The Equations of Incompressible Fluid Dynamics I
The Navier-Stokes Equations

Xinwei Yu

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- The Navier-Stokes equations
  - Derivation.
    - $\nabla \cdot u = 0$: Incompressibility (Conservation of mass).
    - $u_t + u \cdot \nabla u = -\nabla p + \nu \triangle u$: Newton's second law. (Conservation of momentum).
  - Vorticity.

    $$\omega \equiv \nabla \times u$$

    Describes rotation in the flow. Example: $u = \frac{1}{2}\Omega \times x$, Ω constant vector, then $\omega = \Omega$.
  - Energy conservation.
    - Kinetic energy $\int |u|^2 \ dx$. Since density $\rho \equiv 1$.
    - $\frac{d}{dt} \int |u|^2 \ dx + \nu \int |\nabla u|^2 \ dx = 0$. Or

    $$\left(\int |u|^2 \ dx \right)(t) + \nu \int_0^t \int |\nabla u|^2 \ dx = \left(\int |u|^2 \ dx \right)(0).$$
  - Rescaling. Make $\nu = 1$ through rescaling in x and t.
- Function Analysis approach.
  - The linearized Navier-Stokes

    $$\begin{aligned} u_t &= \triangle u - \nabla p \\ \nabla \cdot u &= 0. \end{aligned}$$

    Projection to divergence-free vector field.

    $$u_t = (P\triangle)\, u \equiv Au$$

    A: Stokes operator. Properties of A depend on the domain (e.g., regularity of the domain).
  - Solution to the linearized system:

    $$u(t) = e^{tA} u(0).$$
  - For the original NSE:

    $$u(t) = e^{tA} u(0) - \int_0^t e^{(t-s)A} \left(u \cdot \nabla u \right)(s) \ ds.$$

    So the key issue is to study $e^{(t-s)A} \left(u \cdot \nabla u \right)$ and try to invoke some fixed point theorem.
- Leray weak solutions.

  - Defined after energy conservation.

    $$u \in L^\infty([0,T), L^2), \quad \nabla u \in L^2([0,T), L^2).$$

    Notation: $u \in L^p\left([0,T), L^q\right)$: L^q norm of u as a function of t is L^p in $[0,T)$.
  - Existence for all time T: proved by Leray in the 1930s.
  - Why study regularity then?
    - To prove uniqueness, need

      $$\nabla u \in L^{\frac{n}{4-n}}\left([0,T), L^2\right).$$

      OK for 2D, i.e. $n = 2$ by definition.
    - Related to the correctness of the NSE itself. Since the derivation of the NSE requires some regularity in u.

- Regularity results.
  - Small initial value/large viscosity.
    - Why such conditions work. Example: $R_{j+1} \le R_j^2$. Generally cannot bound R_j. But if $R_0 \le 1$, then by induction $R_{j+1} \le R_j \le R_0$, uniformly bounded.
    - Question. What is the weakest norm one can find such that $||u_0||$ small $\Rightarrow$regularity? Koch-Tataru 2001.
  - Small scaling in one dimension for the region.
    - Example. $\Omega : (x_1, x_2, x_3) \mid |x_3| < \varepsilon$.
    - Flow is almost 2D.
  - Serrin type results.

    $$u \in L^p([0,T), L^q), \quad \frac{2}{p} + \frac{n}{q} \le 1.$$

    then the weak solution u is regular.
    - For Leray weak solutions, only have

      $$\frac{2}{p} + \frac{n}{q} \le \frac{3}{2}.$$

      A big gap.
    - Similar conditions on ∇u or $\omega = \nabla \times u$. Basically

      $$\frac{1}{p} + \frac{n}{q} \le 1$$

      would be OK.

  - Novotny & Penel type results.
    - e is any direciton, fixed. $u \cdot e \in L^p([0,T), L^q)$ for $\frac{4}{p} + \frac{n}{q} \leq 1$, then u is regular.
    - Similar results for ∇u. e.g. $\nabla u_3 \in L^p(L^q)$, $\frac{2}{p} + \frac{3}{q} \leq \frac{3}{2}$. Pokorný 2003.
    - Similar results for ω. e.g. $\omega \in L^1(L^\infty)$. Beale-Kato-Majda 1984.
  - CKN type results.
    - Also called "partial regularity" results.
    - Hausdorff dimension of the points where u is not bounded ≤ 1 in $\mathbb{R}^3 \times \mathbb{R}$ (space-time). Caffarelli-Kohn-Nirenberg 1982.
    - Replace $\triangle u$ by $-(-\triangle)^\alpha u$. Then the Hausdorff dimension $\leq 5 - 4\alpha$. Katz-Pavlovic 2001.
  - CF type results.
    - $\xi \equiv \frac{\omega}{|\omega|}$. If $\nabla \xi$ is bounded where $|\omega| \geq K$ for some K. Then no finite time singularity. Constantin-Fefferman 1996.
    - Improved to $\xi \in C^{1/2}$. da Veiga 2002.
  - Large initial vorticity.
    - $u_0 = \tilde{u}_0 + \Omega \times x$. Ω large enough, then solutio regular for all time. Babin-Mahalov-Nicolaenko 2001.

- Further reference (Review articles).
  - M. Wiegner, The Navier-Stokes Equations - A Neverending Challenge? Jber. d. Dt. Math. - Verein. 101 (1999), 1-25.
  - C. Bardos, A Basic Example of Nonlinear Equations: The Navier-Stokes Equations.