

A Basic Example of Non Linear Equations : The Navier-Stokes Equations

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1 Scaling, hierarchies and formal derivations.

Many systems of partial differential equations, linear and non linear, are used to describe physical phenomenons such as electromagnetism, elasticity, etc... In this section we have chosen to describe the Navier-Stokes equations which govern the flow of a viscous fluid: they are of first importance in fluid mechanics, and exhibit by themselves all the main features and difficulties of non linear equations.

There are several reasons why the study of the Navier Stokes and other closely related equations has been central in the activities of mathematicians for more than two centuries. This started probably with Euler and involved the contributions of such diverse personalities as, Leray, Kolmogorov, Arnold and others. The Navier Stokes equations are perfectly well defined mathematical objects and are paradigms of nonlinear equations. The solutions exhibit in their behaviour many characteristics of genuinely non linear phenomena.

In view of the needs of practical applications in engineering sciences success have been limited. However the results that have been obtained contribute to our understanding of the program and this is the main idea that I would like to describe in these notes.

Fluid mechanics is in the range of our capacity of observations since the beginning of modern science. It is usual, as is done in the book of Uriel Frisch [23], to quote some notes written by Leonardo da Vinci about turbulence in fluids. The mechanics of fluids has been used as a model for the description of phenomena that in the 18th and 19th centuries were quite mysterious, like electromagnetism (cf. Helmholtz who also made important contributions in vortex theory). Eventually the study of fluid mechanics contributed in an essential way, with the work of Boltzmann and Maxwell, to the understanding of the notion of atoms.

There is no question about the validity of the equations. Nothing has to be discovered from them concerning the intimate nature of the physics. They are just consequence of the incontestable Newton law of mechanics either applied directly to the molecules of the fluid or applied, at a more macroscopic level to elementary volumes of fluid (even if it requires some non obvious work to go from the atomic description to the continuous one). The present problems are: *how can one describe the phenomena with adequate equations, how can one compute them, and use and visualize the results in spite of their complexity.*

The equations involve some physical parameters and turn out to be relevant when these parameters have certain values. Therefore as an introduction it is natural to consider a “chain” of equations, hoping, as is often the case, that the next equation will become relevant when the

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structure of the phenomena becomes too complicated to be computed by the previous one. The Navier Stokes equations appear to be one of the main links in this chain:

I Hamiltonian system of particles ,

↓

II Boltzmann equation ,

↓

III Navier Stokes equations ,

↓

IV Models of turbulence .

Each step is deduced from the previous one with the introduction of hierarchy of equations and a process of closure which in some cases leads to the appearance of irreversibility.

According to the classical Newton law, the evolution of N particles is described by a hamiltonian system defined in the phase space $\mathbf{R}^{3N} \times \mathbf{R}^{3N}$:

$$(1) \quad H_N(x_1, x_2, \dots, x_N, v_1, v_2, \dots, v_N) = \sum_{1 \leq i \leq N} m \frac{|v_i|^2}{2} + \sum_{1 \leq i < j \leq N} V(|x_i - x_j|),$$

N is the Avogadro number, of the order of 10^{24} . One introduces σ , the range of action of the interacting potential V (or the diameter of the molecules when instead of (1) one uses the dynamic of elastic collisions for the evolution of the system).

To realize a connection between {I} and {III}, Boltzmann and Maxwell had the idea of introducing a function $f(x, v, t)$ which describes the density of particles which at the point x and at the time t have velocity v ; this is a solution of the so called Boltzmann equation:

$$(2) \quad \partial_t f + v \nabla_x f = C(f).$$

In (2) the left hand side represents the evolution of the function f under the sole action of the proper velocity v of the particles. The right hand side is a collision operator which models the interaction between the particles.

Formally (i.e. without proof of convergence) one proceeds as follows: first for the connection between {I} and {II} one introduce the density function

$$f_N(x_1, x_2, \dots, x_N, v_1, v_2, \dots, v_N)$$

which describes the probability of having at time t the first molecule at the point x_1 with velocity v_1 , the second at the point x_2 with velocity v_2 and so on. This function is a solution of the Liouville equation:

$$(3) \quad \partial_t f_N + \{H_N, f_N\} = 0.$$

Assume that the particles are indistinguishable which means that at time $t = 0$ and therefore at any time t and for any permutation σ of the set $\{1, 2, \dots, N\}$ one has:

$$(4) \quad f_N(x_1, x_2, \dots, x_N, v_1, v_2, \dots, v_N, t) = f_N(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(N)}, v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(N)}, t).$$

Consider the limit of the first marginal when $N \rightarrow \infty$, $\sigma \rightarrow 0$, $N\sigma^2 \rightarrow \lambda$:

$$(5) \quad f(x, v, t) = \lim f_N^1(x, v, t) = \lim \int f_N(x, x_2, \dots, x_N, v, v_2, \dots, v_N, t) dx_2 \dots dx_N dv_2 \dots dv_N.$$

One proves that this function is a solution of the Boltzmann equation. To analyze this limit one must integrate (3) with respect to the variables $x_2, x_3, \dots, x_N, v_2, v_3, \dots, v_N$ and obtain, using (4), an equation of the form:

$$(6) \quad \partial_t f_N^1 + L_N^1 f_N^1 = M_N^1 f_N^2$$

with

$$(7) \quad f_N^2 = f_N^2(x, x_2, v, v_2, t) = \int f_N(x, x_2, \dots, x_N, v, v_2, \dots, v_N, t) dx_3 \dots dx_N dv_3 \dots dv_N$$

and L_N^1 and M_N^2 are suitable operators.

To analyze the second marginal f_N^2 one integrates (6) and obtains an equation for the third marginal f_N^3 defined in a similar manner. Eventually one has a hierarchy (called the BBGKY hierarchy) of N equations

$$(8) \quad \partial_t f_N^l + L_N^l f_N^l = M_N^l f_N^{l+1}, \quad 1 \leq l \leq N-1, \quad \partial_t f_N^N + \{H_N, f_N^N\} = 0$$

for the marginals:

$$\begin{aligned} f_N^l &= \int f_N(x, x_2, \dots, x_N, v, v_2, \dots, v_N, t) dx_{l+1} \dots dx_N dv_{l+1} \dots dv_N, \quad 1 \leq l \leq N-1 \\ f_N^N &= f_N. \end{aligned}$$

Letting N go to infinity one obtains an infinite hierarchy which is called the Boltzmann hierarchy, formally written as:

$$(9) \quad \partial_t f^l + L^l f^l = M^l f^{l+1}.$$

and one observes that if $f(x, v, t)$ is a solution of the corresponding Boltzmann equation

$$(10) \quad f^l(x_1, x_2, \dots, x_l, v_1, v_2, \dots, v_l, t) = \prod_1^l f(x_l, v_l, t)$$

produces a solution of the hierarchy (9). A uniqueness argument (of the Cauchy Kowalewskaya type) plus the fact that the initial data are assumed to be factorized leads to the conclusion that

$$(11) \quad f(x, v, t) = \lim f_N^1(x, v, t).$$

is a solution of the Boltzmann equation.

Going from {II} to {III} is simpler and probably the part of the theory which is by now the best established both at formal and rigorous levels. One observes that the collision operator of

the Boltzmann equation satisfies the following invariance (inherited from the underlying Liouville equation)

$$(12) \quad \int_{\mathbf{R}_v^3} C(f)\Xi(v)dv = 0, \text{ for } \Xi(v) = 1, v_1, v_2, v_3 \text{ and } |v|^2$$

and the entropy condition

$$(13) \quad \int_{\mathbf{R}_v^3} C(f) \log f(v)dv \leq 0$$

with equality if and only if f , as a function of v , is Maxwellian i.e. is given by a formula of the following type:

$$(14) \quad f(v) = \frac{\rho}{(2\pi\theta)^{\frac{d}{2}}} e^{-\frac{|v-u|^2}{2\theta}}.$$

In (14) d is the dimension of the space, ρ is the macroscopic density, u the macroscopic velocity and θ the macroscopic temperature.

The derivation of {II} from {I} corresponds to a regime where:

$$(15) \quad \lambda = \lim N\sigma^2, \quad 0 < \lambda < \infty.$$

The inverse of this number has the dimension of a length, called the mean free path or Knudsen number. On the other hand the total volume occupied by the gas is of the order of $N\sigma^3$ therefore in the above derivation this volume is very small. The term *rarefied gas* is used in this context and to go from {II} to {III} one should let $\lambda = \epsilon^{-1}$ go to infinity. Therefore the Boltzmann equation is rescaled according to the formula:

$$(16) \quad \partial_t f_\epsilon + v \nabla_x f_\epsilon = \frac{1}{\epsilon} C(f_\epsilon).$$

The quantities

$$(17) \quad \rho_f = \int f_\epsilon(x, v, t)dv, \quad \rho_f u_f = \int v f_\epsilon(x, v, t)dv, \quad \rho_f \left(\frac{|u_f|^2}{2} + \frac{d}{2} \theta_f \right) = \int \frac{|v|^2}{2} f_\epsilon(x, v, t)dv$$

define the macroscopic density of momentum, the internal energy and the temperature.

At the level of equation (16) their computation would require the knowledge of higher moments, according to the formulas

$$(18) \quad \partial_t \int f_\epsilon(x, v, t)dv + \nabla_x \int v f_\epsilon(x, v, t)dv = 0,$$

$$(19) \quad \partial_t \int v f_\epsilon(x, v, t)dv + \nabla_x \int v \otimes v f_\epsilon(x, v, t)dv = 0,$$

$$(20) \quad \partial_t \int |v|^2 f_\epsilon(x, v, t)dv + \nabla_x \int v |v|^2 f_\epsilon(x, v, t)dv = 0,$$

and so on, as in the first derivation an infinite hierarchy of moments. However, due to the relaxation property contained in equation (13), the fact that ϵ goes to zero forces f_ϵ to become a maxwellian and this leads to an explicit computation of the moments in term of ρ, u and θ .

In this way not only the compressible Euler equation but all the equations of macroscopic fluid dynamics for a perfect gas can be deduced, with some other convenient scaling, from the Boltzmann equation ([7], [8], [9], [1]).

Much more difficult and completely unsolved questions arise for the relation between {III} and {IV}. This corresponds to situations where the macroscopic fluid becomes turbulent and when some type of averaging is necessary for quantitative or qualitative results. In spite of being the very end of the chain, this step shares in common some points with the previous one.

It is an averaging process and the “turbulent model” starts to be efficient when the original Navier Stokes are out of reach by direct numerical simulations.

In this averaging appears a hierarchy of moments which has been studied “per se” (cf. section (3) and [24]). However this is not sufficient for the following reasons:

There is up to now no well defined notion of equilibrium and relaxation to this equilibrium, with something that would play the role of the entropy as it appears in the derivation of {III} from {II} - not even an indirect proof as in the derivation of {II} from {I} by a uniqueness argument (which does not fully explain how things happen).

The parameters that would lead to turbulent phenomena are not as clearly identified as in the previous steps of the hierarchy. In some sense they are less universal and more local.

In conclusion there is up to now no case where a proof (even formal) of the validity of a derivation of {IV} from {III} is available. The arguments when they exist rely on phenomenological considerations and engineering experiments. *In spite of this lack of justification, such are the equations used to design the airplanes in which you fly!*

It is an experimental fact (not a theorem) that no new mathematical results can be obtained at level n of the chain of equations without the knowledge of its counterpart at level $n + 1$. A tentative explanation would be that the equations at level n contains in their asymptotic behaviour the properties of the equation at level $n + 1$. However as said above the derivation of the model of turbulence is not for the time being accessible by first principles from the macroscopic equations and this may be a reason why theorems at the level of the macroscopic equation remain incomplete. Even the macroscopic equations are the cornerstone of the theory and this is the object of the next section, where comments will be made on the following issue:

- 1) The existence of a smooth solution of the compressible Euler equation for “short time” before the appearance of singularities due to the generation of shocks.
- 2) The existence of a weak solution of the incompressible Navier Stokes equation.

In agreement with the results one observes that the derivation from {I} to {II} has been fully proved by O. Lanford [28] for the hard spheres model. But this proof is valid only for short time. The derivation of the compressible Euler equation from the Boltzmann equation has also been proved for short time ([37], [51] and...). Eventually with the introduction of a convenient scaling one derives with proofs which are almost complete the Leray weak solution from the renormalized solution introduced by di Perna and Lions for the Boltzmann equation. In some cases such a derivation is valid with no restriction on time or smallness assumption on the initial data and under “physical hypotheses” - [7], [8], [9], [10], [34] and [44].

2 Stabilities and instabilities of macroscopic solutions

The stability properties of the macroscopic fluid dynamic equations depend on the size of intrinsic parameters which describe the average state of the fluid. In particular are involved the Reynolds number

$$(21) \quad Re = \frac{\text{Characteristic speed of the Fluid} \times \text{Characteristic length of the flow}}{\text{viscosity}}$$

and the Mach number:

$$(22) \quad Ma = \frac{\text{Characteristic speed of the Fluid}}{\text{Sound speed}}$$

In an adimensional form this leads to the consideration of the following three closely related equations:

- the compressible Euler equation which involve the density, the velocity (or moment), and the temperature (or internal energy)

$$(23) \quad \rho, u, m = \rho u, \theta \text{ and } e = \rho \left\{ \frac{1}{2} |u|^2 + \frac{3}{2} \theta \right\} :$$

$$(24) \quad \partial_t \rho + \nabla_x \cdot (\rho u) = 0,$$

$$(25) \quad \rho (\partial_t + u \cdot \nabla_x) u + \nabla_x (\rho \theta) = 0,$$

$$(26) \quad \frac{3}{2} \rho (\partial_t + u \cdot \nabla_x) \theta + \rho \theta \nabla_x \cdot u = 0.$$

- the incompressible Euler equation:

$$(27) \quad \partial_t u + \nabla(u \otimes u) + \nabla p = 0, \nabla u = 0.$$

- and the incompressible Navier Stokes equation

$$(28) \quad \partial_t u + \nabla(u \otimes u) - \nu \Delta u + \nabla p = 0, \nabla u = 0.$$

The incompressible Euler equation is formally deduced from the incompressible Navier Stokes equation as the viscosity goes to zero or rather with a convenient change of scale as the Reynolds number goes to infinity. The incompressible Euler equation is deduced from the compressible one as the Mach number goes to zero.

As a consequence, stability results for these three equations should be closely related. The compressible Euler equation is the “prototype” of non linear hyperbolic problem. Classical theorems (the Cauchy Lipschitz Theorem) shows that the problem is well posed for a finite time with smooth initial data in $H^s(\mathbf{R}^n)$ with $s > \frac{n}{2} + 1$. However after a finite time singularities appear. This correspond to the generation of shocks by compression. A mathematical formulation of this process can be found in Sideris ([48]).

However written in conservative form the compressible Euler equation *should* admit weak solutions which satisfy an entropy condition and correspond to the propagation of shock or rarefaction waves. Such a result is so far out of reach. It has been proven by a very clever method in *one space dimension* by Glimm in (1965). Since then there has been no serious improvement. Among other things Glimm’s method uses the stability of the solution in the space of functions with bounded

variation. Such a space seems to be well adapted to the description of solutions with discontinuities. These discontinuities correspond to the propagation of a shock front and with the space of functions of bounded variation, the values of the solution before and after the shock are well defined. However it was proven by Littman that in more than one space variable the acoustic equation (which is a linearized version of the compressible Euler equations) is not well posed. J. Rauch extended the result to the Euler equation itself [42]. Therefore there is no hope of extending the method of Glimm to more than one space variable dimension.

The situation is even more complex for the incompressible Euler equation in three space variables. The proof of the local (short time) existence of smooth solution for “regular” initial data e.g. in $H^s(\mathbf{R}^n)$ with $s > \frac{n}{2} + 1$ can be done along the same line as in the compressible case, using pseudodifferential operators; in a less formal way it was done by Lichtenstein in 1925 [31]. On the other hand due to the incompressibility there is no genuine shock and the appearance poses a much more subtle problem which up to now remains basically open. According to the physical intuition it would be the increase of vorticity that would be responsible for the appearance of singularities. Along this line the most important result is probably the result of Beale, Kato and Majda [13], which goes as follows:

Theorem 2.1 (Beale, Kato, Majda): *Let $u \in C^0([0, T_*; H^3(\mathbf{R}^3 \setminus \mathbb{Z}^3))$ be a solution of the three dimensional incompressible Euler equation. Suppose that there exists a time T_* such that the solution cannot be continued up to $T = T_*$ and assume that T_* is the first such time. Then one has for $\omega(x, t) = \nabla \times u(x, t)$,*

$$(29) \quad \int_0^{T_*} \|\omega(t)\|_{L^\infty} dt = \infty,$$

and in particular

$$(30) \quad \limsup_{t \uparrow T_*} \|\omega(t)\|_{L^\infty} = \infty.$$

The above theorem and the absence of proof for the apparition of singularities should be completed by the following observation due to Lions and Di Perna (cf. [33] page 150-153.)

Theorem 2.2 *In space dimension 3, and for any p (the enstrophy which is the space integral of the square of norm of the vorticity*

$$\int |\omega(x, t)|^2 dx$$

corresponds to $p = 2$) there exists no function

$$\phi(Z, t), \quad \lim_{Z \rightarrow 0} \phi(Z, t) = 0$$

such that the vorticity of the incompressible Euler equation satisfies the estimate:

$$(31) \quad \|\omega(t)\|_{L^p} \leq \phi(\|\omega(0)\|_{L^p}, t).$$

The proof consists in the observation of classical examples (which are called Kolmogorov flows); the simplest one which emphasizes the role of the frequencies is obtained on the torus $\mathbf{T} = (\mathbf{R}/2\pi\mathbb{Z})^3$. Observe that the vector field

$$(U(y), 0, W(x, y, z, t)), \quad W(x, y, z, t) = W(x - tU(y), y, z, 0)$$

is an explicit solution of the incompressible Euler equation with zero pressure. In particular one can consider

$$(32) \quad U(y) = \sin ky, V(y) = 0, W(x, y, z, t) = \sin l(x - t \sin(ky))$$

which gives for the vorticity a time increment which is frequency dependent.

$$(33) \quad \omega(x, y, t) = (-tkl \cos l(x - t \sin(ky)) \cos(ky), \quad -l \cos(x - t \sin(ky)), \quad -k \cos(ky)).$$

Theorem 2.2 has several consequences important for our purpose.

It shows the existence of solutions that do not blow up in finite time but whose vorticity increases arbitrarily fast. Such phenomena may be present in practical turbulence. *The name “quasisingularities” should be appropriate.* The existence of quasisingularities reduces seriously the hope of proving the appearance of blow up by numerical simulations, as was tried by several authors. *Numerical simulations cannot discriminate between quantities that blow in a finite time and quantities that increase fast enough to produce a numerical overflow in finite time.* In the mean time this also shows that a priori estimates (other than the one already known) for the incompressible Navier Stokes equation and that would be independent of the Reynolds number (or the viscosity) have very little chance to exist, if they existed they would lead to similar estimates for $\nu = 0$ which may contradict the statement of the above theorem. Personally I do not believe that the solutions of the incompressible Euler or Navier Stokes equation blows up but it may well be that there are no other general estimates than those presently found.

Multiplying by u the incompressible Navier Stokes equation:

$$(34) \quad \partial_t u + u \nabla u - \nu \Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 \text{ in } \Omega \times \mathbf{R}_t^+, \quad u(x, t) = 0 \text{ on } \partial\Omega \times \mathbf{R}_t^+$$

$$(35) \quad u(x, 0) = u_0(x), \quad u_0(x) \in (L^2(\Omega))^3, \quad \nabla \cdot u_0 = 0, \quad u_0(x) \cdot n(x) \text{ on } \partial\Omega,$$

one obtains, if the solution is smooth, the local and global “formal” energy relations:

$$(36) \quad \partial_t \frac{|u|^2}{2} + \nabla(u\{\frac{|u|^2}{2} + p\}) + \nu |\nabla u|^2 - \Delta \frac{|u|^2}{2} = 0$$

and

$$(37) \quad \partial_t \int_{\Omega} \frac{|u|^2}{2} dx + \nu \int_{\Omega} |\nabla u|^2 dx = 0$$

which were at the origin of Leray’s proof of the existence of a weak solution satisfying the estimate:

$$(38) \quad \int_{\Omega} \frac{|u(x, t)|^2}{2} dx + \nu \int_0^t \int_{\Omega} |\nabla u(x, s)|^2 dx ds \leq \int_{\Omega} \frac{|u_0(x)|^2}{2} dx.$$

The followers of Leray tried very hard to improve this result and to obtain with some minimal regularity assumption the existence of a smooth solution, which then would easily be shown to be unique and stable with respect to the initial data. *Even if they are very incomplete the present results are of practical importance because in some sense they show where and how “quasi singular solutions” may not be well controlled or what is sufficient to imply that the solution is indeed well controlled.*

With functional analysis one shows (cf. [46], [49] and others) that a weak solution which is in $L^r(0, T; (L^s(\Omega))^3)$ with smooth initial data is in fact a smooth solution provided (r, s) satisfy the relation:

$$(39) \quad r \in [2, +\infty[, \quad s \in]3, \infty[, \quad \frac{2}{r} + \frac{3}{s} \leq 1$$

Refining the notion of weak solutions, Scheffer [45], Caffarelli, R. Kohn and L. Nirenberg [15] and Sohr and von Wahl [49] proved that it is possible to construct weak solutions globally defined in time which satisfy for any $T > 0$ the following relations:

$$(40) \quad u \in L^\infty(0, T; (H_0^1(\Omega))^3), \quad p \in L^{\frac{5}{3}}([0, T] \times \Omega)$$

and

$$(41) \quad \partial_t \frac{|u|^2}{2} + \nabla \cdot (u \{ \frac{|u|^2}{2} + p \}) + \nu |\nabla u|^2 - \Delta \frac{|u|^2}{2} \leq 0.$$

Such solutions are called *admissible weak solutions*. Notice that for weak solutions or suitable weak solutions the equality in (36) and (38) is not proved. Solutions are not smooth enough to validate multiplications or integrations by part.

If it exists, the set of singularities is of Hausdorff measure 1 in space time. The proof of this began with the work of Leray, continuing with the contribution of Scheffer and culminating with the result of Caffarelli, Kohn and Nirenberg [15]. A more direct and simpler proof has been provided by Fanghua Lin [32], using the pressure estimate of Sohr and von Wahl [49]. *From a practical point of view these theorems evaluate the (small) measure of the set where a quasi singular solution may have large vorticity*

In space dimension two (cf below) the situation concerning the finite time stability of the solution is much better. This implies that to become large (out of control and so on) the 3d solutions have to be genuinely three dimensional - in the language of engineers the flow cannot be *laminar*: this means that there should be no direction in which the velocity is small and that the vorticity should “visit” all directions in arbitrarily small neighborhood of a singular point.

The complement of this observation leads to a series of theorems quoted below.

Theorem 2.3 (Neustupa Novotny and Penel [39]) (i) *Let u be an admissible weak solution of the Navier Stokes equation in a domain $\Omega \times]0, T[$. Assume (compare with (39)) that there exists a sub domain $D \times]T_1, T_2[\subset \subset \Omega \times]0, T[$ and a space direction $e \in S^2(\mathbf{R}^3)$ such that*

$$(42) \quad u \cdot e|_{D \times]T_1, T_2[} \in L^r(]T_1, T_2[, L^s(D)) \text{ with } r \in [4, \infty[, s \in]3, \infty[, 2/r + 3/s \leq \frac{1}{2}$$

Then u is smooth in $D \times]T_1, T_2[$.

Theorem 2.4 (Constantin and Fefferman [20]) *Let us write the Navier Stokes equation in terms of the vorticity:*

$$(43) \quad \partial_t \omega + u \cdot \nabla_x \omega - \nu \Delta_x \omega = \omega \cdot \nabla_x u$$

and introduce the quantity:

$$(44) \quad \xi(x, t) = \frac{\omega(x, t)}{|\omega(x, t)|} \text{ if } \omega(x, t) \neq 0,$$

$$(45) \quad \xi(x, t) = 0 \text{ if } \omega(x, t) = 0.$$

Then for any Leray solution of the 3d Navier Stokes equation one has:

$$(46) \quad \int_0^T dt \int_{|\omega(x, t)| > M} |\nabla_x \xi(x, t)|^2 dx dt \leq \frac{C}{\nu M} \int_{\mathbf{R}^3} \{|u_0(x)|^2 + |\nabla \times u_0(x)|\} dx.$$

ii) Assume that the direction of the vorticity of u is uniformly lipschitz with respect to x when the modulus of this vorticity is large; this means that there exist two positive finite constants C and ρ such that one has

$$(47) \quad \forall (x, y, t) \in (\mathbf{R}^3)^2 \times \mathbf{R}_t^+ \quad \{|\omega(x, t)| > M \text{ and } |\omega(y, t)| > M\} \Rightarrow |\sin \phi(x, y, t)| \leq C \frac{|x - y|}{\rho}$$

with $\phi(x, y, t)$ denoting the angle of the two vectors $\omega(x, t)$ and $\omega(y, t)$. Then if the vorticity is bounded for $t = 0$, in $L^2(\mathbf{R}^3)$, it remains bounded in the same space for $t \geq 0$ and therefore the solution is “regular”.

With other more sophisticated ingredients the proof of theorem 2.3 uses the fact that some regularity on ω_3 can be controlled by the regularity on u_3 according to equation (43) which gives:

$$(48) \quad \partial_t \omega_3 + u \cdot \nabla_x \omega_3 - \omega \cdot \nabla_x u_3 - \nu \Delta_x \omega_3 = 0$$

The regularity of the pair (u_3, ω_3) goes over to the other velocity component according to the formula:

$$(49) \quad \partial_{x_1} u_1 + \partial_{x_2} u_2 = -\partial_{x_3} u_3$$

$$(50) \quad \partial_{x_1} u_2 - \partial_{x_2} u_1 = \omega_3.$$

The proof of theorem 2.4 uses the following explicit representation of the right hand side of equation (43) by the principal value of an integral:

$$(51) \quad |(\omega \cdot \nabla_x u, \omega)(x, t)| = |\omega(x, t)|^2 \left\{ \frac{3}{4\pi} \text{PrincVal} \int \det(\hat{y}, \xi(x), \xi(x + y, t)) |\omega(x + y, t)| \frac{dy}{|y|^3} \right\}$$

Remark- The significance of relation (46) in theorem 2.4 is the following: in regions of high vorticity the direction of vorticity is regular, in an averaged sense, but uniformly with respect to the initial data, and with a $\frac{1}{\nu}$ dependence with respect to the viscosity.

The significance of the assertion ii) is that singularities (or loss of control of the regularity) only appear with large vorticity (in modulus) together with large oscillations of the direction of the vorticity.

Theorem (2.4) has its counterpart at the level of the Euler equation; a similar result can be proven but with the extra assumption that the velocity is bounded in L^∞ .

The significance of both theorems is that the loss of control of regularity can appear only when the velocity and the vorticity of the fluid oscillate in all directions and this emphasises the role of dimension 3, in agreement with the fact (cf. section 4) that in two dimensions the fluid remains smooth. It seems important to observe that it is the volume density:

$$\det(\hat{y}, \xi(x), \xi(x + y, t))$$

which commands the singularities of the system. Whenever this volume density remains smooth (say of the order of $|y|^\alpha$) the fluid remains smooth. This may also be a diagnostic of “turbulent” phenomena.

One can observe similarities between the term involved in the proof of the theorem 2.4

$$\text{PrincVal} \int \det(\hat{y}, \xi(x), \xi(x + y, t)) |\omega(x + y, t)| \frac{dy}{|y|^3}$$

and the helicity

$$(52) \quad \mathcal{H} = \int \omega(x, t) \wedge u(x, t) dx = \frac{1}{4\pi} \iint (\hat{y}, \xi(x), \xi(x+y, t)) \frac{dy}{|y|^2} dx dy$$

which is an invariant of the flow describing in particular the asymptotic crossing number (cf. [2] page 141 - I have no idea how to use such similarities).

Up to now the fact that “big” vortices should stabilize the phenomena has not been fully used. Some preliminary results can be found in the contributions of Babin, Mahalov & Nicolaenko [4], and [5]. The idea goes as follow: write the Euler or Navier Stokes equation in the form

$$(53) \quad \partial_t u + u \wedge \omega - \nu \Delta u + \nabla_x \left(\frac{|u|^2}{2} + p \right) = 0$$

Then according to ([13]) singularities should appear when the vorticity becomes infinite therefore as a first step in the analysis one considers in equation (53) situations where at time $t = 0$ we have

$$(54) \quad \omega(x, 0) = \frac{M}{\epsilon} + \tilde{\omega}_0$$

with M constant and ϵ small. We introduce the Poincaré propagator defined by the equation

$$(55) \quad \partial_t \psi + \frac{M}{\epsilon} \wedge \psi - \nu \Delta \psi + \nabla p = 0, \quad \nabla \psi \equiv 0, \quad \psi(x, 0) = \psi_0(x) \quad \psi(x, t) = e^{-\frac{t}{\epsilon} \mathcal{P}} \psi_0$$

and write equation (53) in Duhamel form:

$$(56) \quad u(x, t) = e^{-\frac{t}{\epsilon} \mathcal{P}} u_0(x, t) + \int_0^t e^{-\frac{t-s}{\epsilon} \mathcal{P}} g(x, s) ds,$$

$$(57) \quad g(x, s) = u \wedge \left(\frac{\Omega}{\epsilon} - \nabla \wedge u \right).$$

For the Poincaré operator there are averaging processes which have a regularizing effect and (cf. Babin, Mahalov and Nicolaenko [4], [5] who, motivated by the geophysical applications, consider the Navier Stokes equation with a large Coriolis force) one obtains that the solution of the Navier Stokes equation is smooth for all times, for ϵ small with respect to ν , with no hypothesis on the size of $\tilde{\omega}_0$). In the case of the Euler equation ($\nu = 0$) one obtains that the solution remains smooth on a time interval $]0, T_\epsilon[$ with

$$(58) \quad \lim_{\epsilon \rightarrow 0} T_\epsilon = \infty.$$

This leads to two types of considerations

1) A scenario for the proof of regularity for the solution of the Navier Stokes equation: if the solution has some tendency to become singular then oscillation in every direction of the velocity and or of the vorticity would by an averaging process stabilize the phenomenon.

2) An approach for turbulent regime: fluids with large Reynolds numbers should allow some type of averaging, validating turbulence models as described below.

3 Turbulence, weak convergence and Wigner measures.

It is a common fact that definitions of turbulence and intermittency are extremely diverse. At the level of the present contribution I would like to support the following point of view concerning a “mathematical definition of turbulence”. *The question is the prediction, computation and representation of phenomena which contain no “mystery” in their generation but which are too complex in their realization to be accessible to classical computations.* With this idea in mind one should observe that two approaches compete:

- A statistical approach in which the velocity of the fluid is a random variable; this is *the statistical theory of turbulence*

- Situations where a L^2 -bounded family of solutions does not strongly converge, but nevertheless one can find an equation (or a system of equations) describing adequately the weak limit.

The randomness in the solution may be generated by randomness in the initial data, or by a forcing term (e.g. f in the right hand side of the Navier Stokes equation).

The weak convergence may be generated by letting the viscosity ν go to zero (or taken identically equal to zero), with initial data, forcing term or boundary conditions (even in two space dimension) that prevent strong convergence. In particular the conjunction of the zero viscosity limit and of viscous boundary condition $u_\nu = 0$ on the boundary of the domain may generate (cf section 4) some genuine weak convergence. In the two approaches the essential role is played by the Reynolds stress tensor:

$$\nabla(u \otimes u).$$

In statistical theory the average $\langle u \otimes u \rangle$ is usually not $\langle u \rangle \otimes \langle u \rangle$; likewise the weak limit $\overline{u \otimes u}$ of $u \otimes u$, if it exists, is usually not $\bar{u} \otimes \bar{u}$. Computation of $\langle u \otimes u \rangle$, resp. $\overline{u \otimes u}$, will usually successively involve all “moments” $\langle u^{\otimes n} \rangle$, resp. $\overline{u^{\otimes n}}$, leading to an infinite hierarchy of equations as in the previous closures. However at variance with the previous derivation, the problem of closing the hierarchy is at the mathematical level completely open.

A refined description of the state of the art can be found in ([24]) where it is shown in particular that a finite hierarchy (up to moments of order N) will produce a good approximation of the infinite hierarchy.

In both cases, at least for computational and predictability reasons under reasonable hypotheses, one should obtain

$$(59) \quad \langle u \otimes u \rangle - \langle u \rangle \otimes \langle u \rangle = -\nu_{turb}(x, t) \{ \nabla \langle u \rangle + {}^t \nabla \langle u \rangle \}$$

$$(60) \quad \overline{u \otimes u} - \bar{u} \otimes \bar{u} = -\nu_{turb}(x, t) \{ \nabla \bar{u} + {}^t \nabla \bar{u} \}$$

leading to an averaged equation of the form:

$$(61) \quad \partial_t U + \nabla(U \otimes U) - \nu \Delta U - \nabla(\nu_{turb}(x, t) \{ \nabla U + {}^t \nabla U \}) + \nabla p = 0, \quad \nabla u = 0$$

with $\nu_{turb}(x, t)$ being a local (x, t) dependent quantity. For the equation (61) to be well posed and “stable for numerical computation” one should assume that $\nu_{turb}(x, t)$ is positive. It is believed that this *turbulent viscosity*, which corresponds to strong averaging processes ν_{turb} , will often be much bigger than the original viscosity ν which is due to intrinsic molecular effects.

The form of the right hand side of (59) or (60) is a direct consequence of an ‘*isotropy, or frame invariance*’ hypothesis (made by Taylor and Kolmogorov) (cf. [36] page 34). This hypothesis has never been proved but it is very natural; it involves the small scales discussed below. The positivity and more generally the value of the turbulent diffusion is a more subtle question. It has no reason

to be a “markovian” quantity and should depend on the “history” of the flow as indicated by one of the most commonly used model the ϵ - k model (cf. also [36] page 51).

In the absence of proof for the derivation of models of turbulence (like the ϵ - k model or others...), even under very restrictive hypotheses, information concerning the statistical properties of turbulent flow become valuable, at least to check the validity of numerical simulations. As it is well known this approach, mostly due to Kolmogorov ([27]), concerns the turbulence spectra defined through space (one could also use time correlation) according to the formula

$$(62) \quad \langle \hat{R}(x, k, t) \rangle = \int_{\mathbf{R}^3} e^{-ir \cdot k} \langle u(x + \frac{r}{2}, t) \otimes u(x - \frac{r}{2}, t) \rangle dr.$$

Assuming for the average frame invariance (isotropy) and homogeneity, i.e. that the quantity

$$\langle u(x + \frac{r}{2}, t) \otimes u(x - \frac{r}{2}, t) \rangle$$

does not depend on (x, t) , one obtains the standard formula:

$$(63) \quad \hat{R}(x, k, t) = \frac{E(|k|)}{4\pi|k|^2} (I - \frac{k \otimes k}{|k|^2}).$$

$E(|k|)$ is the turbulent spectrum, for which qualitative properties like the famous $\frac{5}{3}$ law are predicted.

In the spirit of this paper one should observe that formula (62) makes sense without the isotropy and homogeneity hypothesis and the introduction of randomness

$$(64) \quad \hat{R}(x, k, t) = \int_{\mathbf{R}^3} e^{-ir \cdot k} u(x + \frac{r}{2}, t) \otimes u(x - \frac{r}{2}, t) dr$$

is nothing else than the Wigner transform. To the best of my knowledge this point of view which seems very natural appears in the turbulence literature only in the contribution of D.C. Besnard, F.H. Harlow, R.M. Rauenzahn and C. Zemach [14]. It provides a local and non isotropic definition of the spectrum and with the inverse Fourier transform one deduces the relation:

$$(65) \quad u(x, t) \otimes u(x, t) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbf{R}^3} \hat{R}(x, k, t) dk.$$

Formula (62) leads to a natural connection between the notion of spectra in the statistical theory of turbulence, and the issues of weak convergence through the H measures (cf. [50], [25]) which were introduced to evaluate the following type of expression

$$\overline{u \otimes u} - \bar{u} \otimes \bar{u}.$$

In particular for a family of solutions u_ν of the Navier Stokes equation with viscosity ν going to zero, with initial data and the forcing term (right hand side) uniformly bounded respectively in L^2 and in $L^\infty(0, T, L^2)$, one has:

$$(66) \quad \frac{1}{2} \|u(t, \cdot)\|^2 + \nu \int_0^t |\nabla u(x, s)|^2 dx ds \leq C$$

With the estimate (66) it turns out that the H measure is related to the Wigner measure at the scale ν :

$$(67) \quad dW(x, k) =$$

$$\lim_{\nu \rightarrow 0} \int_{\mathbf{R}^3} e^{-ir \cdot k} \left(u_\nu \left(x + \frac{\sqrt{\nu}r}{2}, t \right) - \bar{u} \left(x + \frac{\sqrt{\nu}r}{2}, t \right) \right) \otimes \left(u_\nu \left(x - \frac{\sqrt{\nu}r}{2}, t \right) - \bar{u} \left(x - \frac{\sqrt{\nu}r}{2}, t \right) \right) dr.$$

by the formula (cf [25])

$$(68) \quad \lim_{\nu \rightarrow 0} (u_\nu \otimes u_\nu) = \bar{u} \otimes \bar{u} + \lim_{\nu \rightarrow 0} (u_\nu - \bar{u}) \otimes (u_\nu - \bar{u}) = \bar{u} \otimes \bar{u} + \int dW(x, k).$$

Therefore one could at least formulate, and maybe in some cases prove at the level of the Wigner measure, the classical issues of turbulence theory: isotropy and universality.

Observe that the Wigner measure

$$\int dW(x, k)$$

is a “microlocal” object which naturally should not depend on the macroscopic characteristic of the fluid. The isotropy would be in agreement with the argument of Kolmogorov which says that it is the smallest of the scales which generate isotropy.

In terms of correlation, frame invariance (or isotropy) is currently investigated and discussed and it seems (cf. for instance [19]) that this property, valid for two point correlations, is no longer valid for higher order correlations. This does not seem to matter too much for the present analysis because the Reynolds stress tensor is computed by means of the Wigner measure, which is a two point correlation.

4 Some special properties of the dimension 2

In two dimensions things are much simpler, mostly because the vorticity, being perpendicular to the plane where the fluid evolves, is conserved along the trajectories:

$$(69) \quad \partial_t \omega + u \nabla \omega = 0.$$

The mechanism analyzed in section 2 that relate the emergence of large vortices to the fact that their direction oscillates in a non controlled fashion, and which therefore may also lead to some type of average, is not present here.

However there are, in space dimension two, open problems where the question of turbulence as introduced above does appear. These problems may have their counterpart in space dimension 3, but they seem either more natural or slightly more tractable in dimension two. I intend to describe briefly two questions: The formation of large scale structures (In genuinely 3d turbulence as in the section 2 one use the term “cascade” to describe the generation of small scale structures. At variance for large scale structures generated in 2d one would speak of “Inverse cascade”) and the effect of the boundary layer. Eventually one may consider the combination of these two effects.

In a bounded domain $\Omega \subset \mathbf{R}^2$ one considers the equation:

$$(70) \quad \partial_t u + \nabla(u \otimes u) + \nabla p = 0, \quad \nabla \cdot u = 0, \quad u(x, t) \cdot n = 0 \text{ on } \partial\Omega, \quad u(x, 0) = u_0(x)$$

and defines a weak solution as a function:

$$u(x, t) \in L^\infty(0, T; (L^2(\Omega))^2), \quad \nabla u(x, t) \equiv 0, \quad u(x) \cdot n(x) = 0 \text{ on } \partial\Omega$$

which for any smooth test function $\phi \in \mathcal{D}(\Omega \times \mathbf{R}_t)$ satisfies the relation

$$(71) \quad \iint \langle \phi_t u + \nabla \phi \mid u \otimes u \rangle dx dt = 0$$

with the initial value condition (which make sense with (71))

$$(72) \quad u(x, 0) = u_0(x).$$

Theorem 4.1 *Assume that there exist a positive Radon measure ω'_0 with compact support and a function $\omega'' \in L^p, p > 1$ such that the initial vorticity is of the form*

$$(73) \quad \omega_0 = \omega'_0 + \omega''_0$$

Then there exists a weak solution of the corresponding 2d Euler equation.

The proof uses the conservation of the L^p norm of the vorticity plus a clever argument of Delort [21] using the positivity of the non L^p part to prove the absence of concentration (cf. also [52] for a follow up).

The general form of the initial data (73) is justified by the need to treat initial data with vorticity concentrated on a curve. This is the famous Kelvin Helmholtz problem known to exhibit instabilities.

The uniqueness of the weak solution has only been established with the assumption $\omega_0 \in L^\infty$ (Yudovich [54]). It is known not to be always true if one only assumes $u_0 \in (L^2(\omega))^2$. In fact Scheffer and Shnirelman (cf.[47]) did construct weak solutions

$$u(x, t) \in L^2(\mathbf{R}_t \times \mathbf{R}_x^2)$$

with compact support in space time.

The regularity of the solution with smooth initial data (e.g. $u_0 \in C^{1,\sigma}(\Omega)$) is a subtle question. With the conservation of the vorticity it is a simple adaptation of the theorem 2.1 (Beale, Kato, Majda). However the original proof given by Wolibner [53] with the analysis of pair dispersion problems which describes the evolution of the distance between two particles of fluid remains interesting. The question comes from the fact that the boundedness of a solution $\omega(x, t)$ in L^∞ does not imply that the gradient $\nabla u(x, t)$ is also bounded in L^∞ (this would easily imply the regularity of the solution). At variance one can show the following “log lipschitz” estimate:

$$(74) \quad |u(x, t) - u(y, t)| \leq C \|\omega(\cdot, t)\|_{L^\infty(\Omega)} |x - y| \log \left(\frac{D}{|x - y|} \right) \text{ with } D \text{ the diameter of } \Omega.$$

From (76) one deduces for a pair $x(t), y(t)$ of particle trajectories:

$$(75) \quad \dot{x}(t) = u(x(t), t), \dot{y}(t) = u(y(t), t),$$

by a Gronwall type argument, the relation

$$(76) \quad \left(\frac{|x(0) - y(0)|}{D} \right)^{e^{-Ct|\nabla \times u|_{L^\infty(\Omega)}}} \geq \frac{|x(t) - y(t)|}{D} \geq \left(\frac{|x(0) - y(0)|}{D} \right)^{e^{Ct|\nabla \times u|_{L^\infty(\Omega)}}}$$

This is sufficient to show that the velocity and the vorticity satisfy the following uniform bound:

$$(77) \quad |\nabla u(\cdot, t)|_{L^\infty(\Omega)} \leq \frac{C}{\alpha} e^{Ct\rho_u} \|\omega(\cdot, t)\|_{C^{0,\alpha}e^{-Ct\rho_u}} \leq \frac{C}{\alpha} e^{Ct\rho_u} \|\omega(\cdot, 0)\|_{C^{0,\alpha}} \text{ with } \rho_u = |\nabla \times u|_{L^\infty(\Omega)}.$$

Then, again by a standard Gronwall argument, it follows that the solution will be as regular as the initial data for any finite time, but the “measure” of this regularity may blow up with $t \rightarrow \infty$.

In fact an example with an initial vorticity in L^∞ but not in $C^{0,\sigma}$, which “saturates” the relation (76) has been given by Bahouri and Chemin [6].

The above consideration leads to the idea that for large time and a sequence of initial data ω_ϵ bounded in L^∞ converging in L^∞ for the weak-* topology but not strongly, exotic phenomena may show up. This is a way to access to an explanation of the formation of coherent structures as observed in the Jupiter red spot or in the anticyclone of the Acores. In both cases the problem is $3d$ but due to the smallness of the thickness of the atmosphere it is mostly driven by two dimensional dynamics.

Consider a sequence of solutions of the $2d$ Euler equation written in vorticity form:

$$(78) \quad \epsilon \partial_t \omega_\epsilon + u_\epsilon \nabla \omega_\epsilon = 0, \nabla \cdot u_\epsilon = 0, \nabla \wedge u_\epsilon = \omega_\epsilon, \text{ in } \Omega,$$

$$(79) \quad u_\epsilon \cdot n = 0 \text{ on } \partial\Omega, \lim_{\epsilon \rightarrow 0} \omega_\epsilon(x, 0) = \bar{\omega}_o \text{ in weak } L^\infty^* .$$

which converges say in $L^\infty([0, T] \times \Omega)$ for the weak star topology to a stationary solution $\bar{u}, \bar{\omega}$

$$(80) \quad \bar{u} \nabla \bar{\omega} = 0 .$$

For simplicity we assume that this solution is smooth. Then (80) implies that the level surfaces of the current function $\bar{\psi}$ ($\bar{u} = \nabla \bar{\psi}^\perp$) and of the vorticity $\bar{\omega} = -\Delta \bar{\psi}$ coincide. Therefore there exists a function F (possibly multivalued) such that:

$$(81) \quad -\Delta \bar{\psi} = \bar{F}(\bar{\psi}) .$$

The determination of this function F is the main issue of the rest of the program (cf. [43] and [35]). In the absence of any “dynamical” proof the present discussion relies on three points:

1) The introduction of Liouville measures (or Young measures). This is fully consistent with the weak convergence of the vorticity both for $t = 0$ and for $0 \leq t \leq T$, which may be as in section 3 a mathematical formulation of turbulence.

2) The emphasis on the quantities which are conserved and which pass to the limit.

3) An argument borrowed from statistical mechanics, which does not seem to have a dynamical counterpart, and which is a follow up of an idea of Onsager [38].

Denote by $\mu_0(x, \lambda)$ the Liouville measure associated to the weak convergence of the initial data and by $\mu(x, \lambda)$ the Liouville measure associated to the convergence to $\bar{\omega}$. The following relations are direct consequences of :

- i) The fact that $\overline{\mu(x, \lambda)}$ is a Liouville measure.
- ii) The fact that the flow is measure preserving, and the vorticity is constant along its trajectories.
- iii) The fact that the solution is uniformly bounded in $L^\infty(\mathbf{R}_t; H^1(\Omega))$.

$$(82) \quad \int_{\mathbf{R}} \bar{\mu}(x, \lambda) d\lambda = 1 ,$$

$$(83) \quad \int_{\Omega} \bar{\mu}(x, \lambda) dx = \sigma(\lambda) \text{ prescribed by the initial data),}$$

$$(84) \quad \frac{1}{2} \int_{\Omega} \left((-\Delta)^{-1} \int_{\mathbf{R}} \lambda \bar{\mu}(x, \lambda) d\lambda \right) \left(\int_{\mathbf{R}} \lambda \bar{\mu}(x, \lambda) d\lambda \right) dx = E \text{ (prescribed by the initial data)}$$

Then comes an argument borrowed from statistical mechanics which says that, with the above constraints, the “most probable” Liouville measure is the one which minimizes the “Kullback” entropy with respect to $\mu_0(x, \lambda)$. It is given by the formula

$$(85) \quad \bar{\mu}(x, \lambda) = \rho(x, \lambda)\mu_0(x, \lambda)$$

and minimize the quantity

$$\iint_{\mathbf{R} \times \Omega} \rho(x, \lambda) \log \rho(x, \lambda) \mu_0(x, \lambda) dx d\lambda$$

under the constraints (82),(83),(84). A variational computation now gives:

$$(86) \quad \begin{aligned} & \iint_{\mathbf{R} \times \Omega} (1 + \log \rho(x, \lambda)) \delta \rho(x, \lambda) \mu_0(x, \lambda) dx d\lambda \\ &= \beta \iint_{\mathbf{R} \times \Omega} \overline{\psi(x)} \delta \rho(x, \lambda) \mu_0(x, \lambda) dx d\lambda + \iint_{\mathbf{R} \times \Omega} (\alpha(\lambda) + \gamma(x)) \delta \rho(x, \lambda) \mu_0(x, \lambda) dx d\lambda. \end{aligned}$$

In (86), $\gamma(x)$ is the Lagrange multiplier of the constraint (82), $\alpha(\lambda)$ the Lagrange multiplier of the constraint (83) and β the Lagrange multiplier of the energy constraint (84). Therefore one obtains eventually the formula:

$$(87) \quad \rho(x, \lambda) = \frac{e^{-\alpha(\lambda) - \beta \lambda \overline{\psi(x)}}}{\int e^{-\alpha(\lambda) - \beta \lambda \overline{\psi(x)}} \mu_0(x, \lambda) d\lambda}$$

and with

$$(88) \quad \bar{\omega} = \int \lambda \rho(x, \lambda) \mu_0(x, \lambda) d\lambda$$

the “mean field equation”:

$$(89) \quad -\Delta \bar{\psi}(x) = \frac{1}{\int e^{-\alpha(\lambda) - \beta \lambda \overline{\psi(x)}} \mu_0(x, \lambda) d\lambda} \int \lambda e^{-\alpha(\lambda) - \beta \lambda \overline{\psi(x)}} \mu_0(x, \lambda) d\lambda.$$

The above analysis gives an example of derivation of averaged model of the type {IV} from equations of the type {III}. Following this analysis and considering not only the minimization of entropy but the maximization of dissipation of entropy, one can reintroduce the time to obtain time dependent turbulent models with a “turbulent viscosity” as this is done in [18].

An other unsolved problem both in 2 and 3 space variables is the nonviscous limit of the solutions of the incompressible Navier Stokes when the viscosity goes to 0 and when a viscous boundary layer is prescribed.

The fact that the problem is still open is even more striking in dimension 2 where the existence of regular solutions is known, for the Navier-Stokes equations as well as for the Euler equations. This problem is challenging, both for applications (estimation of the drag on a wing and of the vortices in the wake of a plane at takeoff) and for fundamental questions in turbulence, because it is the action of the boundary which turns out to be the most natural physical tool to generate turbulence that will spread in the media.

Let u_ν denote the solution of the incompressible Navier Stokes equation with viscous boundary condition.

$$(90) \quad \partial_t u_\nu + \nabla(u_\nu \otimes u_\nu) - \nu \Delta u_\nu + \nabla p_\nu = 0, \quad \nabla u_\nu = 0, \quad u_\nu(x, t) = 0 \quad \text{on } \partial\Omega \times]0, T[$$

then the question is: does the solution u_ν converge for $\nu \rightarrow 0$ to the solution of the Euler equation with the natural impermeability boundary condition.

$$(91) \quad \partial_t u + \nabla(u \otimes u) + \nabla p = 0, \quad \nabla u = 0, \quad n \cdot u(x, t) = 0 \quad \text{on } \partial\Omega \times]0, T[$$

The type of boundary condition has changed, instead of $u_\nu = 0$ one has only the condition $u \cdot n = 0$ and the tangential component of the limit $u \wedge n$ does not vanishes in general. Assuming convergence this generates a boundary layer which in some cases is analyzed with the Prandtl equation. In fact one observes that it is the tangential variation of the pressure on the boundary which generates vorticity according to the formula:

$$(92) \quad \frac{\partial \omega_\nu}{\partial n} = - \frac{\partial p_\nu}{\partial \tau}.$$

However since the problem is not linear nothing prevents the boundary layer to move into the media and to introduce oscillations or turbulent behaviour of the solution that may end with

$$(93) \quad \lim_{\nu \rightarrow 0} (u_\nu \otimes u_\nu) - (\lim_{\nu \rightarrow 0} u_\nu) \otimes (\lim_{\nu \rightarrow 0} u_\nu) \neq 0.$$

In fact such a phenomenon is supposed to happen, and detachment of the boundary layer may produce non trivial turbulent viscosity.

Once again the only estimate available comes from the energy

$$(94) \quad \frac{1}{2} \int_\Omega |u_\nu(x, t)|^2 dx + \nu \int_0^t \int_\Omega |\nabla \wedge u_\nu(x, s)|^2 dx ds = \frac{1}{2} \int_\Omega |u_\nu(x, 0)|^2 dx$$

From (94), with a convenient localizing argument one can deduce:

Theorem 4.2 (Kato 76) [26] *The following facts are equivalent:*

1) *The solution $u_\nu(x, t)$ of the Navier Stokes equation (90) converges strongly (in $L^2(0, T : (L^2(\Omega))^2)$) to the solution of the Euler equation (91).*

2) *One has*

$$(95) \quad \lim_{\nu \rightarrow 0} \nu \int_0^T \int_\Omega |\nabla \wedge u_\nu(x, s)|^2 dx ds = 0.$$

3) *One has*

$$(96) \quad \lim_{\nu \rightarrow 0} \nu \int_0^T \int_{\Omega \cap d(x, \partial\Omega) \leq c\nu} |\nabla \wedge u_\nu(x, s)|^2 dx ds = 0.$$

Observe that the region

$$\Omega \cap \{d(x, \partial\Omega) \leq c\nu\}$$

is much smaller than the standard region for boundary layer in the “parabolic equation” analysis which is of the order of $\sqrt{\nu}$ (with the change of variable

$$X = \frac{x}{\sqrt{\nu}}$$

the coefficient of the Laplacian remains independent of ν). The theorem above gives an indication on the amount of vorticity needed in a very small neighborhood of the boundary to create a detachment of the boundary layer. It is important to observe that this statement is not in contradiction with the analysis made under the assumption of the existence of a boundary layer well localized near the boundary. In this case one should have

$$(97) \quad \nu \int_0^T \int_{\Omega \cap d(x, \partial\Omega) \leq c\nu} |\nabla \wedge u_\nu(x, s)|^2 dx ds \leq C\nu$$

which implies (95). This shows that the validity of the boundary layer analysis needs stronger conditions than the *simple convergence to the Euler flow, which one could qualify as the absence of turbulence*.

Furthermore, in the left hand side of (93) the difference comes from small scales which are “far away” (with respect to the size of the boundary layer) from the region where the vorticity is generated and it would be natural to assume that in this configuration the Wigner measure (cf. section 3) is frame invariant even if the whole problem, due to the presence of the boundary, is not (this has not yet been proven). *As a result one should prove, it is not yet done that the limit of the solution of Navier Stokes equation satisfy the following equation:*

$$(98) \quad \partial_t \bar{u} + \nabla(\bar{u} \otimes \bar{u}) - \nu_{turb}(x, t) \Delta \bar{u} + \nabla \bar{p} = 0,$$

$$(99) \quad \nabla \cdot \bar{u} \equiv 0,$$

$$(100) \quad u \cdot n = 0 \text{ on } \partial\Omega, \nu_{turb}(x, t) \geq 0, u \wedge n(x, t) = 0 \text{ for } x \in \partial\Omega \text{ and } \nu(x, t) > 0.$$

In these equations $\nu_{turb}(x, t)$ represents the *turbulent viscosity* which should be active in some part of the wake of the obstacle.

Eventually the fact that detachments may appear and that the boundary layer analysis may fail is in full agreement with the results which are at present at our disposal. For instance in the half space the Prandtl equation is derived from the 2d Navier Stokes equation as follows: in the Navier-Stokes equation with viscosity $\nu = \epsilon^2$

$$(101) \quad \partial_t u_1^\epsilon - \epsilon^2 \Delta u_1^\epsilon + u_1^\epsilon \partial_{x_1} u_1^\epsilon + u_2^\epsilon \partial_{x_2} u_1^\epsilon + \partial_{x_1} p^\epsilon = 0,$$

$$(102) \quad \partial_t u_2^\epsilon - \epsilon^2 \Delta u_2^\epsilon + u_1^\epsilon \partial_{x_1} u_2^\epsilon + u_2^\epsilon \partial_{x_2} u_2^\epsilon + \partial_{x_2} p^\epsilon = 0,$$

$$(103) \quad \partial_{x_1} u_1^\epsilon + \partial_{x_2} u_2^\epsilon = 0, \quad u_1^\epsilon(x_1, 0) = u_2^\epsilon(x_1, 0) = 0, \quad \forall x_1 \in \mathbf{R}$$

the following scaling, consistent with the persistence of the boundary condition $u_2 \equiv 0$, on the boundary are made:

$$(104) \quad X_1 = x_1, \quad X_2 = \frac{x_2}{\epsilon}, \quad \tilde{u}_1(x_1, X_2) = u_1(x_1, X_2), \quad \tilde{u}_2(x_1, X_2) = \epsilon u_2(x_1, X_2)$$

and the problem becomes the “Prandtl boundary layer” equation:

$$(105) \quad \partial_t \tilde{u}_1 - \nu \partial_{x_2}^2 \tilde{u}_1 + \tilde{u} \partial_{x_1} u_1 + \tilde{u}_2 \partial_{x_2} \tilde{u}_1 + \partial_{x_1} \tilde{p} = 0$$

$$(106) \quad \partial_{x_2} \tilde{p} = 0, \quad \tilde{p}(x_1, x_2, t) = \tilde{p}(x_1, t), \quad \partial_{x_1} \tilde{u}_1 + \partial_{x_2} \tilde{u}_2 = 0$$

$$(107) \quad \tilde{u}_1(x_1, 0) = \tilde{u}_2(x_1, 0) = 0 \text{ for } x_1 \in \mathbf{R}, \quad \lim_{x_2 \rightarrow \infty} \tilde{u}_2(x_1, x_2) = 0, \quad \lim_{x_2 \rightarrow \infty} \tilde{u}_1(x_1, x_2) = U(x_1, t)$$

One can prove that the existence for all $\epsilon > 0$ of a solution of the Prandtl equation (with suitable data) with uniform ϵ uniform estimates implies the convergence of the solution of the Navier Stokes equation to the solution of the Euler equation. It would be interesting to explore if the condition described above is sufficient for the validity of a “Prandtl type boundary layer” a positive answer would mean that the question of the absence of “turbulence” in this problem is related to the analysis of the solutions of the “Prandtl” equations and the following facts have been proven:

1) For analytic initial data, the problem has a well defined solution [3] for finite time, [16] and [17]. The proof given in ([17]) uses only the analyticity with respect to the tangential variable. *This may be related with formula (92) which exhibit loss of derivative in the tangential direction.*

2) For special type of profile which would exclude ejection from the boundary layer, global existence is proven (cf [40]).

3) For some special initial data that correspond to a strong recirculation (initial data having the opposite horizontal direction) a blow up of the solution of the Prandtl equation in finite time is proven ([22]).

4) There are configurations (cf. [12]) where the solutions of the Navier Stokes equations do not converge to the solutions of the Prandtl equation in the boundary layer.

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