The Equations of Incompressible Fluid Dynamics II
The Euler Equations

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- The Euler Equations

$$\begin{aligned} u_t + u \cdot \nabla u &= -\nabla p \\ \nabla \cdot u &= 0. \end{aligned}$$

  Differ from the Navier-Stokes equations by one term: $\nu \triangle u$.

- Relation to turbulence

  Evidence: strictly positive energy dissipation rate

$$\left\langle \nu \left|\nabla u\right|^2 \right\rangle > \varepsilon_0 > 0.$$

  at high Reynolds number, i.e., small ν. If

  1. This is true for $\nu \to 0$.
  2. The limiting u is a solution to the Euler equations.

  Then Euler singularity, if any, is related to turbulence.

- Functional Analysis approach as for NSE?

  - Not clear how to define weak solution.
  - Natural (Naive) way: $u \in L^2$ such that equation is satisfied in a distributional sense. Doesn't work. Even in 2D such weak solution is not unique. Counterexamples:
    * V. Scheffer. 1993.
    * A. Shnirelman. 1997. Simpler construction.

    Constructed weak solution for 2D Euler with finite support in space-time. By no means physical.

    * Highly discontinuous.
    * Unbounded.
    * Actually solves

$$u_t + u \cdot \nabla u = -\nabla p + f$$

      with the forcing f oscillating at infinitely high frequency, so $= 0$ as a distribution.

  - Improved definition for weak solutions.
    * Shnirelman 1997, 2000.
    * Requires strict decreasing energy, i.e., $\int |u|^2 \; dx$ monotonely decreasing. Why: strictly positive dissipation rate $\left\langle \nu \left|\nabla u\right|^2 \right\rangle > \varepsilon_0 > 0$.
    * Existence proved for $\mathbb{T}^3$, i.e., periodical boundary condition in 3D. (Shnirelman 2000).

  - This weak solution is highly irregular. Reason: $u \in C^\alpha$ for $\alpha > 1/3$, then $\int |u|^2 \; dx$ =constant. (Onsager's conjecture. Onsager 1949. Proof: Constantin-E-Titi 1994, Eyink 1994.)

- Vorticity form.
  - $\omega = \nabla \times u$.

$$\omega_t + u \cdot \nabla \omega = \begin{cases} 0 & 2D \\ \omega \cdot \nabla u & 3D \end{cases} .$$

    May be more natural for Euler. At least in 2D. All L^p estimates are trivial for not-too-irregular initial values, since ω is transported and the flow is area-preserving.
  - Still not very successful. No 3D existence known. Even 2D not complete.
    - Existence and uniqueness for $\omega_0 \in L^\infty$. (V. Yudovich, 1963).
    - Existence and uniqueness for $\omega_0 \in L^p$ for all $p \in (p_0, \infty)$. With careful control over these L^p norms. Such control almost gives $\|\omega_0\|_p \leq \ln p$, which allows point singularities like $\log\log(1/x)$. (V. Yudovich, 1995.)
    - Existence for some $\omega_0 \in \mathcal{M}$, bounded measure.
      1. $\omega_0 \geq 0$. J. M. Delort, 1991.
      2. $\omega_0 \geq 0$ in right half plane, $\omega_0(x_1, x_2) = -\omega_0(-x_1, x_2)$. Lopes-Lopes-Xin 2001.
    - In 3D. Not possible to find a priori bound for $\|\omega\|_p$, i.e., $\|\omega\|_p$ can not be bounded by any function of $\|\omega_0\|_p$. R. J. DiPerna, P. L. Lions, 1989.

- Do Navier-Stokes approaches work?
  - Small initial value. Not known.
  - Small scaling in one dimension for the region. Not known.
  - Serrin type results. No such result for u. For ω in special cases.
    - Beale-Kato-Majda 84. $\omega \in L^1([0,T), L^\infty)$. Note $\frac{1}{1} + \frac{1}{\infty} = 1$.
    - Improvements on BKM.
      Kozono-Taniuchi: $\omega \in L^1([0,T), BMO)$. 2000.
      Chae: $\omega \in L^1([0,T), F^0_{\infty,\infty})$. 2002.
  - Novotny & Penel type results. No such result for u. For ω in special cases.
    - Chae: $(\omega_1, \omega_2) \in L^2([0,T), \dot{B}^0_{\infty,1})$. 2004.
  - CKN type results. No such result. The Laplacian $\triangle$ is crucial to all partial regularity results for NSE.

    - CF type results. Similar but with more assumptions. 1. $\nabla\xi \in L^2([0,T), L^\infty)$, 2. $u \in L^\infty([0,T), L^\infty)$, 3. Some other conditions. Constantin-Fefferman-Majda 1996.
    - Large initial vorticity. Babin-Mahalov-Nicolaenko 2001. Similar but with restriction on the dimensions of the period box.

- Numerical computation and other theoretical approaches.
    - Numerical computation has been done to search for Euler singularity, mainly by Robert M. Kerr. Favoring singularity.
    - Main observation: $\|\omega\|_\infty \sim (T-t)^{-1}$, $\|u\|_\infty \sim (T-t)^{-1/2}$. Maximum vorticity in some shrinking region, looks like two vortex sheets meeting at an angle. Thickness $(T-t)$, other two directions $(T-t)^{1/2}$. $\nabla\xi \sim (T-t)^{-1/2}$ inside this region.
    - None of the above theoretical works can be applied to such flows.
    - Deng-Hou-Yu 2004: Exclude blow-up in a large class of flows with these "blow-up" rates.

- Further reading.
    - Peter Constatin, *Euler Equations, Navier-Stokes Equations and Turbulence*, C.I.M.E. Lectures, 2003.
    - M. C. Lopes Filho, H. J. Nussenzveig Lopes, Y. Zheng, *Weak solutions for the equations of incompressible and inviscid fluid dynamics.* 1999.
    - T. Y. Hou, X. Yu, *Lecture Notes on Incompressible Inviscid Flow.* 2004.