(ψ) onto Mod(ϕ). Now assume Mod(L) $\le_{FS} \mathbb{L}$ via Φ. Let σ be a sentence all of whose models are linear orderings and such that Mod(σ) $= [\Phi\text{''}\mathrm{Mod}(\phi)]_{S_\infty}$. Then by Theorem 4.1.1, $\cong\restriction$ Mod(σ) has the Glimm-Effros property. By Theorem 7.2.2 (twice) $\cong\restriction$ Mod(ψ) also has the Glimm-Effros property, a contradiction. The proofs of (ii) and (iii) are the same, using Corollaries 5.3.1 and 6.1.4 instead of Theorem 4.1.1. $\dashv$

Intuitively, Theorem 7.2.3 says that the classes $\mathbb{L}$, $\mathbb{T}_\omega$ and $\mathbb{U}$ are too small to be able to faithfully interpret all countable structures.

7.3 Borel isomorphism relations among simple trees

In this section we show that $\mathbb{T}_\omega$ and $\mathbb{U}$ are able to faithfully interpret invariant Borel classes whose isomorphism relations are Borel. First note that $\mathbb{T}_\omega$ is an invariant Borel subclass of $\mathbb{U}$, hence $\mathbb{T}_\omega \le_{FS} \mathbb{U}$. Therefore we can focus on the result about $\mathbb{T}_\omega$ and the corresponding result about $\mathbb{U}$ would follow. Throughout the section we will freely use the terminology about simple trees defined in Chapter 5.

We will make use of the proof of Theorem 7.2.1 (i), which we give below. *Proof of 7.2.1 (i):* Let $\mathcal{G}$ be the class of all countable graphs. We define a Borel reduction $F : \mathcal{G} \to \mathbb{T}_\omega$. Let T_0 be the subtree of $\omega^{<\omega}$ such that $<> \in T_0$ and $u = < x_1, \cdots, x_n > \in T_0$ iff $x_i \ne x_j$, $\forall 1 \le i, j \le n$. Let R be a graph with domain ω. Then $F(R)$ is a simple tree obtained by adding at most one new terminal node to each node of T_0. For $m, n \in \omega$ let $< m, n >= 2^m 3^n$. Then for $u =< x_1, \cdots, x_{<m,n>} > \in T_0$, a new terminal node is added iff $R(x_m, x_n)$.

It is easy to see that F is invariant. To show that $F(R_1) \cong F(R_2) \Rightarrow R_1 \cong R_2$, use a back and forth argument. $\dashv$

Next we recall a game theoretic characterization for infinitary equivalence between two models, due to Benda and Karp.

Let M and N be two models and α be an ordinal. The game $G_\alpha(M, N)$ is played as follows. Let $\alpha_0 = \alpha$. During the k-th move I plays an ordinal $\alpha_k < \alpha_{k-1}$ and an element of one of the models; II then plays an element of the other model. Eventually I plays $\alpha_n = 0$ and the game finishes after this move.

$$\begin{array}{lcccc}
\mathrm{I} & \alpha_1, x_1 & \alpha_2, x_2 & & \alpha_n = 0, x_n \\
 & & & \cdots & \\
\mathrm{II} & y_1 & y_2 & & y_n
\end{array}$$

Let $a_1, \cdots, a_n$ be the elements of M played and $b_1, \cdots, b_n$ be the corresponding elements of N played. Then II wins the play iff $\varphi_0^{a_1,\cdots,a_n,M} = \varphi_0^{b_1,\cdots,b_n,N}$. $M \equiv_\alpha N$ iff II has a winning strategy for $G_\alpha(M, N)$.

We use this characterization to do a detailed analysis for the proof of 7.2.1 (i) above.

Lemma 7.3.1 *Let $F : \mathcal{G} \to \mathbb{T}_\omega$ be given by the proof of 7.2.1 (i) above. Then for any ordinal $\alpha < \omega_1$,*

(i) if $R_1 \equiv_{\omega\alpha} R_2$ then $F(R_1) \equiv_\alpha F(R_2)$, and,

(ii) if $F(R_1) \equiv_{\omega+\alpha} F(R_2)$ then $R_1 \equiv_\alpha R_2$.

Therefore, if $\alpha = \omega\alpha$, then $R_1 \equiv_\alpha R_2$ iff $F(R_1) \equiv_\alpha F(R_2)$.

Proof. (i) Let Γ be a winning strategy of II for $G_{\omega\alpha}(R_1, R_2)$. We demonstrate a winning strategy for II for $G_\alpha(F(R_1), F(R_2))$. During the k-th move suppose I played α_k and (for definiteness) $s_k \in F(R_1)$. We will obtain a $t_k \in F(R_2)$ after playing at most $l(s_k)$ many moves in $G_{\omega\alpha}(R_1, R_2)$, as follows. Let $s' = p(s_k)$, if s_k is a terminal node and $s' = s_k$ otherwise. Let $s' =< x_1, \cdots, x_l > \in T_0$. In the next l moves of $G_{\omega\alpha}(R_1, R_2)$ let I gradually play ordinals $\omega \cdot \alpha_k + i$ for $i = l-1, \cdots, 0$ and elements of R_1 $x_1, \cdots, x_l$. Let II respond according to Γ. We obtain corresponding elements of R_2 $y_1, \cdots, y_l$. Let $t' =< y_1, \cdots, y_l > \in T_0$. Finally let $t_k = t'$, if $s' = s_k$ and $t_k =$ the terminal node whose parent is t', if $s' = p(s_k)$. This makes sense because if $s' = p(s_k)$, s_k is a terminal node, then $l =< m, n >$ for some $m, n \in \omega$ and $R_1(x_m, x_n)$. Since Γ is a winning strategy for II for $G_{\omega\alpha}(R_1, R_2)$, we have $R_2(y_m, y_n)$. So during the construction of $F(R_2)$ there was a terminal node added to t'.

The case when I played some element in $F(R_2)$ is symmetric. It is easy to see that the strategy described above is a winning strategy for II for $G_\alpha(F(R_1), F(R_2))$.

(ii) Let Γ be a winning strategy for II for $F(R_1) \equiv_{\omega+\alpha} F(R_2)$. We describe a winning strategy for II for $G_\alpha(R_1, R_2)$. During the k-th move of $G_\alpha(R_1, R_2)$ suppose I played an ordinal α_k and (for definiteness) $x_k \in R_1$. Let $s_{k-1} \in F(R_1)$ be played in the previous move ($s_0 =<>$). Then in $F(R_1) \equiv_{\omega+\alpha} F(R_2)$ let I play $\omega + \alpha_k$ and $s_k = s_{k-1}\hat{\ }x_k$. Let II respond according to Γ. We claim that if $t_{k-1} \in F(R_2)$ was played in the previous move then $l(t_k) = l(t_{k-1}) + 1$. Since $s_{k-1} < s_k$ and Γ is winning, we have $t_{k-1} < t_k$. If $l(t_k) > l(t_{k-1}) + 1$, then let $t' \in F(R_2)$ be any element such that $t_{k-1} < t' < t_k$ and let I play t' in her next move. Player II will have no reasonable response, contradicting that Γ is winning for II. This proves the claim. Now let $t_k = t_{k-1}\hat{\ }y_k$. Then y_k is the next move for II for $G_\alpha(R_1, R_2)$.

We verify that the above strategy is winning for II. Assume $x_1, \cdots, x_k, y_1, \cdots, y_k$ is a play according to this strategy but in which II loses. Without loss of generality assume $\varphi_0^{x_1,\cdots,x_{k-1},R_1} = \varphi_0^{y_1,\cdots,y_{k-1},R_2}$. Assume that I played x_k and II responded y_k. Then there is $j < k$ such that $R_1(x_j, x_k) \not\Leftrightarrow R_2(y_j, y_k)$. For definiteness assume $R_1(x_j, x_k)$ but $\neg R_2(y_j, y_k)$. Let $l =< j, k >$. Then there is a terminal node $u \in F(R_1)$ with $u > s_k$ and $l(u) = l + 1$ and there is no node in $F(R_2)$ on the $l+1$-st level and extending t_k. We continue to play $F(R_1) \equiv_{\omega+\alpha} F(R_2)$ as follows. First let I play ordinal $l \in \omega$ and u in her next move. Player II responds according to Γ and plays some v. Note that $v > t_k$. We claim that v is not a terminal node. For suppose v is terminal, then $l(v) \neq l + 1$. Now for the next $l - k$ moves let I gradually play ordinals $l - 1, \cdots, k$ and all elements of $F(R_1)$ in between s_k and u. Player II responds according to Γ and plays $l - k$ many nodes in $F(R_2)$. In case $l(v) > l + 1$ these $l - k$ many nodes can not be all the nodes in between t_k and v; if I next plays a node in between t_k and v which does not appear in the $l - k$ many nodes previously played by II, then II will have no reasonable response, contradiction. In case $l(v) < l + 1$ among the $l - k$ many nodes played by II there must be at least two identical ones, also a contradiction. Therefore v is not terminal. Let I play any node in $F(R_2)$ extending v; once again II will have no reasonable response to play, contradiction. $\dashv$

The above lemma is sufficient to establish Theorem 7.3.3 (i) below. However, we would like to prove a stronger fact so as to deduce Theorem 7.3.3 (ii).

Lemma 7.3.2 *Let $F : \mathcal{G} \to \mathbb{T}_\omega$ be given by the proof of 7.2.1 (i) above. Then for any $R \in \mathcal{G}$, $sr(F(R)) \leq \omega + sr(R) + \omega$.*

Proof. Fix a graph R. Let $\alpha = \mathrm{sr}(R)$. Suppose $(F(R), \vec{s}) \equiv_{\omega+\alpha+\omega} (F(R), \vec{t})$, where $\vec{s} = (s^1, \ldots, s^n), \vec{t} = (t^1, \ldots, t^n)$. By playing the game $G_{\omega+\alpha+\omega}((F(R), \vec{s}), (F(R), \vec{t}))$ we can see that $(F(R), T^s) \equiv_{\omega+\alpha} (F(R), T^t)$, where T^s is the smallest subtree of $F(R)$ containing all s^i, $i \leq n$, and T^t is the similar tree given by $\vec{t}$. Thus for simplicity of notation we may assume that $(F(R), \vec{s}) \equiv_{\omega+\alpha} (F(R), \vec{t})$ and that $\vec{s}$ and $\vec{t}$

are closed under subsequences. Suppose for each $i \leq n$, $s^i =< a^i_1, \ldots, a^i_{l(i)} >, t^i =< b^i_1, \ldots, b^i_n >$, where $l(i) = l(s^i) = l(t^i)$. (Here $l(s^i) = l(t^i)$ because $(F(R), s^i) \equiv_\omega (F(R), t^i)$.) Let $\vec{a} =< a^i_j \mid i \leq n, j \leq l(i) >$ and $\vec{b} =< b^i_j \mid i \leq n, j \leq l(i) >$. Let $K = \sup_{i \leq n} l(i)$. We show that there is an automorphism τ of $F(R)$ such that $\tau(\vec{s}) = \vec{t}$.

Use induction on K. When $K = 0$ there is nothing to prove. We start with $K = 1$. Assume $a^i_1 \neq a^j_1$, $i \neq j$. Then $b^i_1 \neq b^j_1$, $i \neq j$. It suffices to show that for each $i \leq n$, the subtrees $T_{a^i_1}$ and $T_{b^i_1}$ of $F(R)$ are isomorphic. The problem is now simplified to the following: suppose $(F(R), < a >) \equiv_{\omega+\alpha} (F(R), < b >)$, show that $(F(R), < a >) \cong (F(R), < b >)$. By the similar argument as in the proof of Lemma 7.3.2 (ii), the hypothesis implies that $(R, a) \equiv_\alpha (R, b)$. Since $\mathrm{sr}(R) = \alpha$, there must be an automorphism π of R such that $\pi(a) = b$. Then π induces an automorphism τ of $F(R)$ such that $\tau(< a >) =< b >$. This finishes the proof of the case $K = 1$.

Now consider the general inductive case. Suppose $l(s^i) = K$ exactly when $i \leq m$ for some $m < n$. Then $(F(R), s^{m+1}, \ldots, s^n) \equiv_{\omega+\alpha} (F(R), t^{m+1}, \ldots, t^n)$. By the inductive hypothesis there is an automorphism τ' such that $\tau'(s^j) = t^j$, $m < j \leq n$. It suffices to show that for each j, $m < j \leq n$, there is an isomorphism of T_{s^j} with T_{t^j} such that the sets $\{s^i \mid i \leq m\} \cap T_{s^j}$ and $\{t^i \mid i \leq m\} \cap T_{t^j}$ are matched under the isomorphism. Similar to the $K = 1$ case, the problem is now simplified to the following: suppose $(F(R), < a_1, \ldots, a_n >) \equiv_{\omega+\alpha} (F(R), < b_1, \ldots, b_n >)$, show that $(F(R), < a_1, \ldots, a_n >) \cong (F(R), < b_1, \ldots, b_n >)$. Again, by a similar argument as in the proof of Lemma 7.3.2 (ii), we have that $(R, a_1, \ldots, a_n) \equiv_\alpha (R, b_1, \ldots, b_n)$. Since $\mathrm{sr}(R) = \alpha$, there is an automorphism π of R such that $\pi(\vec{a}) = \vec{b}$. This automorphism then induces an automorphism τ of $F(R)$ such that $\tau(< \vec{a} >) =< \vec{b} >$. $\dashv$

The computation of Scott rank of $F(R)$ may not be optional, but it is good enough to derive the following theorem.

Theorem 7.3.3 *Let A be an invariant Borel subset of Mod(L) for some countable language L such that $\cong \restriction A$ is Borel. Then*

(i) $A \leq_{FS} \mathbb{T}_\omega$, and

(ii) There is an invariant Borel class of simple trees B with $A \sim_B B$.

Proof. By Theorem 7.1.1 we may assume A is an invariant Borel class of graphs. Then there is a countable ordinal α such that $\mathrm{sr}(R) \leq \alpha, \forall R \in A$. Let $F : \mathcal{G} \to \mathbb{T}_\omega$ be given by the proof of 7.2.1 (i) above. Then by Lemma 7.3.1, $\mathrm{sr}(F(R)) \leq \omega + \alpha + \omega$, $\forall R \in A$. It follows that F gives rise to $\cong \restriction A \leq_B \equiv_{\omega+\alpha+\omega} \restriction \mathbb{T}_\omega$, where $\equiv_{\omega+\alpha+\omega}$ is a Borel equivalence relation. By Theorem 7.1.2, F gives rise to $\cong \restriction A \leq_{FS} \equiv_{\omega+\alpha+\omega} \restriction \mathbb{T}_\omega$. Let $B = [F``A]_{S_\infty}$. In particular we have that B is Borel. Note that $\cong \restriction B$ is the same as $\equiv_{\omega+\alpha+\omega} \restriction B$, therefore it is Borel. Then by Theorem 7.1.2 $A \leq_{FS} B$ and $A \sim_B B$. It then follows that $A \leq_{FS} \mathbb{T}_\omega$. $\dashv$

Corollary 7.3.4 *Let A be an invariant Borel subset of Mod(L) for some countable language L such that $\cong\restriction A$ is Borel. Then*

(i) $A \leq_{FS} \mathbb{U}$, and

(ii) There is an invariant Borel subclass B of $\mathbb{U}$ with $A \sim_B B$.

7.4 Borel isomorphism relations among linear orderings

The results in the preceding section remain true when simple trees are replaced by linear orderings. In fact the proofs follow from the same ideas of the proofs in the preceding section. We demonstrate the results below and make some remarks about the reducibility between $\mathbb{T}_\omega$ and $\mathbb{L}$.

As in the preceding section, we first recall the proof of Theorem 7.2.1 (iii).
Proof of 7.2.1 (iii): By Theorem 7.1.1, it suffices to define a Borel reduction F from the class of graphs $\mathcal{G}$ into $\mathbb{L}$. Let $\{S_n\}_{i\in\omega}$ enumerate all the first-order quantifier free types in the language of graphs. Fix a graph R with universe ω. For any tuple $s \in R$, let $n(s)$ be the unique n such that S_n is the quantifier free type of s realized in R. We are to define $F(R)$.

Let T_0 be the subtree of $\omega^{<\omega}$ such that $<>\in T_0$ and $u =< x_1, \cdots, x_n >\in T_0$ iff $x_i \neq x_j$, $\forall 1 \leq i, j \leq n$. We define a linear order M of order type $\mathbb{Q}$ and an identification map $I : M \to T_0$. M will be the direct limit of linear orders M_k, each of which is of order type $\mathbb{Q}$; I will be the direct limit of $I_k : M_k \to \omega^{\leq k} \cap T_0$. In order for the direct limit to make sense, we will see to it that $I_k``(M_k - M_{k-1}) \subseteq \omega^k \cap T_0$. Intuitively, the map I associates to every element in M a non-repeating sequence of natural numbers as its identification.

The construction begins with $M_0 = \mathbb{Q}$ and I_0 being identically $<>$. Having defined M_k and I_k, we let M_{k+1} to be $(1 + \mathbb{Q}) \times M_k$, ordered in the reverse lexicographic order. Denote the left end point of $1 + \mathbb{Q}$ by $-\infty$. Each element x of M_k is identified with the element $(-\infty, x)$ of M_{k+1}. Define I_{k+1} on each $\mathbb{Q} \times \{x\}$ for $x \in M_k$ so that its image is all of the $< I_k(x), m >$, with each $I_{k+1}^{-1}(< I_k(x), m >)$ dense in $\mathbb{Q} \times \{x\}$.

Finally define $F(R)$ by replacing each $x \in M$ by the linear ordering of order type $\mathbb{Q} + n(I(x)) + 2 + \mathbb{Q}$. One can check that F is invariant and by a back and forth argument that F is a reduction. ⊣

The notation involved in this proof is more complicated than that of 7.2.1 (i). It takes even more notation to analyze the proof. Fix $R \in \mathcal{G}$. For each $x \in F(R)$, let $C(x)$ be the copy of $\mathbb{Q} + N + \mathbb{Q}$ for some N which contains x. The copy of N in $C(x)$ is called the *concrete part* of $C(x)$. If y is not in the concrete part of $C(x)$, then it is in the *dense part.* The dense part of $C(x)$ is further divided into *left dense part* and *right dense part*, with obvious reference. Note that $C(x)$ together with its various parts are all first order definable from x. Extend the identification map I to $F(R)$ in the obvious way, so that I is constant on $C(x)$ for each x. Then the concrete part of $C(x)$ has exactly $n(I(x))+2$ many elements. For any $x \in F(R)$, let T_x be the maximal segment of $F(R)$ such that $C(x) \subseteq T_x$ and that for any $y \in T_x$, either $y \in C(x)$ or $y > x$, and that for any $y \in T_x$, $l(I(y)) \geq l(I(x))$. Intuitively, T_x is obtained by taking the subtree of T_0 below $I(x)$ and then finding the linear order segment in $F(R)$ correspondent with this subtree. This is the reason to use the notation T_x reminiscent of the subtree notion for simple trees. There will be no confusion since this terminology is only used in this section and only for linear orderings. For each M_k and $x \in M_k$, let $T_x^{M_k}$ be the maximal segment of M_k such that x is the smallest element of $T_x^{M_k}$ and for any $y \in T_x^{M_k}$, $y \in M_{k-1}$. This is a one level analog of T_x.

We now have the following parallel lemma to Lemma 7.3.1.

Lemma 7.4.1 *Let $F : \mathcal{G} \to \mathbb{L}$ be given by the proof of 7.2.1 (iii) above. Then for any ordinal $\alpha < \omega_1$,*

(i) if $R_1 \equiv_{\omega\alpha} R_2$ then $F(R_1) \equiv_\alpha F(R_2)$, and,

(ii) if $F(R_1) \equiv_{\omega+\alpha} F(R_2)$ then $R_1 \equiv_\alpha R_2$.

Therefore, if $\alpha = \omega\alpha$, then $R_1 \equiv_\alpha R_2$ iff $F(R_1) \equiv_\alpha F(R_2)$.

Proof. (i) Let Γ be a winning strategy of II for $G_{\omega\alpha}(R_1, R_2)$. We demonstrate a winning strategy for II for $G_\alpha(F(R_1), F(R_2))$. We keep a side board on which the game $G_{\omega\alpha}(R_1, R_2)$ is simultaneously played, and we describe how II plays in an arbitrary move in $G_\alpha(F(R_1), F(R_2))$. During the k-th move suppose I played α_k and (for definiteness) $x_k \in F(R_1)$. Let $s_k = I(x_k)$. In the next $l(s_k)$ moves of $G_{\omega\alpha}(R_1, R_2)$ let I gradually play ordinals $\omega \cdot \alpha_k + i$ for $i = l(s_k) - 1, \ldots, 0$ and elements of s_k. Let II respond according to Γ. We obtain a corresponding tuple t_k of elements of R_2. Since Γ is a winning strategy, the quantifier free type of t_k in R_2 is the same as that of s_k in R_1. We are to choose a $y_k \in F(R_2)$ so that the following conditions are fulfilled:

(a) $I(y_k) = t_k$, and

(b) letting x_i, y_i, $i < k$ be the previous moves, we have that $y_k < y_i$ iff $x_k < x_i$, $y_k > y_i$ iff $x_k > x_i$, and

(c) If x_k is in the concrete part of $C(x_k)$, say x_k is the m-th element in the concrete part of $C(x_k)$, then y_k is also the m-th element in the concrete part of $C(y_k)$; If x_k is in the left (or right) dense part of $C(x_k)$, then y_k is also in the left (respectively right) dense part of $C(y_k)$.

Once we verify that such an y_k can be chosen, the strategy is obviously a winning strategy for II. We make the inductive assumption that (a)-(c) are true for the previous moves.

If $t_k = I(t_i)$ for some $i < k$, then choose y_k so that $C(y_k) = C(y_i)$ and so that (b) and (c) are fulfilled. A moment reflection gives that this can be done.

Suppose $t_k \neq I(t_i)$ for any $i < k$. Suppose $l = l(t_k)$. We find some element of $I_l^{-1}(t_k)$ in M_l so that the analog of (b) holds on that level of the construction. This can be done by a Cantor type argument, given the construction of I_l. Now we have chosen $C(y_k)$. Then y_k can be chosen in $C(y_k)$ so that (c) is fulfilled.

(ii) Let Γ be a winning strategy for II in $G_{\omega+\alpha}(F(R_1), F(R_2))$. We define a winning strategy for II in $G_\alpha(R_1, R_2)$. During the k-th move of $G_\alpha(R_1, R_2)$ suppose I played an ordinal α_k and (for definiteness) $a_k \in R_1$. Let $a_i \in R_1, b_i \in R_2$, $i < k$, be the previous moves. If $a_k = a_i$ for some $i < k$, then let $b_k = b_i$ be II's move. Otherwise, assume without loss of generality that $a_i \neq a_j$, for any $i \neq j$, $i, j \leq k$. Let $s_k =< a_1, \ldots, a_k >$. Let $x_k \in F(R_1)$ be an arbitrary element in the concrete part of $C(x_k)$ such that $I(x_k) = s_k$. Let I play x_k in $G_{\omega+\alpha}(F(R_1), F(R_2))$. Let II respond according to Γ. Suppose y_k is the element of $F(R_2)$ played by II. Let $t_k = I(y_k)$. Then $n(t_k) = n(s_k)$ and $l(t_k) = l(s_k) = k$, by a similar argument as in the proof of Lemma 7.3.2 (ii). Let $t_k =< b'_1, \ldots, b'_k >$. Then we must have $b'_i = b_i$ for all $i < k$. Let $b_k = b'_k$ be II's move. We have described a strategy for II in $G_\alpha(R_1, R_2)$. It is obviously winning for II since $n(t_k) = n(s_k)$, that is, the quantifier free type of t_k is the same as the quantifier free type of s_k. $\dashv$

Again, Lemma 7.4.1 is sufficient for proving Theorem 7.4.3 (i). Nevertheless, we show the following stronger fact, parallel to Lemma 7.3.2.

Lemma 7.4.2 *Let $F : \mathcal{G} \to \mathbb{L}$ be given by the proof of 7.2.1 (iii) above. Then for any $R \in \mathcal{G}$, $sr(F(R)) \leq \omega + sr(R) + \omega$.*

Proof. Fix $R \in \mathcal{G}$. Let $\alpha = \mathrm{sr}(R)$. Suppose that

$$(F(R), x_1, \ldots, x_n) \equiv_{\omega+\alpha+\omega} (F(R), y_1, \ldots, y_n).$$

By extending the sequences and by the similar argument as in the proof of Lemma 7.3.2, we may assume that

(1) $(F(R), x_1, \ldots, x_n) \equiv_{\omega+\alpha} (F(R), y_1, \ldots, y_n)$, and

(2) For any x_i, say $I(x_i) = s$, there is x_j such that $I(x_j) = s \upharpoonright (l(s) - 1)$ and that for any $y \in F(R)$, if $I(y) = I(x_j)$, then either $y > x_i$ or $y \in C(x_j)$ or $y < x_j$. In particular, the set of $\{I(x_i) \mid i \leq n\}$ is a subtree of T_0 by condition (2). It is easy to see that $n(I(x_i)) = n(I(y_i))$, for any $i \leq n$. Let $K = \sup_{i \leq n} l(I(x_i))$. We show by induction on K that there is an automorphism τ of $F(R)$ such that τ is induced by some automorphism of M_K and $\tau(\vec{x}) = \vec{y}$.

If $K = 0$, then $x_i, y_i \in M_0$ for all $i \leq n$. Since the quantifier free type of $\vec{x}$ is the same as quantifier free type of $\vec{y}$, by a back and forth argument there is an automorphism as required.

Now consider the general inductive case. Suppose $l(I(x_i)) = K$ exactly when $i \leq m$ for some $m < n$. Then the sequences $(x_{m+1}, \ldots, x_n)$ and $(y_{m+1}, \ldots, y_n)$ satisfy the conditions (1) and (2) above. By inductive hypothesis there is an automorphism π of M_{K-1} such that $\pi(x_j) = y_j$ for $m < j \leq n$. We are to extend *pi* to some automorphism τ of M_K such that $\tau(x_i) = y_i$ for $i \leq m$. Without loss of generality we may assume that $m + 1 = n$. It follows from the construction of $F(R)$ and the usual back and forth argument that we may assume without loss of generality that $n(I(x_i)) = n(I(x_j))$ for all $i \neq j$, $i, j < n$. And it suffices to construct an isomorphism ρ between $T^{M_K}_{x_n}$ and $T^{M_K}_{y_n}$ such that

(a) $\rho(x_i) = y_i$, for $i < n$, and

(b) ρ matches the set $I_K^{-1}(I(x_i))$ with $I_K^{-1}(I(y_i))$, for each $i < n$.

Again, by the construction of $F(R)$ and keeping the back and forth argument in mind, we may assume without loss of generality that $I(x_i) = I(x_j)$, for all $i < n$. Since the quantifier free type of $\vec{x}$ is the same as the quantifier free type of $\vec{y}$, this isomorphism is easy to obtain on the M_K level by the usual back and forth argument. Finally, to extend all the automorphism of M_K to $F(R)$ we need to see that T_{x_i} is isomorphic to T_{y_i} for all $i < n$. So the problem is now simplified to the following: suppose $(F(R), x) \equiv_{\omega+\alpha} (F(R), y)$, show that $(F(R), x) \cong (F(R), y)$. Letting $I(x) = s$ and $I(y) = t$. By the similar argument as in the proof of Lemma 7.4.1 (ii) we have that $(R, s) \equiv_\alpha (R, t)$. Since $\mathrm{sr}(R) = \alpha$, there is an automorphism of R moving s to t. This isomorphism induces an automorphism of $F(R)$ moving x to y, hence moving T_x to T_y. This shows that T_x and T_y are isomorphic. $\dashv$

What makes the proof of Lemma 7.4.2 more complicated than that of Lemma 7.3.2 is that the automorphism of the structure is no longer an obvious combination of isomorphisms of its substructures. Here all the automorphisms or isomorphisms are to be constructed using the back and forth argument. It is still possible to freely combine the isomorphisms of substructures because in the construction of the structure all relevant occurrences were made dense. Now by exactly the same proof of Theorem 7.3.3 we have the following theorem.

Theorem 7.4.3 *Let A be an invariant Borel subset of Mod(L) for some countable language L such that $\cong \upharpoonright A$ is Borel. Then*

(i) $A \leq_{FS} \mathbb{L}$, and

(ii) There is an invariant Borel class of linear orderings B with $A \sim_B B$.

This theorem rules out the possibility of an analog of Schirmann's Theorem (Theorem 4.1.2) for infinitary logic.

As to the relationship between $\mathbb{L}$ and $\mathbb{T}_\omega$, we do not know anything about the FS-reducibility or non-reducibility, except those implied by Theorems 7.3.3 and 7.4.3. However, we do know that there is no strong FS-reducibility notion from $\mathbb{U}$ to $\mathbb{L}$. This is the content of the following theorem.

Theorem 7.4.4 $\mathbb{U} \not\leq_{sFS} \mathbb{L}$.

Proof. Suppose θ is such a reduction. Let A be a first-order axiomatizable class of countably infinitely many simple trees. Such classes exist by the remarks of Section 5.3. Consider $B = [\theta``A]_{S_\infty}$. Since θ is a strong FS-reduction, B is an invariant class of linear orderings also axiomatizable by a first-order theory T. Also B has countably infinitely many models. But by Rubin's Theorem (or Theorem 4.1.2), B has either one or $2^{\aleph_0}$ many models, contradiction. $\dashv$

So far all the positive FS-reducibility results we have for pairs of non-Borel isomorphism relations are derived from classical interpretability results, namely, in such a way that atomic formulas are first interpreted and then interpretation of other formulas are obtained by induction on their complexity. This approach always yield strong FS-reducibility. The above simple theorem indicates that we need much sharper techniques and new methods to answer questions like whether $\mathbb{U} \leq_{FS} \mathbb{L}$.

CHAPTER 8

The O Notation and Related Equivalence Relations

8.1 Background and definitions

The O notation is widely used in computer science (algorithm analysis) and in mathematical analysis (estimation of orders). When a function comes up in these fields, it is usually compared with some well understood functions first. Examples of such canonical functions include n^a, $n^a(\log n)^b$, $n^a e^{bn}$, etc. Sometimes the analyzed function can be shown to have exactly the same order as one of these canonical functions; but more often no such equivalence can be obtained. Of course, obtaining the exact order of a function might not be very important in these fields, depending on the motive of the study and the applications in mind. However, from a theoretical point of view, it is certainly very desirable if the list of canonical functions can be completed. Intuitively, it would mean that we understand the functions so well that, as far as its order is concerned, we have captured all possibilities in this list. The list itself may be long, even infinite, and each form above is actually representing uncountably many different functions, so the current list is already uncountable. We are willing to understand the word "list" in its weakest sense as long as an empirical such list shows up and is proved to be complete. Probably the only restriction is that all of these have to be done in some definable fashion, since one can immediately think of an abstract set of representatives by applying the Axiom of Choice. One way to formulate the ideal here in rigorous terms is the following: find a Polish space X such that each point in the space is coding some function and for any function there is a point in X provably in the same order. Note that the above listed functions fit in this framework. For example, the functions n^a can be coded by $\mathbb{R}$ (from which the constant a is chosen), and similarly $n^a(\log n)^b$ can be coded by a copy of $\mathbb{R}^2$. Therefore, $\mathbb{R}^3$ is enough to code all of the functions of the form n^a and $n^a(\log n)^b$. This formulation is still vague in two senses. First, it is not clear what is in general an acceptable coding for the functions. Second, it is not clear in what sense a function is provably equivalent to another function. The existence of such a list is dubious no matter how the question is understood. The point is to give a *proof* in some sense that there is no such list. In this chapter we try to formulate this question in a workable form and give an answer to it. Several interesting definable equivalence relations arise in this study, and we will survey the results.

Recall the definition of O. For two functions f and g, we say that f is $O(g)$, denoted by $f \in O(g)$, if there is some constant $c > 0$ such that $f(x) \leq cg(x)$ for sufficiently large x. To avoid trivialities and unnecessary notational complexity we make the following more strict definition. Consider the Polish space $(\mathbb{R}^+)^\omega$, where $\mathbb{R}^+$ is the space of all positive real numbers. For $f, g \in (\mathbb{R}^+)^\omega$, $f \in O(g)$ if there is $c > 0$ such that $f(n) \leq cg(n)$ for all $n \in \omega$. We define the equivalence relation Θ on $(\mathbb{R}^+)^\omega$ by

$$f\Theta g \ \Leftrightarrow f \in O(g) \wedge g \in O(f),\ \forall f, g \in (\mathbb{R}^+)^\omega.$$

Conventionally $f\Theta g$ is written as $f \in \Theta(g)$. Here we intend to emphasize that it is an equivalence relation.

We would like to understand the above question as whether Θ is smooth. We will not argue that this is the only reasonable formulation of the above question. Nevertheless, it is very close in spirit and its answer will give us in some sense a deep understanding of the situation. We will show in the next section that Θ is not smooth. In fact, our results give more information about how high above Θ is located in the Borel reducibility hierarchy of Borel equivalence relations.

Let us remark that the above definition of Θ also makes sense for the space $(\mathbb{N}^+)^\omega$ instead of $(\mathbb{R}^+)^\omega$, where $\mathbb{N}^+$ is the set of positive natural numbers. The space $(\mathbb{N}^+)^\omega$ is essentially the same as the Baire space ω^ω. We denote this alternate equivalence relation $(\mathbb{N}^+)^\omega/\Theta$. If a distinction needs to be made, the original equivalence relation will be denoted $(\mathbb{R}^+)^\omega/\Theta$.

8.2 Θ and ℓ^∞

Consider the Polish space $\mathbb{R}^\omega$. Define

$$\ell^\infty = \{f \in \mathbb{R}^\omega \mid \exists M > 0 \forall n |f(n)| < M\}.$$

$(\ell^\infty, +)$ is a Borel subgroup of the abelian Polish group $(\mathbb{R}, +)$. We will denote the groups ℓ^∞ and $\mathbb{R}^\omega$ also. Let ℓ^∞ act on $\mathbb{R}^\omega$ by shift: for any $g \in \ell^\infty$, $f \in \mathbb{R}^\omega$,

$$g.f = g + f$$

where $g + f$ is the pointwise addition. The orbit equivalence relation will be denoted by $\mathbb{R}^\omega/\ell^\infty$ or simply ℓ^∞, if there is no danger of confusion. When in the above definition all occurrences of $\mathbb{R}^\omega$ is replaced by $\mathbb{Z}^\omega$ the definition still makes sense, and we will denote the resulting equivalence relation $\mathbb{Z}^\omega/\ell^\infty$. The structure $(\ell^\infty \cap \mathbb{Z}^\omega, +)$ is again a Borel subgroup of $(\mathbb{Z}^\omega, +)$. In fact, there is another variation of the definition for the Baire space. We denote the equivalence relation by $\omega^\omega/\ell^\infty$. There is no Borel group associated with this equivalence relation.

Proposition 8.2.1 *The following equivalence relations are Borel equivalent:*
(i) $(\mathbb{N}^+)^\omega/\Theta$,

(ii) $(\mathbb{R}^+)^\omega/\Theta$,
(iii) $\mathbb{R}^\omega/\ell^\infty$,
(iv) $\mathbb{Z}^\omega/\ell^\infty$,
(v) $\omega^\omega/\ell^\infty$.

Proof. We show that (i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(i).

(i)⇒(ii) is obvious.

(ii)⇒(iii): Define $\theta : (\mathbb{R}^+)^\omega \to \mathbb{R}^\omega$ by

$$\theta(f)(n) = \log f(n), \ \forall f \in (\mathbb{R}^+)^\omega.$$

Now for $f, g \in (\mathbb{R}^+)^\omega$,

$$\begin{aligned} f\Theta g &\Leftrightarrow f \in O(g) \wedge g \in O(g) \\ &\Leftrightarrow \exists c > 0 \forall n (f(n) \leq cg(n) \wedge g(n) \leq cf(n)) \\ &\Leftrightarrow \exists M > 0 \forall n (|\log f(n) - \log g(n)| \leq M) \\ &\qquad \text{(by taking } M = \max(1, \log c)) \\ &\Leftrightarrow \theta(f) - \theta(g) \in \ell^\infty. \end{aligned}$$

Therefore θ is a reduction from $(\mathbb{R}^+)^\omega/\Theta$ to $\mathbb{R}^\omega/\ell^\infty$.

(iii)⇒(iv): Let $\theta : \mathbb{R}^\omega \to \mathbb{Z}^\omega$ be defined by

$$\theta(f)(n) = \lfloor f(n) \rfloor, \ \forall f \in \mathbb{R}^\omega,$$

where $\lfloor x \rfloor$ is the largest integer $\leq x$. It is easy to verify that θ works.

(iv)⇒(v): Define $\theta : \mathbb{Z}^\omega \to \omega^\omega$ by

$$\begin{aligned} \theta(f)(2n) &= \begin{cases} f(n) & \text{if } f(n) \geq 0 \\ 0 & \text{otherwise} \end{cases} \\ \theta(f)(2n+1) &= \begin{cases} 0 & \text{if } f(n) \geq 0 \\ -f(n) & \text{otherwise} \end{cases} \end{aligned}$$

for $f \in \mathbb{Z}^\omega$. Then when $f(n)g(n) > 0$, we have $|\theta(f)(n) - \theta(g)(n)| = |f(n) - g(n)|$; when $f(n)g(n) < 0$, we have

$$|\theta(f)(n) - \theta(g)(n)| \leq \max(|f(n)|, |g(n)|) \leq |f(n)| + |g(n)| = |f(n) - g(n)|.$$

These imply that $f\ell^\infty g \Rightarrow \theta(f)\ell^\infty\theta(g)$. For the reverse direction, suppose $M > 0$ is such that $|\theta(f)(n) - \theta(g)(n)| \leq M$ for all n. Then by the definition of θ we have that $|f(n)|, |g(n)| < M$ for all n. Therefore $|f(n) - g(n)| < 2M$ for all n.

(v)⇒(i): Let $\theta : \omega^\omega \to (\mathbb{N}^+)^\omega$ be given by

$$\theta(f)(n) = 2^{f(n)},\ \forall f \in \omega^\omega.$$

A similar argument as in the proof of (ii)⇒(iii) shows that θ works. $\dashv$

This proposition allows us to shift our focus from Θ to ℓ^∞. As the notation suggests, the equivalence relation ℓ^∞ is related to ℓ^p, for $1 \leq p < \infty$. Recall that for $1 \leq p < \infty$,

$$\ell^p = \{f \in \mathbb{R}^\omega \mid \sum_{n\in\omega} |f(n)|^p < \infty\}.$$

Each ℓ^p is a Borel subgroup of $\mathbb{R}^\omega$; in fact they are Polish when given the metric from the p-norm $\|\ \|_p$. Thus they induce Borel equivalence relations $\mathbb{R}^\omega/\ell^p$ by the shift action. We simply denote the equivalence relations ℓ^p. These equivalence relations was studied in, for example, [DH].

Theorem 8.2.2 (Dougherty-Hjorth) *For* $1 \leq p < q < \infty$, $\ell^p <_B \ell^q$.

We extend the result by taking ℓ^∞ into account.

Theorem 8.2.3 *For* $1 \leq p < \infty$, $\ell^p <_B \ell^\infty$.

Proof. By Theorem 8.2.2 it suffices to show that for $1 \leq p < \infty$, $\ell^p \leq_B \ell^\infty$. Fix $1 \leq p < \infty$. Let $\{s_m\}_{m\in\omega}$ enumerate the sequences in $\mathbb{Q}^{<\omega}$. We define $\theta : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by

$$\theta(f)(m) = (\sum_{n<l(s_m)} |f(n) - s_m(n)|^p)^{\frac{1}{p}}$$

for any $f \in \mathbb{R}^\omega$. We verify that θ works.

First, suppose $\theta(f)\ell^\infty\theta(g)$, that is, $\theta(f)-\theta(g) \in \ell^\infty$. Let $\{m_k\}_{k\in\omega}$ be a sequence such that for each k, $l(s_{m_k}) = k$ and

$$|g(n) - s_{m_k}(n)| \leq \frac{1}{2^{n+1}},\ \forall n < k.$$

Then for each k,

$$\begin{aligned}
\theta(g)(m_k) &= (\sum_{n<k} |g(n) - s_{m_k}(n)|^p)^{\frac{1}{p}} \\
&\leq (\sum_{n<k} 2^{-(n+1)p})^{\frac{1}{p}} \\
&\leq (\sum_{n=0}^{\infty} \frac{1}{2^{n+1}})^{\frac{1}{p}} = 1;
\end{aligned}$$

and by Minkowski's inequality,

$$\begin{aligned}(\sum_{n<k}|f(n)-g(n)|^p)^{\frac{1}{p}} &\leq (\sum_{n<k}|f(n)-s_{m_k}(n)|^p)^{\frac{1}{p}}+(\sum_{n<k}|f(n)-s_{m_k}(n)|^p)^{\frac{1}{p}}\\ &= \theta(f)(m_k)+\theta(g)(m_k)\\ &= (\theta(f)(m_k)-\theta(g)(m_k))+2\theta(g)(m_k)\\ &\leq |\theta(f)(m_k)-\theta(g)(m_k)|+2.\end{aligned}$$

It follows that the (increasing) sequence $(\sum_{n<k}|f(n)-g(n)|^p)^{\frac{1}{p}}$ is bounded, hence $\|f-g\|_p<\infty$, or $f-g\in\ell^p$.

For the reverse direction, suppose $f-g\in\ell^p$. Then by Minkowski's inequality again, for each m,

$$\begin{aligned}|\theta(f)(m)-\theta(g)(m)| &= |(\sum_{n<l(s_m)}|f(n)-s_m(n)|^p)^{\frac{1}{p}}-(\sum_{n<l(s_m)}|g(n)-s_m(n)|^p)^{\frac{1}{p}}|\\ &\leq (\sum_{n<l(s_m)}|f(n)-g(n)|^p)^{\frac{1}{p}}\\ &\leq \|f-g\|_p.\end{aligned}$$

It follows that $\theta(f)-\theta(g)$ is bounded, hence $\theta(f)-\theta(g)\in\ell^\infty$. $\dashv$

The above proof is inspired by Oliver's proof ([Ol]) that c_0 is bi-reducible to the density ideal. The proof for $\ell^1\leq_B\ell^\infty$ was known to Casevitz ([Ca]) by a similar argument.

Corollary 8.2.4 *Θ is not smooth.*

Proof. Because $\mathrm{id}(2^\omega)\leq_B\ell^1<_B\ell^\infty\sim_B\Theta$. $\dashv$

We actually have a strictly increasing sequence

$$\mathrm{id}(2^\omega)<_B E_0<_B\ell^1<_B\ldots<_B\ell^\infty.$$

So Θ is high above in the Borel reducibility hierarchy; in fact, there are $2^{\aleph_0}$ many Borel degrees below it which are non-smooth. The first result in the chain is more or less folklore. There is nothing in between $\mathrm{id}(2^\omega)$ and E_0 ([DJK]). The second follows from Hjorth's turbulence theory ([Hj2]). Hjorth ([HK2]) also showed that all the equivalence relation $<_B\ell^1$ are essentially countable.

8.3 More about ℓ^∞

It seems that the study of the equivalence relation ℓ^∞ is poorly documented. There are several facts about ℓ^∞ that are well-known but hard to find in the literature. In this section we discuss these facts. We are not trying to give a historical account. Instead, we focus on gathering the facts and proving them as directly as possible.

First, we give a proof of the well-known fact that ℓ^∞ is not Polishable. Recall that a Borel subgroup of a Polish group is Polishable if it admits a Polish topology preserving its Borel structure.

Proposition 8.3.1 *The Borel subgroups of* $\mathbb{R}^\omega$ ℓ^∞ *and* $\ell^\infty \cap \mathbb{Z}^\omega$ *are not Polishable.*

Proof. We show the result for ℓ^∞. The proof is the same for $\ell^\infty \cap \mathbb{Z}^\omega$.

Assume ℓ^∞ is Polishable. Work under the Polish topology of ℓ^∞ that gives the same Borel sets as before. For each $N \in \omega$, let

$$B_N = \{f \in \ell^\infty \mid \forall n(|f(n)| \leq N)\}.$$

Then $\ell^\infty = \bigcup B_N$. Note that B_N is Borel, and hence have the Baire property. By Baire Category Theorem, there is some N such that B_N is non-meager. By a theorem of Pettis (Theorem 1.2.5 of [BK]) $B_N - B_N$ contains an open neighborhood of the identity. Since $B_N - B_N \subseteq B_{2N}$, B_{2N} contains an open neighborhood of the identity. Then from the separability of the Polish topology in assumption it follows that there are countably many elements $f_0, f_1, \ldots, f_k, \ldots$ such that

$$\ell^\infty = \bigcup_k f_k + B_{2N}.$$

We then construct a $g \in B_{5N}$ such that $g \notin f_k + B_{2N}$ for any k, arriving at a contradiction. The construction of g is by diagonalization:

$$g(k) = \begin{cases} 5N & \text{if } |f_k(k)| \leq 2N \\ 0 & \text{if } |f_k(k)| > 2N \end{cases}$$

For any k, since $|g(k) - f_k(k)| > 2N$, $g \notin f_k + B_{2N}$. ⊣

Recall the definition of equivalence relation E_1 on the space $\mathbb{R}^\omega$:

$$f E_1 g \Leftrightarrow \exists N \forall n > N(f(n) = g(n)).$$

The first noticeable properties about E_1 appeared in [KL].

Theorem 8.3.2 (Kechris-Louveau)
(i) *Let E be a Borel equivalence relation. If $E <_B E_1$, then $E \leq_B E_0$.*
(ii) *Let G be a Polish group and X be a Borel G-space. Then $E_1 \not\leq_B E_G^X$.*

In fact a conjecture of Kechris says that if E is an arbitrary Borel equivalence relation, then either $E_1 \leq_B E$ or else there is an orbit equivalence relation E_G^X such that $E \leq_B E_G^X$. This has been partially verified by Solecki (see, e.g., [So]).

Theorem 8.3.3 (Solecki)

(i) *Let* I *be an analytic ideal on* ω. *Let* E_I *be the equivalence relation on* 2^ω *defined by* $xE_Iy \Leftrightarrow x\triangle y \in I$. *Then either* I *is Polishable or else* $E_1 \leq_B E_I$.

(ii) *Let* G *be an abelian Polish group and* H *a* $\mathbf{\Pi}^0_3$ *subgroup of* G. *Let* G/H *be the orbit equivalence relation of the shift action of* H *on* G. *Then either* H *is Polishable or else* $E_1 \leq_B G/H$.

Coming back to ℓ^∞ we have the following immediate corollary.

Corollary 8.3.4 $E_1 <_B \ell^\infty$.

Proof. Note that ℓ^∞ is a $\mathbf{\Sigma}^0_2$ subgroup of $\mathbb{R}^\omega$. It then follows from Proposition 8.3.1 and Theorem 8.3.3 (ii) that $E_1 \leq_B \ell^\infty$. We must have that $E_1 <_B \ell^\infty$: we saw in the preceding section that $E_0 <_B \ell^1 <_B \ell^\infty$; if $E_1 \sim_B \ell^\infty$ there would be a contradiction to Theorem 8.3.2 (i). $\dashv$

It is also possible to construct a reduction directly for $E_1 \leq_B \ell^\infty$. In [Ca] the following reduction was noticed:

$$\theta(f)(n) = nf(n),\ \forall f \in \mathbb{R}^\omega.$$

Also notice that, given $E_1 \leq_B \ell^\infty$, Proposition 8.3.1 can be viewed as an immediate corollary of Theorem 8.3.2 (ii). This is probably the intrinsic reason for the difficulty to attribute the results about ℓ^∞: it is hard to tell who first realized which part of the cycle.

In fact there are also $2^{\aleph_0}$ many Borel degrees between E_1 and ℓ^∞. Mazur defined a system of $2^{\aleph_0}$ many Borel equivalence relations arising from ideals resembling ℓ^∞ ([Maz]). Oliver has shown that all of them are below ℓ^∞. The argument in Proposition 8.2.1 can be modified to show that all the ideals in the Mazur system are not Polishable, hence by Theorem 8.3.3 (i), all these equivalence relations are above E_1.

Before the end of this chapter we remark that there is something in between ℓ^∞ and the ℓ^p's. Therefore we have the following diagram of Borel degrees.

The equivalence relation ℓ^ω is defined on a direct sum $X = \bigoplus_{i\in\mathbb{N}^+} X_i$ where each X_i is a copy of $\mathbb{R}^\omega$. For $x, y \in X$,

$$x\ell^\omega y \Leftrightarrow \exists i(x, y \in X_i \wedge \|x-y\|_i < \infty).$$

It is easy to see that for any $1 \leq p < \infty$, $\ell^p <_B \ell^\omega$ (by Theorem 8.2.2).

Proposition 8.3.5 $\ell^\omega <_B \ell^\infty$.

Proof. We first show that $\ell^\omega \leq_B \ell^\infty$. For each $i \geq 1$, let θ_i be the reduction defined in the proof of Proposition 8.1.2 for $\ell^i \leq_B \ell^\infty$. Let $A_1, A_2, \ldots, A_i, \ldots$ be a

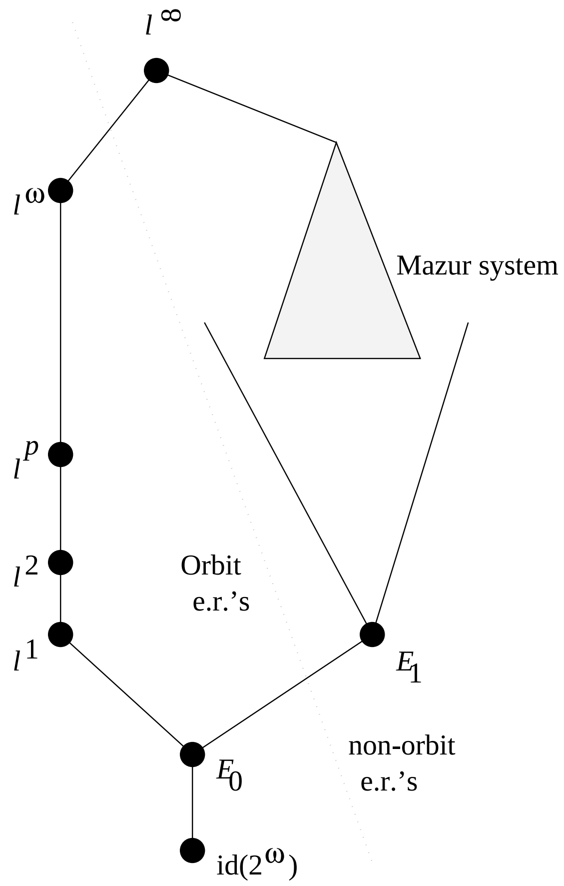

Figure 8.1: Borel degrees below ℓ^∞.

decomposition of ω into disjoint infinite sets. For each A_i and each $n \in \omega$, let n^{A_i} be the n-th number in A_i. We define $\theta : X \to \mathbb{R}^\omega$ by

$$\theta(x)(m) = \begin{cases} m + \theta_i(x)(n) & \text{if } x \in X_i \text{ and } m = n^{A_i} \\ 0 & \text{otherwise} \end{cases}$$

We verify that $x\ell^\omega y$ iff $\theta(x)\ell^\infty\theta(y)$.

Consider two cases. First suppose that there is i with $x, y \in X_i$. Then for $m \notin A_i$, $\theta(x)(m) = \theta(y)(m) = 0$. For $m \in A_i$ and $m = n^{A_i}$, $\theta(x)(m) - \theta(y)(m) = \theta_i(x)(n) - \theta_i(y)(n)$. Therefore $\theta(x)\ell^\infty\theta(y)$ iff $\theta_i(x)\ell^\infty\theta_i(y)$. But then since $\|x-y\|_i < \infty$ and θ_i is a reduction of ℓ^i to ℓ^∞, it follows that $\theta(x)\ell^\infty\theta(y)$ iff $x\ell^\omega y$. If $x \in X_i$ and $y \in X_j$ for $i \neq j$, then for $m \in A_i$, $\theta(x)(m) - \theta(y)(m) \geq m$, thus is unbounded.

Now to see that $\ell^\infty \not\leq_B \ell^\omega$, notice that ℓ^ω can be induced by a Polish group action on X. The acting group is $G = \prod_i \ell^i$. For $g = (g_i) \in G$ and $x \in X$,

$g.x = g_i.x$, if $x \in X_i$. It is easy to see that this is a Borel action. Now by Theorem 8.3.2 (ii) and Corollary 8.3.4, $\ell^\infty \not\leq_B \ell^\omega$. $\dashv$

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