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FURTHER REMARKS ON FIRST-ORDER INFINITESIMAL MECHANISMS

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Abstract—This paper analyses a particular structural assembly that has two independent inextensional mechanisms and two independent states of self-stress. It is shown that, for certain values of a variable bar length, this assembly is a first-order infinitesimal mechanism; and yet it cannot be stiffened by the imposition of a state of prestress.

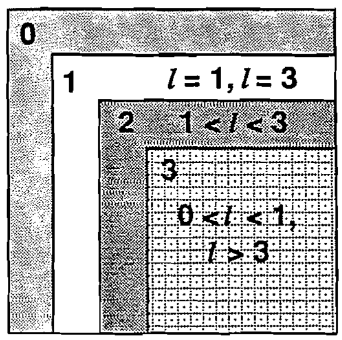
1. INTRODUCTION

The classical *equilibrium matrix* can tell us a lot about the structural characteristics of an assembly of bars and simple swivel joints, of the kind illustrated in Fig. 2. In particular, standard matrix techniques can be used to find the four fundamental vector sub-spaces; and these reveal not only the number m ($\geqslant 0$) of independent mechanisms and the number s ($\geqslant 0$) of independent states of self-stress, but also the details of the inextensional modes and self-stress patterns concerned.

All of these calculations are made with respect to the initial geometry of the assembly; and so they obviously cannot reveal whether mechanisms (when $m > 0$) are *finite*—as in a simple four-bar chain, for example—or merely *infinitesimal*; or indeed whether infinitesimal mechanisms are of first- or higher-order, to use terms that we shall define below.

In many practical problems it is important for the engineer to know, for an assembly having $m > 0$, what is the order of its mechanisms. Thus, in a recent paper (Calladine and Pellegrino, 1991a) we set out to devise a general test which would indicate whether or not the deformation modes of a given assembly were first-order infinitesimal. We started from the observation that in some assemblies having $m > 0$ and $s = 1$, the state of self-stress can impart first-order stiffness to all possible modes. We were interested in generalizing this route of analysis to cases where $s \geqslant 2$; and we produced an algorithm which shows, for a given assembly, whether it is possible to find a single state of self-stress which imparts first-order stiffess to all of the mechanisms. If such a state of self-stress does exist, then all of the mechanisms are indeed of first order. What we did not realize was that although satisfaction of our test was *sufficient* to prove that all mechanisms are of first order, it was not *necessary*: failure of our test does not rule out the possibility that all mechanisms are indeed of first order.

Figure 1 shows a "map" in which the various possible kinds of behaviour which an assembly may display occupy different regions. Thus, region 0 corresponds to assemblies



0	finite
1	higher-than-first-order infinitesimal
2 + 3	first-order infinitesimal
2	first-order infinitesimal which cannot be stiffened by a single state of self-stress
3	first-order infinitesimal for which all mechanisms are stiffened by one state of self-stress

Fig. 1. Graphic illustration of different types of infinitesimal mechanisms. The assembly shown in Fig. 2 lies in different regions, for different values of l. The algorithm described by Calladine and Pellegrino (1991a) identifies assemblies which belong to region 3.

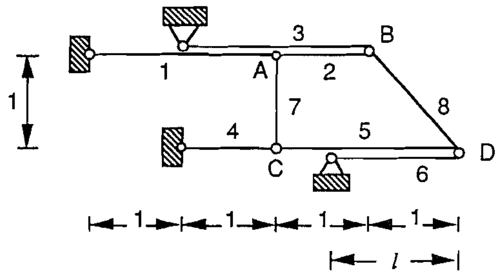


Fig. 2. Planar assembly with two independent inextensional mechanisms, $m = 2$, and two independent states of self-stress, $s = 2$. Bar 6 has variable length l. It is assumed that bar 6 is always to the left of joint D.

with finite mechanisms, while regions 2 and 3 together correspond to assemblies having only first-order infinitesimal mechanisms. The only other possibility, region 1, is that an assembly has higher-than-first-order infinitesimal mechanisms. Our previous test revealed assemblies in region 3; but it did not reveal assemblies in region 2—of which, at the time, we knew no examples.

In this paper we clarify the situation by discussing the particular assembly shown in Fig. 2, which was brought to our attention by Kuznetsov (1991a). The lengths of all bars but one are fixed, but bar 6 has variable length l. Kuznetsov has pointed out that for any value of l other than 1 or 3 the system shown in Fig. 2 is a first-order infinitesimal mechanism.

2. GENERAL CASE ($l > 0$)

In a previous paper (Calladine and Pellegrino, 1991a) we have considered a general assembly with m independent inextensional mechanisms and s independent states of self-stress, where m, s and the corresponding sets of independent mechanisms $\mathbf{D}$ and states of self-stress $\mathbf{T}$ are found by analysing the equilibrium matrix for the assembly. Let $\mathbf{D}\boldsymbol{\beta}$ be a general inextensional mechanism and $\mathbf{T}\boldsymbol{\alpha}$ be a general state of self-stress ($\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are respectively m- and s-dimensional vectors of arbitrary parameters). We have shown that the work W done when the assembly is prestressed with $\mathbf{T}\boldsymbol{\alpha}$ and then displaced by $\mathbf{D}\boldsymbol{\beta}$ is given by:

$$W = \boldsymbol{\beta}^T\left(\sum_{i=1}^{s} \mathbf{P}_i^T \mathbf{D}\alpha_i\right)\boldsymbol{\beta}$$

where $\mathbf{P}_i$ is the matrix of product-forces which correspond to the state of self-stress i, i.e. column i of $\mathbf{T}$.

If we describe a particular assembly as being a "first-order infinitesimal mechanism" we mean that for *all* small $\boldsymbol{\beta}$s the changes in length of the bars are second-order infinitesimal. If, however, for all but some $\boldsymbol{\beta}$s the length changes are second-order infinitesimal in $\boldsymbol{\beta}$ but for that mechanism or those mechanisms they are third-order infinitesimal, then the assembly is described as a "second-order infinitesimal mechanism". Indeed it is possible to find an assembly which has the characteristics of a first-order infinitesimal mechanism for all but one single mechanism which is of second order; in which case the assembly as a whole is described as being of second order. This agrees with Definition 1 of an nth order infinitesimal mechanism, where $n \geqslant 1$, in Tarnai (1989).

In a first-order infinitesimal mechanism W must be positive for any $\boldsymbol{\beta} \neq 0$. For the assembly shown in Fig. 2

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{bmatrix}^T$$

hence

$$W = [\beta_1 \quad \beta_2]\left(\begin{bmatrix} 1.5 & -1 \\ -1 & 0.5 \end{bmatrix}\alpha_1 + \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5-1/l \end{bmatrix}\alpha_2\right)\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

In our previous paper we have argued that, in order to make $W > 0$ for any $\boldsymbol{\beta} \neq 0$, one needs to find a particular state of self-stress, and hence an $\boldsymbol{\alpha}$, for which the quadratic form W is positive definite. This approach assumes that, as in the special and well-studied case $s = 1$, there should be a single $\boldsymbol{\alpha}$ for the entire family of $\boldsymbol{\beta}$. However, it turns out that in the example of Fig. 2 with $0 < l < 1$ and with $l > 3$ there is one $\boldsymbol{\alpha}$ which makes $W > 0$ for all $\boldsymbol{\beta}$s; but in the range $1 < l < 3$ different $\boldsymbol{\alpha}$s are needed to cover the entire range of $\boldsymbol{\beta}$s. Thus, our search for a single $\boldsymbol{\alpha}$, following the scheme of our previous paper, leads to the conclusion that in the range $1 < l < 3$ there is no state of self-stress which will stiffen the two-dimensional infinity of inextensional mechanisms; but it is not correct to deduce from this that the assembly is not a first-order infinitesimal mechanism. An assembly may still be a first-order infinitesimal mechanism if for any mechanism, $\boldsymbol{\beta}$, there is at least one state of self-stress, $\boldsymbol{\alpha}$, for which $W > 0$.

In this case, an equivalent and more productive approach to the question of the existence of a higher-than-first-order mechanism is to seek mechanisms for which *W vanishes for all possible states of self-stress*. Thus, for the example of Fig. 2, we can have higher-than-first-order mechanisms only if both quadratic forms in the expression for W vanish simultaneously:

$$\left\{\begin{array}{l} [\beta_1 \quad \beta_2]\begin{bmatrix} 1.5 & 1 \\ -1 & 0.5 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0 \\ [\beta_1 \quad \beta_2]\begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5-1/l \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0. \end{array}\right. \tag{1}$$

The first equation in the system (1) has the solutions $\beta_1/\beta_2 = 1, \frac{1}{3}$, regardless of l, while the second equation has the solutions

$$\frac{\beta_1}{\beta_2} = \frac{1 \pm \sqrt{\frac{6}{l} - 2}}{3}.$$

Hence the system has one solution only (viz. $\beta_1 = \beta_2$) for $l = 1$, and one solution only (viz. $\beta_1 = \frac{1}{3}\beta_2$) for $l = 3$. This calculation indicates that for each of $l = 1$ and $l = 3$ there is one particular higher-than-first-order mechanism. For all other values of $l > 0$ there are *no* solutions to system (1), and therefore the assembly is a first-order infinitesimal mechanism. This result agrees with Kuznetsov (1991a) and the above analysis is equivalent to that in Kuznetsov (1991b), although it was done independently.

It is interesting to note that the same system of quadratic equations (1) can be obtained as two second-order compatibility conditions for the top and bottom parts of the given linkage, respectively. For example, the second-order compatibility condition for bars 1, 2 and 3, which refers to the horizontal components of displacement, may be assembled from the schematic diagram of second-order displacements shown in Fig. 3. It is

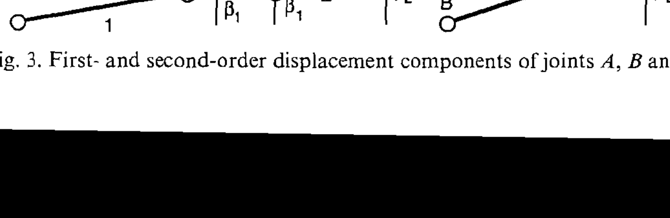


Fig. 3. First- and second-order displacement components of joints A, B and C.

$$\frac{\beta_1^2}{4}+\frac{(\beta_2-\beta_1)^2}{2}=\frac{\beta_2^2}{4},$$

which coincides with the first equation in the system (1).

3. TWO SPECIAL CASES: $l = 1$, $l = 3$

Calladine and Pellegrino (1991b) have shown that the particular case of an assembly with $l = 1$ is a second-order infinitesimal mechanism, i.e. that the mechanism having $\beta_1 = \beta_2$, which is the only mechanism of higher-than-first-order, is actually a second-order infinitesimal mechanism. This result has been checked independently by means of a perturbation analysis of strain energy (Salerno, 1990).

We have not studied the case $l = 3$ in any detail. The calculation described above reveals a single mechanism which is higher-than-first-order (viz. $\beta_1 = \frac{1}{3}\beta_2$), which Kuznetsov (1991a) has found to be third-order infinitesimal.

4. CONCLUSIONS

The example described in this paper demonstrates that only some, and not all, assemblies with $s > 1$ which are first-order-infinitesimal mechanisms can be endowed with first-order stiffness against all inextensional modes by a single state of self-stress. This is a crucial consideration for designers of Tensegrity domes, cable nets, etc., who require their assemblies to have first-order stiffness in all possible modes.

The condition that W should be a positive definite quadratic form, introduced by Calladine and Pellegrino (1991a) is necessary and sufficient for the existence of a single state of self-stress which stiffens all mechanisms, in which case the assembly is a first-order infinitesimal mechanism. However, if the test in that paper is not satisfied, the assembly may or may not be a first-order infinitesimal mechanism: and to find out if it is higher-than-first-order, it is necessary to perform further calculations.

A necessary and sufficient condition for an assembly to be a first-order infinitesimal mechanism is that there is no solution to the system of s quadratic equations: $\boldsymbol{\beta}^T\mathbf{P}_i^T\mathbf{D}\boldsymbol{\beta} = \mathbf{0}$ where $i = 1, \ldots, s$. Indeed, this appears to agree with a suggestion by Kötter: see Section 5 of our previous paper and also Kuznetsov (1991b).

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