Chapter 6
Introduction to Return and Risk

Road Map

Part A Introduction to Finance.

Part B Valuation of assets, given discount rates.

Part C Determination of risk-adjusted discount rates.
  • Introduction to return and risk.
  • Portfolio theory.
  • CAPM and APT.

Part D Introduction to derivative securities.

Main Issues

• Defining Risk

• Estimating Return and Risk

• Risk and Return - A Historical Perspective
Asset returns over a given period are often uncertain:

\[
\tilde{r} = \frac{\tilde{D}_1 + \tilde{P}_1 - P_0}{P_0} = \frac{\tilde{D}_1 + \tilde{P}_1}{P_0} - 1
\]

where

- \( \sim \) denotes an uncertain outcome (random variable)
- \( P_0 \) is the price at the beginning of period
- \( \tilde{P}_1 \) is the price at the end of period - uncertain
- \( \tilde{D}_1 \) is the dividend at the end of period - uncertain.

Return on an asset is a random variable, characterized by

- all possible outcomes, and
- probability of each outcome (state).

**Example.** The S&P 500 index and the stock of MassAir, a regional airline company, give the following returns:

<table>
<thead>
<tr>
<th>State</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.20</td>
<td>0.60</td>
<td>0.20</td>
</tr>
<tr>
<td>Return on S&amp;P 500 (%)</td>
<td>-5</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Return on MassAir (%)</td>
<td>-10</td>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>
Risk in asset returns can be substantial.


• Expected rate of return on an investment is the discount rate for its cash flows:

\[ \bar{r} \equiv E[\tilde{r}] = \frac{E_0[\tilde{D}_1 + \tilde{P}_1]}{P_0} - 1 \]

or

\[ P_0 = \frac{E_0[\tilde{D}_1 + \tilde{P}_1]}{1 + \bar{r}} \]

where \( \bar{\cdot} \) denotes an expected value.

• Expected rate of return compensates for time-value and risk:

\[ \bar{r} = r_F + \pi \]

where \( r_F \) is the risk-free rate and \( \pi \) is the risk premium

- \( r_F \) compensates for time-value
- \( \pi \) compensates for risk.

Questions:

1. How do we define and measure risk?

2. How are risks of different assets related to each other?

3. How is risk priced (how is \( \pi \) determined)?
2 Defining Risk

Example. Moments of return distribution. Consider three assets:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean (%)</th>
<th>StD (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{r}_0$ (%)</td>
<td>10.0</td>
<td>0.00</td>
</tr>
<tr>
<td>$\tilde{r}_1$ (%)</td>
<td>10.0</td>
<td>10.00</td>
</tr>
<tr>
<td>$\tilde{r}_2$ (%)</td>
<td>10.0</td>
<td>20.00</td>
</tr>
</tbody>
</table>

- Between Asset 0 and 1, which one would you choose?
- Between Asset 1 and 2, which one would you choose?

Investors care about expected return and risk.
Key Assumptions On Investor Preferences for 15.401

1. Higher mean in return is preferred:
\[ \bar{r} = E[\tilde{r}] \]

2. Higher standard deviation (StD) in return is disliked:
\[ \sigma = \sqrt{E[(\bar{r} - \bar{r})^2]} \]

3. Investors care only about mean and StD (or variance).

Under 1-3, standard deviation (StD) gives a measure of risk.

Investor Preference for Return and Risk

Expected return (\( \bar{r} \))

Risk (\( \sigma \))

increasing return

decreasing risk
3 Historical Return and Risk

Three central facts from history of U.S. financial markets:

1. Return on more risky assets has been higher on average than return on less risky assets:

   Average Annual Total Returns from 1926 to 2005 (Nominal)

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean (%)</th>
<th>StD (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-bills</td>
<td>3.8</td>
<td>3.1</td>
</tr>
<tr>
<td>Long term T-bonds</td>
<td>5.8</td>
<td>9.2</td>
</tr>
<tr>
<td>Long term corp. bonds</td>
<td>6.2</td>
<td>8.5</td>
</tr>
<tr>
<td>Large stocks</td>
<td>12.3</td>
<td>20.2</td>
</tr>
<tr>
<td>Small stocks</td>
<td>17.4</td>
<td>32.9</td>
</tr>
<tr>
<td>Inflation</td>
<td>3.1</td>
<td>4.3</td>
</tr>
</tbody>
</table>

   Average Annual Total Returns from 1926 to 2005 (Real)

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean (%)</th>
<th>StD (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-bills</td>
<td>0.7</td>
<td>4.0</td>
</tr>
<tr>
<td>Long term T-bonds</td>
<td>2.9</td>
<td>10.4</td>
</tr>
<tr>
<td>Long term corp. bonds</td>
<td>3.2</td>
<td>9.7</td>
</tr>
<tr>
<td>Large stocks</td>
<td>9.1</td>
<td>20.3</td>
</tr>
<tr>
<td>Small stocks</td>
<td>13.9</td>
<td>32.3</td>
</tr>
</tbody>
</table>
Return Indices of Investments in the U.S. Capital Markets

Real returns from 1926 to 2004

<table>
<thead>
<tr>
<th>Security</th>
<th>Initial</th>
<th>Total Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-Bills</td>
<td>$1.00</td>
<td>1.74</td>
</tr>
<tr>
<td>Long Term T-Bonds</td>
<td>$1.00</td>
<td>6.03</td>
</tr>
<tr>
<td>Corporate Bonds</td>
<td>$1.00</td>
<td>8.86</td>
</tr>
<tr>
<td>Large Stocks</td>
<td>$1.00</td>
<td>242.88</td>
</tr>
<tr>
<td>Small Stocks</td>
<td>$1.00</td>
<td>1,208.84</td>
</tr>
</tbody>
</table>
2. Returns on risky assets can be highly correlated to each other:

**Cross Correlations of Annual Nominal Returns (1926 – 2005)**

<table>
<thead>
<tr>
<th></th>
<th>Bills</th>
<th>T-bonds</th>
<th>C-bonds</th>
<th>L. stocks</th>
<th>S. stocks</th>
<th>Inflation</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-bills</td>
<td>1.00</td>
<td>0.23</td>
<td>0.20</td>
<td>-0.02</td>
<td>-0.10</td>
<td>0.41</td>
</tr>
<tr>
<td>L.T. T-bonds</td>
<td>1.00</td>
<td>0.93</td>
<td>0.12</td>
<td>-0.02</td>
<td>-0.14</td>
<td></td>
</tr>
<tr>
<td>L.t. C-bonds</td>
<td>1.00</td>
<td>0.19</td>
<td>0.08</td>
<td>-0.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Large stocks</td>
<td>1.00</td>
<td>0.79</td>
<td></td>
<td>-0.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small stocks</td>
<td>1.00</td>
<td></td>
<td></td>
<td>0.04</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inflation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
</tr>
</tbody>
</table>

**Cross Correlations of Annual Real Returns (1926 – 2005)**

<table>
<thead>
<tr>
<th></th>
<th>Bills</th>
<th>T-bonds</th>
<th>C-bonds</th>
<th>L. stocks</th>
<th>S. stocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-bills</td>
<td>1.00</td>
<td>0.57</td>
<td>0.57</td>
<td>0.11</td>
<td>-0.06</td>
</tr>
<tr>
<td>L.T. T-bonds</td>
<td>1.00</td>
<td>0.95</td>
<td>0.20</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>L.t. C-bonds</td>
<td>1.00</td>
<td>0.26</td>
<td>0.11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Large stocks</td>
<td></td>
<td>1.00</td>
<td>0.79</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small stocks</td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>
3. Returns on risky assets are serially uncorrelated.

Serial Correlations of Annual Asset Returns (1926 – 2005)

<table>
<thead>
<tr>
<th>Asset</th>
<th>Serial Correlation</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Nominal return</td>
<td>Real return</td>
</tr>
<tr>
<td>T-bills (&quot;risk-free&quot;)</td>
<td>0.91</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>Long term T-bonds</td>
<td>-0.08</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>Long term corp. bonds</td>
<td>0.08</td>
<td>0.19</td>
<td></td>
</tr>
<tr>
<td>Large stocks</td>
<td>0.03</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>Small stocks</td>
<td>0.06</td>
<td>0.03</td>
<td></td>
</tr>
</tbody>
</table>

(Note: The main source for the data in this subsection is *Stocks, bonds, bills and inflation, 2006 Year Book*, Ibbotson Associates, Chicago, 2006.)
4 Risk and Horizon

Previous discussions focused on return and risk over a fixed horizon. Often, we need to know:

- How do risk and return vary with horizon?
- How do risk and return change over time?

We need to know how successive asset returns are related.

IID Assumption: Asset returns are IID when successive returns are independently and identically distributed.

For IID returns, \( \tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t \) are independent draws from the same distribution.

\( P_t \) is the asset price (including dividend). The continuously compounded return is

\[
\frac{P_t}{P_{t-1}} = e^{\tilde{r}_t} \quad \text{or} \quad \log \frac{P_t}{P_{t-1}} = \log P_t - \log P_{t-1} = \tilde{r}_t.
\]

Definition: Asset price (in log) follows a Random Walk when its changes are IID.
**Example.** An IID return series — a binomial tree for prices:

```
115.76
107.49
110.25
105
```

```
107.49
102.38
```

```
110.25
105
```

```
102.38
```

```
107.49
```

```
102.38
```

```
110.25
105
```

```
102.38
```

```
107.49
```

```
99.82
```

```
102.38
```

```
107.49
```

```
99.82
```

```
97.5
```

```
102.38
```

```
99.82
```

```
95.06
```

```
99.82
```

```
92.68
```

```
97.5
```

```
102.38
```

```
99.82
```

```
95.06
```

```
99.82
```

```
92.68
```

where

1. price can go up by 5% or down by 2.5% at each node
2. probabilities of “up” and “down” are the same at each node.

For the above binomial price process:

- Successive returns are independent and identically distributed.
- If “up” and “down” are equally likely, expected return is

\[
\frac{\log 1.05 + \log 0.975}{2} = 1.17\%.
\]

- Return variance for one-period is

\[
\sigma_1^2 = \left(\frac{1}{2} \log \frac{1.05}{0.975}\right)^2 = (0.0371)^2.
\]

- Return variance over \( T \) periods is \((0.0371)^2 \times T\).
Implications of the IID assumption

(a) Returns are serially uncorrelated.

(b) No predicable trends, cycles or patterns in returns.

(c) Risk (measured by variance) accumulates linearly over time:
   • Annual variance is 12 times the monthly variance.

Advantage of IID assumption:

• Future return distribution can be estimated from past returns.
• Return distribution over a given horizon provides sufficient information on returns for all horizons.
• IID assumption is consistent with information-efficient market.

Weakness of IID assumption:

• Return distributions may change over time.
• Returns may be serially correlated.
• Risk may not accumulate linearly over time.
5 Investment in the long-run

Are stocks less risky in the long-run? — Not if returns are IID.

- Higher expected total return.
- Higher probability to outperform bond.
- More uncertainty about terminal value.

**Example.** Return profiles for different horizons.

- \( r_{\text{bond}} = 5\% \).
- \( r_{\text{stock}} = 12\% \) and \( \sigma_{\text{stock}} = 20\% \).
6 Appendix: Probability and Statistics

Consider two random variables: \( \tilde{x} \) and \( \tilde{y} \)

<table>
<thead>
<tr>
<th>State</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>\cdots</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>( p_1 )</td>
<td>( p_2 )</td>
<td>( p_3 )</td>
<td>\cdots</td>
<td>( p_n )</td>
</tr>
<tr>
<td>Value of ( \tilde{x} )</td>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>( x_3 )</td>
<td>\cdots</td>
<td>( x_n )</td>
</tr>
<tr>
<td>Value of ( \tilde{y} )</td>
<td>( y_1 )</td>
<td>( y_2 )</td>
<td>( y_3 )</td>
<td>\cdots</td>
<td>( y_n )</td>
</tr>
</tbody>
</table>

where \( \sum_{i=1}^{n} p_i = 1 \).

1. Mean: The expected or forecasted value of a random outcome.

\[
E[\tilde{x}] = \bar{x} = \sum_{j=1}^{n} p_j \cdot x_j.
\]

2. Variance: A measure of how much the realized outcome is likely to differ from the expected outcome.

\[
\text{Var}[\tilde{x}] = \sigma_x^2 = E \left[ (\tilde{x} - \bar{x})^2 \right] = \sum_{j=1}^{n} p_j \cdot (x_j - \bar{x})^2.
\]

Standard Deviation (Volatility):

\[
\text{StD}[\tilde{x}] = \sigma_x = \sqrt{\text{Var}[\tilde{x}]}.
\]


\[
\text{Skew}[\tilde{x}] = \frac{3}{\sigma_x} \sqrt{E \left[ (x - \bar{x})^3 \right]}.
\]


\[
\text{Kurtosis}[\tilde{x}] = \frac{4}{\sigma_x} \sqrt{E \left[ (x - \bar{x})^4 \right]}.
\]
Example 1. Suppose that random variables $\tilde{x}$ and $\tilde{y}$ are the returns on S&P 500 index and MassAir, respectively, and

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<tr>
<td>Probability</td>
<td>0.20</td>
<td>0.60</td>
<td>0.20</td>
</tr>
<tr>
<td>Return on S&amp;P 500 (%)</td>
<td>-5</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Return on MassAir (%)</td>
<td>-10</td>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

- Expected Value:
  
  $$\bar{x} = (0.2)(-0.05) + (0.6)(0.10) + (0.2)(0.20) = 0.09$$
  
  $$\bar{y} = 0.12$$

- Variance:
  
  $$\sigma^2_x = (0.2)(-0.05-0.09)^2 +$$
  
  $$\quad (0.6)(0.10-0.09)^2 +$$
  
  $$\quad (0.2)(0.20-0.09)^2$$
  
  $$= 0.0064$$
  
  $$\sigma^2_y = 0.0256$$

- Standard Deviation (StD):
  
  $$\sigma_x = \sqrt{0.0064} = 8.00\%$$
  
  $$\sigma_y = 16.00\%.$$
Covariance and Correlation

1. Covariance: A measure of how much two random outcomes “vary together”.

\[
\text{Cov}[\tilde{x}, \tilde{y}] = \sigma_{xy} = E[(\tilde{x} - \bar{x})(\tilde{y} - \bar{y})] = \sum_{j=1}^{n} p_j \cdot (x_j - \bar{x})(y_j - \bar{y}).
\]


\[
\text{Corr}[\tilde{x}, \tilde{y}] = \rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}.
\]

Note:

(a) \(\rho_{xy}\) must lie between -1 and 1.

(b) The two random outcomes are

- Perfectly positively correlated if \(\rho_{xy} = +1\)
- Perfectly negatively correlated if \(\rho_{xy} = -1\)
- Uncorrelated if \(\rho_{xy} = 0\).

(c) If one outcome is certain, then \(\rho_{xy} = 0\).
Example 1. (Continued.) For the returns on S&P and MassAir:

<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.20</td>
<td>0.60</td>
<td>0.20</td>
</tr>
<tr>
<td>Return on S&amp;P 500 ((\bar{x})) (%)</td>
<td>-5</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Return on MassAir ((\bar{y})) (%)</td>
<td>-10</td>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

with mean and StD:

\[
\bar{x} = 0.09, \quad \sigma_x = 0.08, \\
\bar{y} = 0.12, \quad \sigma_y = 0.16.
\]

We have

- Covariance:

\[
\sigma_{xy} = (0.2)(-0.05-0.09)(-0.10-0.12) + \\
(0.6)(0.10-0.09)(0.10-0.12) + \\
(0.2)(0.20-0.09)(0.40-0.12) \\
= 0.0122.
\]

- Correlation:

\[
\rho_{xy} = \frac{0.0122}{(0.08)(0.16)} = 0.953125.
\]
Computation Rules

Let $a$ and $b$ be two constants.

\[ E[a\tilde{x}] = aE[\tilde{x}] \]

\[ E[a\tilde{x} + b\tilde{y}] = aE[\tilde{x}] + bE[\tilde{y}] \]

\[ E[\tilde{x}\tilde{y}] = E[\tilde{x}] \cdot E[\tilde{y}] + \text{Cov}[\tilde{x}, \tilde{y}] \]

\[ \text{Var}[a\tilde{x}] = a^2\text{Var}[\tilde{x}] = a^2\sigma^2_x \]

\[ \text{Var}[a\tilde{x} + b\tilde{y}] = a^2\sigma^2_x + b^2\sigma^2_y + 2(ab)\sigma_{xy} \]

\[ \text{Cov}[\tilde{x} + \tilde{y}, \tilde{z}] = \text{Cov}[\tilde{x}, \tilde{z}] + \text{Cov}[\tilde{y}, \tilde{z}] \]

\[ \text{Cov}[a\tilde{x}, b\tilde{y}] = (ab)\text{Cov}[\tilde{x}, \tilde{y}] = (ab)\sigma_{xy} \]
Linear Regression

Relation between two random variables $\tilde{y}$ and $\tilde{x}$:

$$\tilde{y} = \alpha + \beta \tilde{x} + \tilde{\epsilon}$$

where

$$\beta = \frac{\text{Cov}[\tilde{y}, \tilde{x}]}{\text{Var}[\tilde{x}]} = \frac{\sigma_{yx}}{\sigma_x^2}$$

$$\alpha = \bar{y} - \beta \bar{x}$$

$\text{Cov}[\tilde{x}, \tilde{\epsilon}] = 0$.

- $\beta$ gives the expected deviation of $\tilde{y}$ from $\bar{y}$ for a given deviation of $\tilde{x}$ from $\bar{x}$.

- $\tilde{\epsilon}$ has zero mean: $\mathbb{E}[\tilde{\epsilon}] = 0$.

- $\tilde{\epsilon}$ represents the part of $y$ that is uncorrelated with $x$: $\text{Cov}[\tilde{x}, \tilde{\epsilon}] = 0$. 

Furthermore:

\[ \sigma^2_y = \text{Var}[\tilde{y}] = \text{Var}[\alpha + \beta \tilde{x} + \tilde{\epsilon}] \]
\[ = \beta^2 \sigma^2_x + \sigma^2_\epsilon \]

Total Variance = Explained Variance

+ Unexplained Variance.

- Explained variance: \( \beta^2 \sigma^2_x \)
- Unexplained variance: \( \sigma^2_\epsilon \).

What fraction of the total variance of \( \tilde{y} \) is explained by \( \tilde{x} \)?

\[ R^2 = \frac{\text{Explained Variance}}{\text{Total Variance}} = \frac{\beta^2 \sigma^2_x}{\sigma^2_y} = \frac{\beta^2 \sigma^2_x}{\beta^2 \sigma^2_x + \sigma^2_\epsilon}. \]
Example 1. (Continued.) In the above example: $\tilde{x}$ is the return on S&P 500 and $\tilde{y}$ is the return on MassAir.

$$\beta = \frac{0.0122}{0.08^2} = 1.9062 \quad \text{and} \quad \alpha = -0.0516.$$  

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<td>Return on MassAir (%)</td>
<td>-10.00</td>
<td>10.00</td>
<td>40.00</td>
</tr>
<tr>
<td>$\bar{\epsilon} = \tilde{y} - (\alpha + \beta \tilde{x})$ (%)</td>
<td>4.69</td>
<td>-3.90</td>
<td>7.03</td>
</tr>
</tbody>
</table>

Moreover,

$$\sigma^2_x = 0.0064, \quad \sigma^2_y = 0.0256, \quad \sigma^2_\epsilon = 0.00234$$

and

$$R^2 = \frac{(1.9062)^2(0.0064)}{0.0256} = 0.9084$$

$$1 - R^2 = 0.0916.$$
7 Homework

Readings:

- BKM Chapter 5.2–5.4.
- BMA Chapter 7.1.