THE HOMEOMORPHISM PROBLEM FOR COUNTABLE TOPOLOGICAL SPACES

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ABSTRACT. We consider the homeomorphism problem for countable topological spaces and investigate its descriptive complexity as an equivalence relation. It is shown that even for countable metric spaces the homeomorphism problem is strictly more complicated than the isomorphism problem for countable graphs and indeed it is not Borel reducible to any orbit equivalence relation induced by a Borel action of a Polish group. We also characterize the relative complexity of some other equivalence relations arising in the study.

1. INTRODUCTION

In this paper we consider the homeomorphism problem for countable topological spaces. The aim is to understand the complexity of this problem and to provide concrete evidence for the common belief that it is complicated. The study was motivated in part by [TU], which does an excellent job in explaining why countable topological spaces can and should be studied from a descriptive set theoretic point of view, and by some questions of Uzcátegui, which we will explain in detail later in this section. In fact the homeomorphism problems for various classes of compact metric spaces have been investigated from the perspective of definable equivalence relations in recent years (c.f., e.g., [CG],[GK],[Hj]), and have been successful examples of applications of the general theory of definable equivalence relations to classification problems arising from various branches of mathematics. Therefore it is very natural to study the homeomorphism problem for countable topological spaces along the same line of thinking.

The theory of definable equivalence relations is based on a very simple setup: a standard Borel space and an equivalence relation on it. The complexity of the equivalence relation is only manifested in comparisons to other equivalence relations. The way to compare two equivalence relations is as follows. Given equivalence relations E on X and F on Y, we say that E is *Borel reducible to* F, denoted $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that, for all $x, y \in X$,

$$xEy \iff f(x)Ff(y).$$

Intuitively, this means that E is no more complex than F. In case $E \leq_B F$ but $F \not\leq_B E$, we denote $E <_B F$ and the intuitive meaning is that E is strictly simpler than F. In case $E \leq_B F$ and $F \leq_B E$ we say that E is *Borel bireducible with* F. Here the intuitive meaning is that E and F are of the same complexity. Classification problems are naturally equivalence relations. In case the objects can be coded by elements of some standard Borel space, it makes sense to compare the

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classification problem with other equivalence relations in terms of Borel reducibility, and hence information about its relative complexity can be obtained.

Another, more traditional, way to characterize the complexity of an equivalence relation is to determine its descriptive complexity. This is sometimes related to Borel reducibility (for example, a $\mathbf{\Sigma}^1_1$-complete equivalence relation can not be Borel reducible to any Borel equivalence relation). More often it is used as the only alternative when information about Borel reducibility is not available. In any case, the descriptive complexity is a mild indicator for the complexity of an equivalence relation.

As a first clue about the complexity of our problem, let us consider the homeomorphic classification of countable compact metric spaces. By the well known Stone duality, this has the same complexity as the isomorphism problem for countable superatomic Boolean algebras (c.f., e.g., [Ko]), and thus can be classified by complete invariants of the form (α, n), where α is a countable ordinal and n is a natural number. It also follows by well known general results that this equivalence relation is not Borel.

For general countable topological spaces, we need first to find ways to code them by elements of standard Borel spaces. Without definability assumptions this is impossible, since there are more than continuum many such spaces. Uzcátegui proposed to consider topologies with closed subbases, since such topologies, especially the Hausdorff ones, seem to form a stable concept by results of [TU]. Formally, we assume without loss of generality that our topological spaces all have underlying space $\mathbb{N}$, the set of natural numbers. Each subset of $\mathbb{N}$ is identified with an element of $2^{\mathbb{N}}$, the Cantor space. Let $\mathcal{K}(2^{\mathbb{N}})$ be the standard Borel space of all closed subsets of $2^{\mathbb{N}}$, with the Effros Borel structure (or equivalently, endowed with the Hausdorff topology). Then for $K, L \in \mathcal{K}(2^{\mathbb{N}})$, let $K \cong_s L$ iff the topology generated by subbasis K is homeomorphic to that by L. Of course some of the closed subsets of $2^{\mathbb{N}}$ code the same topology. Thus there seems to be another equivalence relation involved here. We let $\equiv_s$ be the identity relation of topologies coded by their closed subbases; formally, for $K, L \in \mathcal{K}(2^{\mathbb{N}})$, let $K \equiv_s L$ iff the topology generated by subbasis K is the same as that by L. Uzcátegui asked: what are the complexity of $\equiv_s$ and $\cong_s$ on $\mathcal{K}(2^{\mathbb{N}})$?

A direct computation gives that $\equiv_s$ on $\mathcal{K}(2^{\mathbb{N}})$ is $\mathbf{\Pi}^1_1$ and $\cong_s$ on $\mathcal{K}(2^{\mathbb{N}})$ is $\mathbf{\Sigma}^1_2$. We do not know whether they are respectively $\mathbf{\Pi}^1_1$-complete and $\mathbf{\Sigma}^1_2$-complete. In any case, it seems that these equivalence relations are beyond the scope of the best understood part of the current theory of definable equivalence relations.

It turns out that we can do a lot more with countable topological spaces with countable bases. A countable basis can be coded by a single element of $(2^{\mathbb{N}})^{\mathbb{N}}$ or $2^{\mathbb{N}\times\mathbb{N}}$, and one can define the identity relation and the homeomorphism relation similar to what we did in the preceding paragraph. It is not hard to see that the identity relation is Borel and the homeomorphism relation is $\mathbf{\Sigma}^1_1$, thus they are suitable for comparison to equivalence relations we know a lot about. In section 2 we give definitions for relevant equivalence relations used as milestones in the theory of definable equivalence relations. In section 3 we deal with the identity relation of topologies. In the last section we give two pieces of evidence for the complexity of the homeomorphism problem.

2. Preliminaries on equivalence relations

One of the simplest equivalence relations coming up in the Borel reducibility hierarchy is the identity relation, or the equality relation, of real numbers. More sophisticated equivalence relations can be constructed in a variety of ways, and their mutual Borel reducibility is thus complicated.

Here are some procedures to produce new equivalence relations from given ones. Let I be a finite or countable index set and for each $i \in I$, let E_i be an equivalence relation on a standard Borel space X_i. The *product* of E_i's is the equivalence relation F on the product space $\prod_{i\in I} X_i$ defined by

$$(x_i)F(y_i) \iff \forall i \in I\,(x_i E_i y_i),$$

for $(x_i),(y_i) \in \prod_{i\in I} X_i$. In this case it is clear that each E_i is Borel reducible to F. A special case is when all the E_i's are the same, say as some E, and the index set is $\mathbb{N}$. In this case the product equivalence relation is called the (infinite) *power* of E and is denoted by $(E)^\omega$ (or E^ω when there is no danger of confusion). In general $E \leq_B (E)^\omega$ and we can say no more.

For an equivalence relation E on X, we can define another equivalence relation $(E)^+$ on $X^{\mathbb{N}}$ as follows. For $(x_i),(y_i) \in X^{\mathbb{N}}$,

$$(x_i)(E)^+(y_i) \iff \forall n\exists m(x_n E y_m) \wedge \forall n \exists m(x_m E y_n).$$

Equivalently, (x_i) and (y_i) are $(E)^+$-equivalent iff

$$\{[x_i]_E \,|\, i \in \mathbb{N}\} = \{[y_i]_E \,|\, i \in \mathbb{N}\},$$

where for $x \in X$, $[x]_E = \{y \in X \,|\, yEx\}$. With obvious reasons, $(E)^+$ is often informally refered to as (*the identity of*) *countable sets of E-classes*. If E is Borel, then $E <_B (E)^+$ ([FS]).

Not all equivalence relations can be generated by applying some abstract procedure to simpler equivalence relations. Sometimes we simply have to define them. In the theory of Borel equivalence relations the following equivalence relations are well known (and their names have become standard). The equivalence relation E_0 on $2^{\mathbb{N}}$ is defined by

$$xE_0y \iff \exists n \forall m \geq n\,(x(m) = y(m)),$$

for $x, y \in 2^{\mathbb{N}}$. In a similar fashion, we define E_1 on $2^{\mathbb{N}\times\mathbb{N}}$ by

$$(x_i)E_1(y_i) \iff \exists n \forall m \geq n\,(x_m = y_m),$$

for $(x_i),(y_i) \in 2^{\mathbb{N}\times\mathbb{N}}$.

Although E_0 and E_1 are much alike, in particular they are both F_σ ($\mathbf{\Sigma}^0_2$), they turn out to have very different properties. In order to elaborate on this, we need to look at orbit equivalence relations induced by Borel actions of Polish groups. This is an important, arguably the central, class of equivalence relations investigated in descriptive set theory. Suppose G is a Polish group acting on a standard Borel space X in a Borel manner. The action induces an orbit equivalence relation on X, denoted by E^X_G, in the usual way:

$$xE^X_G y \iff \exists g \in G\,(g \cdot x = y),$$

for $x, y \in X$. For example, the infinite permutation group S_∞, consisting of all permutations of $\mathbb{N}$, is a Polish group. Well known results of model theory and some general results of Becker and Kechris ([BK]) showed that any orbit equivalence relation induced by a Borel action of S_∞ is Borel reducible to the isomorphism relation of countable graphs, thus the isomorphism of countable graphs is a very complex equivalence relation; in particular, it is known to be $\mathbf{\Sigma}^1_1$-complete ([FS]). In general, many natural mathematical equivalence relations are either orbit equivalence relations or Borel reducible to some orbit equivalence relation induced by a Borel action of a Polish group.

But not always. It was shown by Kechris and Louveau that E_1 is not Borel reducible to any orbit equivalence relation induced by a Borel action of a Polish group ([KL]). Thus any equivalence relation more complex than E_1 has the same property. In contrast, E_0 can be induced by a continuous action of $\mathbb{Z}$ on $2^{\mathbb{N}}$.

The main dichotomy theorem in [KL] also implies that $E_1 <_B (E_1)^\omega$.

In the subsequent sections, we will compare the abovementioned equivalence relations to the equivalence relations arising in the study of the homeomorphism problem for countable topological spaces. A majority of our equivalence relations are defined on $2^{\mathbb{N}}$ or some variations such as $2^{\mathbb{N}\times\mathbb{N}}$. We view elements of this space to be either functions from $\mathbb{N}$ to $2 = \{0, 1\}$ or subsets of $\mathbb{N}$; and we freely switch our point of view without explicit mention.

3. Coding countable bases

In this section we focus on countable topological spaces with a countable basis, i.e., which are second countable (equivalently first countable). We consider natural ways to code the topologies by real numbers. The codes of topologies are subject to a natural equivalence relation – the equivalence of the topologies they code. We investigate the relative complexity of the equivalence relations corresponding to various codings.

Without loss of generality we consider topological spaces with $\mathbb{N}$ as the underlying set. A second countable topology on $\mathbb{N}$ can be coded naturally by a basis or a subbasis. Let

$$\mathrm{BS} = \{\vec{x} = (x_0, x_1, \ldots) \in (2^{\mathbb{N}})^{\mathbb{N}} \mid \{x_n\}_{n\in\mathbb{N}} \text{ is a basis}\}$$

and

$$\mathrm{SB} = \{\vec{x} = (x_0, x_1, \ldots) \in (2^{\mathbb{N}})^{\mathbb{N}} \mid \{x_n\}_{n\in\mathbb{N}} \text{ is a subbasis}\}.$$

Then BS is the set of codes for all countable bases of topologies on $\mathbb{N}$ and is a $\mathbf{\Pi}^0_3$ subset of $2^{\mathbb{N}\times\mathbb{N}}$. Similarly, SB is the set of codes for all countable subbases and a G_δ ($\mathbf{\Pi}^0_2$) subset of $2^{\mathbb{N}\times\mathbb{N}}$. Let $\equiv_s$ and $\equiv_b$ be the equivalence relations defined on BS and SB, respectively, by the equivalence of topologies. In symbols, for $x, y \in \mathrm{BS}$, let

$$\begin{aligned} x \leq_b y \quad &\Leftrightarrow \quad \text{the topology coded by } x \text{ is coarser than that coded by } y \\ &\Leftrightarrow \quad \forall m \forall n (x(m,n) = 1 \to \exists k (y(k,n) = 1 \wedge \forall l (y(k,l) = 1 \to x(m,l) = 1)) \end{aligned}$$

and

$$x \equiv_b y \iff x \leq_b y \ \wedge \ y \leq_b x.$$

It is evident that $\equiv_b$ is a $\mathbf{\Pi}^0_3$ equivalence relation on BS. In a similar fashion one can define $\equiv_s$ formally and readily see that it is a $\mathbf{\Pi}^0_3$ equivalence relation on SB. Moreover, it is obvious that $\mathrm{BS} \subset \mathrm{SB}$ and $\equiv_b$ and $\equiv_s$ coincide on BS.

Lemma 3.1. *$\equiv_b$ and $\equiv_s$ are Borel bireducible.*

Proof. We only need to show $(\equiv_s) \leq_B (\equiv_b)$. Given $x \in \mathrm{SB}$, let $\{x_n\}_{n\in\mathbb{N}}$ be the subbasis coded by x. Let $\theta(x) \in 2^{\mathbb{N}\times\mathbb{N}}$ code the collection of subsets of $\mathbb{N}$ consisting of sets of the form

$$x_{n_1} \cap x_{n_2} \cap \cdots \cap x_{n_k}$$

where $n_1, \ldots, n_k \in \mathbb{N}$. Then $\theta(x) \in \mathrm{BS}$ and the topology generated by x is the same as that by $\theta(x)$. This θ is a Borel reduction from $\equiv_s$ to $\equiv_b$. □

If the topology is moreover metrizable we have another alternative way to code the topology, namely by a compatible metric. Let $\mathrm{M} \subset \mathrm{BS}$ be the set of all x coding a metrizable topology on $\mathbb{N}$. Let $D \subset \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ be the (closed) set of all functions $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ which are metrics on $\mathbb{N}$. Let $\equiv_m$ be the equivalence relation on D given by the equivalence of topologies, i.e., for $x, y \in D$, $x \equiv_m y$ iff the topology given by the metric coded by x is compatible with the one given by y.

Lemma 3.2. *$\equiv_b\restriction \mathrm{M}$ and $\equiv_m$ are Borel bireducible.*

Proof. From a metric on $\mathbb{N}$ we can define a countable basis for the topology given by the metric. This definition is Borel and, since it preserves the topology, is a reduction from $\equiv_m$ to $\equiv_b\restriction \mathrm{M}$. For the other direction one can follow the procedure given by the standard proof of the Urysohn metrization theorem. □

Lemmas 3.1 and 3.2 show that different ways to code second countable topologies are essentially equivalent whenever they make sense. From now on we will treat these equivalence relations more formally and for further Borel reducibility results in this section we will pay less attention to their intuitive meanings. For example, later this section we will show that $\equiv_b$ and $\equiv_m$ are in fact Borel bireducible.

We now consider another related equivalence relation arising from the context of coding topologies. For $x, y \in 2^{\mathbb{N}\times\mathbb{N}}$, let

$$x \leq_n y \iff \forall n \exists m \forall k \Big(\bigwedge_{i \leq m} y(i,k) = 1 \to \bigwedge_{j \leq n} x(j,k) = 1 \Big)$$

and

$$x \equiv_n y \iff x \leq_n y \ \wedge\ y \leq_n x.$$

The intuition behind the definition is that we would like to view the sequence $\{x_n\}_{n\in\mathbb{N}}$ given by $x \in 2^{\mathbb{N}\times\mathbb{N}}$ as a neighborhood basis (thus the subscript n) for some imaginary point ∞. Hence $x \equiv_n y$ iff the sequences coded by x and y respectively generate equivalent neighborhood bases.

Lemma 3.3. *$(\equiv_n) \leq_B (\equiv_m) \leq_B (\equiv_b) \leq_B (\equiv_n)^\omega$.*

Proof. (i) $(\equiv_n) \leq_B (\equiv_m)$: First of all it is easy to see that there is a Borel function

$$f : 2^{\mathbb{N}} \to D'$$

where

$$D' = \{d : 2\mathbb{N} \times 2\mathbb{N} \to \mathbb{R} \mid d \text{ is a metric on the set of even natural numbers } 2\mathbb{N}\},$$

such that for every x, the metric $f(x)$ takes values < 1 and for $x, y \in 2^{\mathbb{N}}$, $x \neq y$, the topology generated by $f(x)$ on $2\mathbb{N}$ is incompatible with that generated by $f(y)$.

Now let $x \in 2^{\mathbb{N}\times\mathbb{N}}$ and let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence of subsets of $\mathbb{N}$ coded by x. Without loss of generality we may assume $x_{n+1} \subset x_n$ for all n. Let $x_\infty = \bigcap_{n\in\mathbb{N}} x_n$ and let $x'_n = x_n \setminus x_\infty$ for any n. Then $x'_{n+1} \subset x'_n$ for all n and $\bigcap_{n\in\mathbb{N}} x'_n = \emptyset$.

Now we define a metric $\theta(x)$ on $\mathbb{N}$ as follows. Let $\theta(x)$ on $2\mathbb{N}$ be given by $f(x_\infty)$. For $n, m \in \mathbb{N}$, n even and m odd, let $\theta(x)(n,m) = \theta(x)(m,n) = 1$. It remains to define $\theta(x)$ on the set of odd natural numbers $2\mathbb{N}+1$. For notational simplicity we shall call this part of the metric δ_x. For $n, m \in \mathbb{N}$, define

$$\delta_x(2n+3, 2m+3) = \sup\{\frac{1}{2^k} : n \notin x'_k \text{ or } m \notin x'_k\},$$

and

$$\delta_x(2n+3, 1) = \delta(1, \pi(n)) = \sup\{\frac{1}{2^k} : n \notin x'_k\}.$$

This finishes the definition of $\theta(x)$, for which it is easy to check that it is a metric on $\mathbb{N}$.

Now note that

$$x \equiv_n y \iff x_\infty = y_\infty \text{ and } \forall n \exists m (y'_m \subseteq x'_n) \text{ and } \forall n \exists m (x'_m \subseteq y'_n).$$

By the definition of θ, the restrictions of $\theta(x)$ and $\theta(y)$ on $2\mathbb{N}$, which are respectively $f(x)$ and $f(y)$, generate compatible topologies iff $x_\infty = y_\infty$. The definition of δ_x implies that for any $k \in \mathbb{N}$, $x'_k = \{n \mid \delta_x(2n+3, 1) \leq 1/2^{k+1}\}$. It follows that the neighborhood bases for the imaginary point ∞ given by $\{x'_n\}_{n\in\mathbb{N}}$ and $\{y'_n\}_{n\in\mathbb{N}}$ are equivalent iff the topologies given by δ_x and δ_y on $2\mathbb{N}+1$ are compatible.

(ii) $(\equiv_m) \leq_B (\equiv_b)$: Immediate from Lemma 3.2.

(iii)$(\equiv_b) \leq_B (\equiv_n)^\omega$: Suppose $x \in \mathrm{BS}$ and let $\{x_n\}_{n\in\mathbb{N}}$ be the countable basis coded by x. Then for each $m \in \mathbb{N}$, let $\theta_m(x)$ code the collection of x_n so that $m \in x_n$. Define $\theta(x) = (\theta_0(x), \theta_1(x), \dots)$. Then x and y code equivalent topologies iff for every point $m \in \mathbb{N}$, the neighborhood basis of m given by x, which is coded in $\theta_m(x)$, is equivalent with the corresponding neighborhood basis of m given by y. This means that

$$x \equiv_b y \iff \forall m (\theta_m(x) \equiv_n \theta_m(y)),$$

which finishes our proof of the lemma. □

We now introduce yet another equivalence relation into the picture, which will help us eventually obtain Borel bireducibility of all equivalence relations so far considered.

For equivalence relations E and F on an arbitrary set, we say that E is *of finite index over* F if $F \subseteq E$ and every E-equivalence class is the union of finitely many F-equivalence classes. Now we consider equivalence relations on $\mathbb{N}$. Two equivalence relations E and F on $\mathbb{N}$ are *commensurate* if both E and F are of finite index over $E \cap F$.

We can code commensurability as a Borel equivalence relation. Let $\mathrm{P} \subset 2^{\mathbb{N}\times\mathbb{N}}$ be the set of all codes for partitions of $\mathbb{N}$, i.e.,

$$x \in \mathrm{P} \iff \forall m \exists n (x(n,m) = 1) \ \wedge\ \forall n \forall n' \forall m (x(n,m) = 1 \to x(n',m) = 0).$$

Apparently P is a G_δ subset of $2^{\mathbb{N}\times\mathbb{N}}$. For $x, y \in \mathrm{P}$, let $x \equiv_c y$ iff x and y (as equivalence relations) are commensurate. Then $\equiv_c$ is a $\mathbf{\Pi}^0_3$ equivalence relation on P.

Lemma 3.4. *$\equiv_c$ and $\equiv_n$ are Borel bireducible.*

Proof. First of all it is easy to see that there is a Borel function

$$f : 2^{\mathbb{N}} \to \mathrm{P}$$

such that for all $x, y \in 2^{\mathbb{N}}$, $x \neq y$, $f(x)$ and $f(y)$ are not commensurate. In fact, it is enough to define f so that each $f(x)$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$. A definition of f can be

$$(m,n)f(x)(m',n') \iff m = m' \ \wedge \ (x(m) = 1 \ \vee \ (x(m) = 0 \ \wedge \ n = n')).$$

Now we consider the direction $(\equiv_n) \leq_B (\equiv_c)$. Given $x \in 2^{\mathbb{N}\times\mathbb{N}}$ we can define $\{x_n\}_{n\in\mathbb{N}}$ and x_∞ just as in the proof of Lemma 3.3 (i). The argument there, together with the fact in the preceding paragraph, justifies that we may assume without loss of generality that $x_{n+1} \subset x_n$ ($x_{n+1} \neq x_n$) for all n and that $x_\infty = \emptyset$. Now we define $\tilde{x}_0 = \mathbb{N} \setminus x_0$ and for $n \geq 1$, $\tilde{x}_n = x_{n-1} \setminus x_n$. Then the collection $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ is a partition of $\mathbb{N}$. We let $\rho(x) \in \mathrm{P}$ code this partition. Then ρ is obviously Borel.

Now

$$x \equiv_n y \iff \forall n \exists m (y_m \subseteq x_n) \ \wedge \ \forall n \exists m (x_m \subseteq y_n)$$

and

$$\begin{aligned} \forall n \exists m (y_m \subseteq x_n) &\iff \forall n \exists m (\tilde{x}_n \subseteq \tilde{y}_0 \cup \cdots \cup \tilde{y}_m) \\ &\iff \rho(x) \text{ is of finite index over } \rho(x) \cap \rho(y). \end{aligned}$$

Thus $x \equiv_n y$ iff $\rho(x) \equiv_c \rho(y)$.

It is clear that the process can be reversed to produce a Borel reduction of the opposite direction. □

A moment reflection verifies that $\equiv_c$ is Borel bireducible with $(\equiv_c)^\omega$. Thus putting Lemmas 3.1 through 3.4 together, we have shown the following theorem.

Theorem 3.1. *The equivalence relations $\equiv_s$, $\equiv_b$, $\equiv_m$, $\equiv_n$, $\equiv_c$ and arbitrary finite or infinite products of them are pairwise Borel bireducible.*

The theorem reinforces the stability of the complexity for codings of second countable topologies. Thus intuitively we may refer to the equivalence relations as the "identity of topologies". The following proposition gives an estimate for the complexity of this equivalence relation. As a consequence which is not surprising, this identity of topologies is not reducible to any orbit equivalence relation induced by a Borel action of a Polish group.

Proposition 3.2. $(E_1)^\omega \leq_B (\equiv_b)$.

Proof. Since $\equiv_b$ is Borel bireducible to $(\equiv_b)^\omega$ it suffices to prove $E_1 \leq_B (\equiv_b)$. For $x \in 2^{\mathbb{N}\times\mathbb{N}}$ let B_x be the set of all $y \in 2^{\mathbb{N}\times\mathbb{N}}$ such that

$$\exists n \ (\forall m \geq n \forall k \ (y(m,k) = x(m,k)) \ \wedge \ \forall m < n \exists \text{ finitely many } k \ (y(m,k) = 1)).$$

Then B_x is countable and is in fact a basis over $\mathbb{N} \times \mathbb{N}$. Let τ_x denote the topology on $\mathbb{N} \times \mathbb{N}$ generated by this basis. We claim that xE_1y iff $\tau_x = \tau_y$.

Suppose xE_1y. By symmetry we only show that $\tau_x \subseteq \tau_y$. Let $(i,j) \in \mathbb{N} \times \mathbb{N}$ and $U \in B_x$ with $(i,j) \in U$. Let $N > i$ such that for all $n \geq N$ and k, $x(n,k) = y(n,k)$ and $x(n,k) = U(n,k)$. Then define $V \in 2^{\mathbb{N}\times\mathbb{N}}$ by

$$V(n,k) = \begin{cases} 1 & \text{if } (n,k) = (i,j), \\ 0 & \text{if } n < N \text{ and } (n,k) \neq (i,j), \\ y(n,k) & \text{if } n \geq N. \end{cases}$$

Then $V \in B_y$ and $(i,j) \in V \subseteq U$ as required.

Suppose $\tau_x = \tau_y$. Note that $x \in B_x$, so there is $V \in B_y$ such that $V \subseteq x$. Hence there is $n \in \mathbb{N}$ such that for all $m \geq n$ and all k, $V(m,k) = y(m,k)$. On the other hand, there is also $U \in B_x$ with $U \subseteq V$. Thus there is $N \geq n$ such that for all $m \geq N$ and all k, $U(m,k) = x(m,k)$. It follows that for all $m \geq N$ and all k, $x(m,k) = y(m,k)$, hence xE_1y. □

Note that the topologies τ_x constructed in the above proof are T_1 but not Hausdorff for all x but from one E_1-class. However, it follows abstractly from Theorem 3.1 that $(E_1)^\omega \leq_B (\equiv_m)$. We would like to remark that it is not hard to modify the above proof so as to give a direct reduction to $\equiv_m$.

We do not know if $(E_1)^\omega <_B (\equiv_b)$.

4. The homeomorphism problem

In this section we turn to the homeomorphism problem for countable topological spaces which are second countable. We will follow the notation of the preceding section.

Let $\cong_s$, $\cong_b$ and $\cong_m$ be the homeomorphism relations on SB, BS and D, respectively. Then each of them is a $\mathbf{\Sigma}^1_1$ equivalence relation. In fact, the action of S_∞ on $\mathbb{N}$ induces natural Borel actions of S_∞ on SB, BS and D, so that for $\sigma \in \{s, b, m\}$ and appropriate x and y,

$$x \cong_\sigma y \iff \exists g \in S_\infty\, (g \cdot x \equiv_\sigma y).$$

Lemmas 3.1 and 3.2 establish quickly that $(\cong_m) \leq_B (\cong_b) \leq_B (\cong_s) \leq_B (\cong_b)$.

It is also natural to consider $\cong_n$ and $\cong_c$ from a similar perspective, i.e., with naturally induced actions of S_∞ on $2^{\mathbb{N}\times\mathbb{N}}$ and P respectively, we could define

$$x \cong_n y \iff \exists g \in S_\infty\, (g \cdot x \equiv_n y), \text{ for } x, y \in 2^{\mathbb{N}\times\mathbb{N}},$$

and

$$x \cong_c y \iff \exists g \in S_\infty\, (g \cdot x \equiv_c y), \text{ for } x, y \in \mathrm{P}.$$

However, it is straightforward to check that $\cong_c$ has only 4 equivalence classes and $\cong_n$ has only countably many equivalence classes. Hence these equivalence relations are not interesting. Recall in the proof of Theorem 3.1 that $\equiv_n$ and $\equiv_c$ were instrumental in establishing the bireducibility of the other equivalence relations. Here we do not know if $(\cong_b) \leq_B (\cong_m)$.

In the rest of this section we will prove results which indicate that $\cong_m$ is complex. Since $\cong_m$ is no more complex than $\cong_b$ the same consequences apply to $\cong_b$ as well. We shall argue from two different perspectives the complexity of $\cong_m$. The first is that $\cong_m$ is not Borel reducible to any orbit equivalence relation induced by a Borel action of a Polish group. The second is that the isomorphism relation of countable graphs is Borel reducible to $\cong_m$. By general results of Becker and Kechris, this is equivalent to saying that any orbit equivalence relation induced by a Borel action of S_∞ is Borel reducible $\cong_m$. In particular, $\cong_m$ is $\mathbf{\Sigma}^1_1$-complete. These are implemented by the following theorems.

Theorem 4.1. $(\equiv_m)^+ \leq_B (\cong_m)$.

Proof. By examining the proof of Theorem 3.1 one sees that $\equiv_m$ is Borel bireducible with the identity of topologies whose codes belong to the following set:

$$\tilde{D} = \{d \in D \,|\, d \leq 1 \text{ and } \tau_d \text{ has exactly one limit point } 0\},$$

where τ_d is the topology on $\mathbb{N}$ coded by d. Indeed, one can take the composition of the reductions from $\equiv_c$ to $\equiv_n$ and from $\equiv_n$ to $\equiv_m$. The resulting reduction is from $\equiv_n$ to $\equiv_m$, whose image consists of topologies with at most one limit point.

By trivial but notationally tedious rearrangement of the reduction it is possible to make the limit point, if any, 0. In other words, we then have a reduction from $\equiv_n$ (hence from $\equiv_m$) to $\equiv_m$ restricted on $\tilde{D}$ plus one more topology, namely the discrete one. A further modification can then make the image a subset of $\tilde{D}$, as required by the claim.

Now we define a Borel reduction from $(\equiv_m \restriction \tilde{D})^+$ to $\cong_m$. As a first step we define for each $d \in \tilde{D}$ a countable metric space M_d. The topological space of M_d is obtained by adding to each point $n \neq 0$ in the topological space $(\mathbb{N}, \tau_d)$ a countable sequence L_n of new points and making it converge to n. The sequence L_n is linearly ordered and topologized by its order topology, and the order type of L_n is ω^n. For us it is necessary to see that M_d is metrizable but the exact metric on $\mathbb{N}$ used to code it is irrelevant.

We define a compatible metric δ_d for M_d. Note that the underlying set of M_d is the disjoint union of $\mathbb{N}$ together with disjoint copies of all $L_n = \omega^n$ for $0 \neq n \in \mathbb{N}$. For each $n > 0$ let $\lambda_n \leq 1$ be some fixed canonical metric on $L_n \cup \{n\}$ compatible with its order topology. The following formulae take care of all cases of the definition of δ_d:

$$\begin{aligned}
\delta_d(n,m) &= d(n,m), \text{ for } n,m \in \mathbb{N},\\
\delta_d(\alpha,\beta) &= \lambda_n(\alpha,\beta), \text{ for } \alpha,\beta \in L_n \cup \{n\},\\
\delta_d(\alpha,m) &= \delta_d(m,\alpha) = \lambda_n(\alpha,n) + d(n,m), \text{ for } \alpha \in L_n,\\
\delta_d(\alpha,\gamma) &= \lambda_n(\alpha,n) + d(n,m) + \lambda_m(m,\gamma), \text{ for } \alpha \in L_n,\ \gamma \in L_m.
\end{aligned}$$

Now given a countable set (for notational compactness we abuse the vector sign)

$$\vec{d} = \{d_0, d_1, d_2, \ldots\},$$

we define the countable metric space $M_{\vec{d}}$ to be the disjoint union of

$$M_{d_0}, M_{d_1}, M_{d_2}, \ldots.$$

It is easy to define a compatible metric on $M_{\vec{d}}$; in fact, it suffices to specify that points from different M_{d_n} components have distance 3 (noting that each M_{d_n} has diameter 3). Topologically, each M_{d_n} is a clopen subset of $M_{\vec{d}}$.

The map $\vec{d} \mapsto M_{\vec{d}}$ is obviously Borel. We claim that

$$\vec{d} \text{ is } (\equiv_m)^+\text{-equivalent to } \vec{r} \iff M_{\vec{d}} \text{ is homeomorphic to } M_{\vec{r}}.$$

The "only if" direction is clear. To verify the "if" direction, let f_0 be a homeomorphism from $M_{\vec{d}}$ onto $M_{\vec{r}}$. For each $k \in \mathbb{N}$, let $M_{\vec{d}}^{(k)}$ and $M_{\vec{r}}^{(k)}$ be the k-th Cantor-Bendixson derivative of $M_{\vec{d}}$ and $M_{\vec{r}}$, respectively. Let

$$M_{\vec{d}}^{(\infty)} = \bigcap_{k \in \mathbb{N}} M_{\vec{d}}^{(k)}.$$

Then $M_{\vec{d}}^{(\infty)}$ consists of all $0 \in \mathbb{N} \subset M_{d_n}$ for all $n \in \mathbb{N}$. Similarly one can define $M_{\vec{r}}^{(\infty)}$. Now

$$f_\infty = f_0 \restriction M_{\vec{d}}^{(\infty)} : M_{\vec{d}}^{(\infty)} \to M_{\vec{r}}^{(\infty)}$$

is a homeomorphism, in particular a one-one correspondence. Without loss of generality, we may assume that for each $n \in \mathbb{N}$, f_∞ send $0 \in \mathbb{N} \subset M_{d_n}$ to $0 \in \mathbb{N} \subset M_{r_n}$.

Note that, for each $k \in \mathbb{N}$, the map

$$f_k = f_0 \restriction M_{\vec{d}}^{(k)} : M_{\vec{d}}^{(k)} \to M_{\vec{r}}^{(k)}$$

is also a homeomorphism. By induction on k, we can define an infinite sequence of homeomorphisms $\{g_k\}_{k\in\mathbb{N}}$, each $g_k : M_{\vec{d}} \to M_{\vec{r}}$, so as to satisfy the following conditions:

(i) for each $k \in \mathbb{N}$, $g_k \restriction M_{\vec{d}}^{(k+1)} = f_k$;
(ii) for each $k \in \mathbb{N}$ and $n \in \mathbb{N}$, g_k send $k \in \mathbb{N} \subset M_{d_n}$ to $k \in \mathbb{N} \subset M_{r_n}$;
(iii) for each $k \in \mathbb{N}$, letting V_k be the subset of $M_{\vec{d}}$ where g_k and g_{k+1} disagree, then V_k is clopen and is disjoint from V_l if $k \neq l$.

In fact, let $g_0 = f_0$. Each g_{k+1} is a modification of g_k according to the requirement (ii). In order to maintain that g_{k+1} is a homeomorphism, we need to identify, for each $k+1 \in \mathbb{N} \subset M_{d_n}$ (whose image under g_k is not yet $k+1 \in \mathbb{N} \subset M_{r_n}$), a clopen set $U \subset M_{d_n}$ containing it and then redefine $g_{k+1} \restriction U$ in a natural way. To guarantee (iii) we need only make U disjoint from all V_l for $l < k$, which is possible since by induction the union of V_l for $l < k$ is clopen in $M_{\vec{d}}$.

Now by conditions (i) and (iii) the direct limit of g_k exists. Let g_∞ denote this direct limit. Then g_∞ is a homeomorphism from $M_{\vec{d}}$ onto $M_{\vec{r}}$, which follows from (iii) and the fact that each g_k is a homeomorphism. Moreover, by (ii) we have that for each $n \in \mathbb{N}$, g_∞ send the copy of $\mathbb{N}$ in M_{d_n} onto the copy of $\mathbb{N}$ in M_{r_n}. On these copies, g_∞, being a homeomorphism, is in fact the identity map. Thus the topologies on these copies are identical. In symbols, $d_n \equiv_m r_n$. Finally, it follows that $\vec{d}$ and $\vec{r}$ are $(\equiv_m)^+$-equivalent. □

By now it is clear that $\cong_m$ is not Borel reducible to any orbit equivalence relation induced by a Borel action of a Polish group, since

$$E_1 <_B (E_1)^\omega \leq_B (\equiv_m) <_B (\equiv_m)^+ <_B (\cong_m).$$

The next theorem is our second approach to demonstrating that $\cong_m$ is complex. Its proof closely follows the line of proof for the main theorem in [CG]. As always the proof consists of two parts: a construction and a verification. For the sake of completeness, we repeat here some part of the proof in [CG] so that at least the construction of the metric spaces is self-contained. A warning concerning notation is due: since we try to follow the use of symbols in [CG] the notation in the following proof is not consistent with that in the earlier parts of this paper. For example, τ no longer denotes a topology and d is no longer a metric.

Theorem 4.2. *The isomorphism of countable graphs is Borel reducible to $\cong_m$.*

Proof. We need to associate to each countable graph G a countable metric space M_G so that

$$G \text{ is isomorphic to } G' \iff M_G \text{ is homeomorphic to } M_{G'}.$$

For this we will only consider infinite graphs and assume without loss of generality that they have underlying universe $\mathbb{N}$. The space M_G will be described as a topological space, and it will be clear from the construction that M_G is metrizable. Indeed, each M_G is going to be a subset of the real line with the induced topology, hence there will be no doubt about its metrizability.

To begin with, we assign to each countable graph G a labeled tree which codes the isomorphism type of G. This method of setup is a standard procedure using some concepts of logic. Let $\mathcal{L}_0$ be the language of one binary relation symbol, i.e., the language of graphs. Let $v_0, v_1, \ldots$ be a list of logical variables. Let TY denote the set of all quantifier free types in the variables $v_0, v_1, \ldots$ in the language

$\mathcal{L}_0 \cup \{=\}$. For every $n \in \mathbb{N}$, let TY_n consist of those types in TY in which only variables in $\{v_0, \ldots, v_n\}$ occur. Let e be a non-repetitive enumeration of TY such that for each $n \in \mathbb{N}$ and $i < j \in \mathbb{N}$, $e(j) \in \mathrm{TY}_n$ implies $e(i) \in \mathrm{TY}_n$. The type $e(i)$ is said to be coded by i. Now for $s \in \mathbb{N}^{<\mathbb{N}}$ let $\tau_G(s) \in \mathbb{N}$ be the code of the quantifier free type of s in G. Then τ_G assigns labels to the tree $\mathbb{N}^{<\mathbb{N}}$. The metric space we are to define codes the information presented by this labeled tree.

We next define a core space for all M_G. Eventually each M_G will be constructed uniformly from this space by adding to it some more points. For now, the construction of the core space does not depend on the labels in the tree induced by G. We start with the interval $[0, 1]$ and the standard Cantor $1/3$ set, $E_{1/3}$, obtained by removing infinitely many open intervals from $[0, 1]$. Among these removed intervals we can define a natural order: if I and J are two of such intervals, naturally

$$I < J \iff a < b \text{ for all } a \in I \text{ and } b \in J.$$

Then the set of all removed intervals, ordered by $<$, has the same order type as the rational numbers. We can thus enumerate this set of removed intervals by $\mathcal{I} = \{I_q\}_{q\in\mathbb{Q}}$ so that $q < q' \Leftrightarrow I_q < I_{q'}$. We also fix an enumeration of $\mathbb{Q} = \{q_n\}_{n\in\mathbb{N}}$.

By induction on $s \in \mathbb{N}^{<\mathbb{N}}$ we define subsets $\mathcal{I}_s = \{J_{s^\frown n}\}_{n\in\mathbb{N}}$ of $\mathcal{I}$ so as to satisfy the following conditions:

(i) $\mathcal{I}_\emptyset = \{J_n\}_{n\in\mathbb{N}}$ is a decreasing sequence such that $\lim_{n\to\infty}(\sup J_n) = 0$;

(ii) if $\mathrm{lh}(s) > 0$, $\{J_{s^\frown n}\}_{n\in\mathbb{N}}$ is a decreasing sequence such that $\lim_{n\to\infty}(\sup J_{s^\frown n}) = J_s$;

(iii) for $s, t \in \mathbb{N}^{<\mathbb{N}}$, $J_s < J_t$ iff $s <_{\mathrm{KB}} t$, where $<_{\mathrm{KB}}$ is the Kleene-Brouwer ordering on $\mathbb{N}^{<\mathbb{N}}$;

(iv) if s is not constantly 0, then $J_{s^\frown 0} < J_{s^-}$, where s^- is the $<_{\mathrm{KB}}$-immediate predecessor of s in $\mathbb{N}^{\leq \mathrm{lh}(s)}$;

(v) for each s, $J_{s^\frown 0} = I_{q_l}$ where l is the smallest such that $J_s < I_{q_l}$ and I_{q_l} satisfies (iv) in place of $J_{s^\frown 0}$.

The conditions can be met since $\mathcal{I}$ is a dense linear order, and condition (v) guarantees that

$$\mathcal{I} = \bigcup_{s\in\mathbb{N}^{<\mathbb{N}}} \mathcal{I}_s.$$

Denote $d_\emptyset = 0$, and $d_s = \sup J_s$ for other $s \in \mathbb{N}^{<\mathbb{N}}$. Let $Q = \{d_s \mid s \in \mathbb{N}^{<\mathbb{N}}\}$, which we called the set of critical points in [CG]. This space Q is our countable core space for M_G. Note that its completion is precisely $E_{1/3}$.

Following [CG] we also fix a surjection $a : 2\mathbb{N} \to \mathbb{N}$ with the property that $a^{-1}(\{n\})$ is infinite for every $n \in \mathbb{N}$. This induces a map of $(2\mathbb{N})^{<\mathbb{N}}$ onto $\mathbb{N}^{<\mathbb{N}}$, which we also denote by a. Discriminate the elements of $\mathbb{N}^{<\mathbb{N}}$ by saying s is *good* if $s \in (2\mathbb{N})^{<\mathbb{N}}$, s is *mixed* if $s = t^\frown(2n + 1)$, where t is good and $n \in \mathbb{N}$, and s is *bad* if it is not good nor mixed.

We are finally ready to define our space M_G. M_G is the union of Q with countably many points, which are placed in the removed intervals J_s as follows:

- if s is good then add in J_s an increasing sequence of order type $\omega^{\tau_G(a(s))+2}$ converging to d_s;
- if s is mixed, say $s = t^\frown(2n + 1)$, where t is good and $n \in \mathbb{N}$, then add in J_s an increasing sequence of order type $\omega^{\tau_G(a(t))+2}$ converging to d_s;
- if s is bad then add a middle point in J_s.

The difference between our space M_G here and the space X_G defined in [CG] is merely that we used $E_{1/3}$, instead of $\tilde{Q}$, as the core space for X_G. It is immediate that the completion of M_G is exactly X_G. In [CG] we established that

$$G \text{ is isomorphic to } G' \iff X_G \text{ is homeomorphic to } X_{G'}.$$

Thus if G and G' are not isomorphic, then M_G is not homeomorphic with $M_{G'}$, since otherwise a homeomorphism from M_G onto $M_{G'}$ would extend to a homeomorphism from X_G onto $X_{G'}$. On the other hand, if G and G' are isomorphic, then the argument in [CG] (particularly that of Claim 5 of section 3) actually establishes that there is a homeomorphism of X_G onto $X_{G'}$ which sends $X_G \setminus E_{1/3}$ onto $X_{G'} \setminus E_{1/3}$, the copy of $E_{1/3} \setminus Q$ in X_G onto that in $X_{G'}$, and the copy of Q in X_G onto that in $X_{G'}$. Thus, putting two pieces of these together to form M_G, we obtain a homeomorphism from M_G onto $M_{G'}$. □

Note that the spaces M_G constructed in the above proof are actually countable linear orders endowed with the order topology. Thus we have the following corollary.

Corollary 4.3. *The isomorphism for countable graphs is Borel reducible to the homeomorphism for countable linear orders endowed with the order topology.*

Note that a homeomorphism between linear orders with the order topology need not preserve the linear orders. Thus the above homeomorphism problem is a coarser equivalence relation than the isomorphism of countable linear orders, which is known to be Borel bireducible with the isomorphism of countable graphs ([FS]). We do not know if the homeomorphism for countable linear orders with the order topology is Borel bireducible with the graph isomorphism.

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