11-13 Class 14
Option prices,

The multi-period problem; The Black-Sholes equation and formula

Option prices

- Things to remember from last class
- Need an expected value of stock conditional on the strike price (K=S)
  - If call $E(S_T|S_T>K)$
  - If put $E(S_T|S_T<K)$
- Need a bond price/ interest rate
- Can build a replicating portfolio (borrow b in bonds and buy a*shares of stock)

One period problem

- Find a portfolio of a stock and b bonds Such that
  1. $a(S_o+\Delta)+b(1+r)= \Delta$
  2. $a(S_o-\Delta)+b(1+r)= 0$
- From (2) $b=-a(S_o-\Delta)/(1+r)$
- Replace in (1) $a(S_o+\Delta)-a((S_o-\Delta)/(1+r))(1+r)= \Delta$
  - $a(S_o+\Delta)-a(S_o-\Delta)= \Delta$

    $a2\Delta = \Delta$

  $a= ½$ $\quad$ $b=-0.5(S_o-\Delta)/(1+r)$
- Price of the option

  $C = aS_o+b= 0.5(rS_o+\Delta)/(1+r)$

Another way to look at this is

$aS_o -C=b$ <u>the stock minus the call is a riskless asset</u>

Can make continuous

- Notation (still focus on the call option)
- At time zero the option is sold at price C. The stock x has a price at t=0 of S .
- The stock price at time T S_T has a distribution density f(x) when the option might be exercised.
- The strike price is K=S. B=b(1+r)
- The option is exercised if the price is above and that occurs with a probability F(K).
- Denote the expected value of x conditional on

$$\mathrm{x} > \mathrm{k}\; E(x|x > K) = \int_K^\infty f(x)x\,dx \;\text{ by }\; E(\overline{x})$$

- Denote the expected value of x conditional on x≤k by $E(\underline{x})$

- If the option is exercised profits are

$$C = x - s \text{ or } C = E(\overline{x}) - s$$

- Find the replicating portfolio. It solves

$$\text{a}E(\overline{x}) + B = C \qquad (1)$$

$$aE(\underline{x}) + B = 0 \qquad (2)$$

$$1 - 2 \Rightarrow \text{a}(E(\overline{x}) - E(\overline{\underline{x}})) = E(\overline{x}) - \text{S}$$

$$\text{a} = \frac{(E(\overline{x}) - \text{S})}{E(\overline{x}) - E(\underline{x})}$$

$$\text{B} = -\frac{E(\underline{x})(E(\overline{x}) - \text{S})}{E(\overline{x}) - E(\underline{x})}$$

- The cost of the option is the cost of the replicating portfolio

$$NPV = C = aS + b$$

$$= \frac{(E(\overline{x}) - S)S}{E(\overline{x}) - E(\underline{x})} - \frac{(E(\overline{x}) - S)E(\underline{x})}{E(\overline{x}) - E(\underline{x})}$$

$$= \frac{(E(\overline{x}) - S)(S - E(\underline{x}))}{E(\overline{x}) - E(\underline{x})}$$

Now replace S, $E(\overline{x})$, $E(\underline{x})$ by their values

$$c = \frac{\left(\int_K^{\infty} f(x)xdx - S\right)\left(S - \int_{-\infty}^{K} f(x)xdx\right)}{\int_K^{\infty} f(x)xdx - \int_{-\infty}^{K} f(x)xdx}$$

Pricing the one period option

- Easy if you have risk
- But (1) what is a period
- (2) what about multi periods?
  - Could take a short cut and get the distribution of outcomes at time T (and don't worry about T-1 or T-2)
  - Or worry about inter temporal arbitrage.

Two period problem

- Stock follows a random walk
- Each period stock either goes up or down by some rate $S_{t+1}=S_t(1\pm\Delta)$
- Want to price an option that expires after two period and $K=S_0$.
- S_2 takes on 3 possible values
  - $S_o(1-\Delta)^2$, $S_0(1+\Delta)(1-\Delta)$, $S_o(1+\Delta)^2$
  - Because $S_0(1+\Delta)(1-\Delta)=S_0(1-\Delta^2)<S_0$ option only valuable if $S_2 = S_o(1+\Delta)^2$

Option value at time 1

- Suppose $S_1 = S_o(1+\Delta)$. Build replicating portfolio for option at strike price S_o Not S_1

  1. $a(S_1(1+\Delta)+b(1+r)= S_1(1+\Delta)-S_0$
  2. $a(S_1(1-\Delta))+b(1+r)= 0$

- From (2) $b=-a(S_1(1-\Delta))/(1+r)$
- Replace in (1) $a(S_1(1+\Delta))-a((S_1(1-\Delta))/(1+r))(1+r)= S_1(1+ \Delta)-S_0$

$$a(S_1(1+\Delta))-a(S_1(1-\Delta))= S_1\Delta-S_0$$

$$a2\Delta\ S_1 = S_1\Delta-S_0 \quad \text{Replace } S_1 \text{ with } S_o(1+r)$$

$$a2\Delta(1+ \Delta)S_0 = (1+ 2\Delta+ \Delta^2)S_0 -S_0$$

$$a2\Delta(1+ \Delta)=(2\Delta+ \Delta^2) \Leftrightarrow a2(1+ \Delta)=(2\Delta+\Delta)$$

$$A=(2+\Delta)/(2+2\Delta)$$

$$b=-(2+ \Delta)/(2+2\Delta)(S_1(1-\Delta))/(1+r)$$

- Price of the option at time 1

$$C_{1u} =aS_1+b$$

$$C_{1u} =(2+\Delta)/(2+2\Delta)S_1-(2+\Delta)/(2+2\Delta)(S_1(1-\Delta))/(1+r)$$

$$C_{1u} =(2+\Delta)/(2+2\Delta)S_1[1-(1-\Delta))/(1+r)]$$

$$\mathbf{C_{1u} =(2+\Delta)/(2+2\Delta)S_1[(r+\Delta))/(1+r)]}$$

Exercise the option at t=1?

- Value of exercising the option now
  - $S_1 - K = S_1 - S_o = S_o(1+\Delta) - S_o = \Delta S_o$
- Value of holding on to the option
    - $C_{1u} = (2+\Delta)/(2+2\Delta)S_1[(r+\Delta))/(1+r)]$

  $C_{1u} = S_1(2+\Delta)(r+\Delta)/[(2+2\Delta)(1+r)]$ Replace S_1 with $S_o(1+r)$

    - $C_{1u} = S_o(1+r)(2+\Delta)(r+\Delta))/[(2+2\Delta)(1+r)]$

  $C_{1u} = S_o(r+\Delta)(2+\Delta)/(2+2\Delta)$ if $r = \Delta$

  $C_{1u} = \Delta S_o(2+\Delta)/(1+\Delta) > \Delta s_o$
- C_{1u} is increasing in Δ so If $r<\Delta$ $C_{1u} > \Delta S_o$
  - So you do not exercise the option
  - If $r > \Delta$ then what ever the probability of the upside the return to the stock is less than the risk free interest rate so you do not invest.

Period 0

- Note if $S_1 = S_o(1-\Delta)$, $C_{1u} = 0$. Period stock values are either $S_o(1-\Delta)^2$, or $S_0(1+\Delta)(1-\Delta)$ both of which are less than the call.
- The option at time 1 can take on two values
  - $C_{1u} = S_o(r+\Delta)(2+\Delta)/(2+2\Delta)$
  - $C_{1D} = 0$
- Now we can construct a replicating portfolio

1. $a(S_0(1+\Delta))+b(1+r)= C_{1u}$
2. $a(S_O(1-\Delta))+b(1+r)= 0$

$b=-a(S_o(1-\Delta))/(1+r)$ then

$$a(S_0(1+\Delta))-a(S_o(1-\Delta))(1+r)/(1+r) = C_{1u}$$

$$a(S_0(1+\Delta))-a(S_o(1-\Delta))=S_o(r+\Delta)(2+\Delta)/(2+2\Delta)$$

$$aS_0(2\Delta)=S_o(r+\Delta)(2+\Delta)/(2+2\Delta)$$

$$a=[(r+\Delta)(2+\Delta)]/[2\Delta(2+2\Delta)]$$

Value of option at time 0 is $aS_o-a(S_o(1-\Delta))/(1+r)=aS_o[1-(1-\Delta))/(1+r)]$

Or $S_o[1-(1-\Delta))/(1+r)]\ [(r+\Delta)(2+\Delta)]/[2\Delta(2+2\Delta)]$

Beyond the two period problem

- The CDF of stock prices evolves over time
- Let us think of a binomial problem where Δ is small and the interval of time goes to zero..so that for a unit of time T you are making more and more draws…
- In the limit (by central limit theorem) the value S_t has a normal distribution with some mean growth rate (μ)and variance (σ)
- So the question is can we price option in this context?

Black-Sholes

- Solved this problem
- Consider a Call option on a stock with current price S, strike price K, expected growth μ, volatility σ. Risk free interest rate is r, and W is a random variable with mean o
- $dS = \mu S dt + \sigma S dW$
- What about the call?

Claim (Ito's lemma)

The change derivative of the call is

$dC=[(\partial C/\partial t)+(\partial C/\partial S)\mu S+(1/2(\partial^2 C/\partial S^2)\sigma^2 S^2]dt+(\partial C/\partial S)\sigma S dW$

Consider a portfolio V of h shares minus a call

$V=aS-C$ Remember $aS_o-C=b$

$dV=adS-dC$ and set $a=\partial C/\partial S$

$dV=(\partial C/\partial S)dS-dC$

Now we can replace by the components

- $dS = \mu S dt + \sigma S dW$
- $dC = [(\partial C/\partial t) + (\partial C/\partial S)\mu S + (1/2(\partial^2 C/\partial S^2)\sigma^2 S^2]dt + (\partial C/\partial S)$ $\sigma S dW$
- $dV = (\partial C/\partial S)dS - dC$

So

- $dV = (\partial C/\partial S)(\mu S dt + \sigma S dW) -$ $[(\partial C/\partial t) + (\partial C/\partial S)\mu S + (1/2(\partial^2 C/\partial S^2)\sigma^2 S^2]dt + (\partial C/\partial S)\sigma S dW$
- Cancel the $(\partial C/\partial S)\sigma S dW$ terms
- $dV =$

$(\partial C/\partial S)(\mu S dt) - [(\partial C/\partial t) + (\partial C/\partial S)\mu S + (1/2(\partial^2 C/\partial S^2)\sigma^2 S^2]dt$

- dV=

$(\partial C/\partial S)(\mu S dt) - [(\partial C/\partial t) + (\partial C/\partial S)\mu S + (1/2(\partial^2 C/\partial S^2)\sigma^2 S^2]dt$

- Cancel the $(\partial C/\partial S)(\mu S dt)$ terms.
- That leaves
- $dV = -[(\partial C/\partial t) + +(1/2(\partial^2 C/\partial S^2)\sigma^2 S^2]dt$
- The portfolio is independent of the growth rate (μ) or the random variable (W)
- Its perfectly hedged just like $V_o = aS_o - C$
- But that means its return must be the risk free rate

Implication of the perfect hedge

- $dV=-[(\partial C/\partial t)+ +(1/2(\partial^2 C/\partial S^2)\sigma^2 S^2]dt$
- $dV=rVdt=r(\partial C/\partial S)S - C]dt$
- $(\partial C/\partial t)+ +(1/2(\partial^2 C/\partial S^2)\sigma^2 S^2=-r(\partial C/\partial S)S +rC$
- $(\partial C/\partial t)+ r(\partial C/\partial S)S +(1/2(\partial^2 C/\partial S^2)\sigma^2 S^2=rC$
- And that is the Black-Sholes differential equation for call options when the stock is has an normal distribution (μ,σ^2).
- This has very well behaved solutions.

From the Equation to the formula

- The value of a call given stock price S, strike price K, exercise at T is value of the replicating portfolio

$$C(S,K,T) = SN(x) - \frac{K}{(1+r)^T} N(x - \sigma\sqrt{T})$$

N(x) is the cdf normal $N(X) = N\left(\frac{\ln\left(\frac{S}{K(1+r)^T}\right)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \right)$

$$c(S,K,T) = SN\left(\frac{\ln\left(\frac{S}{K(1+r)^T}\right)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \right)$$

$$- \frac{K}{(1+r)} N\left(\frac{\ln\left(\frac{S}{K(1+r)^T}\right)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2} \right)$$

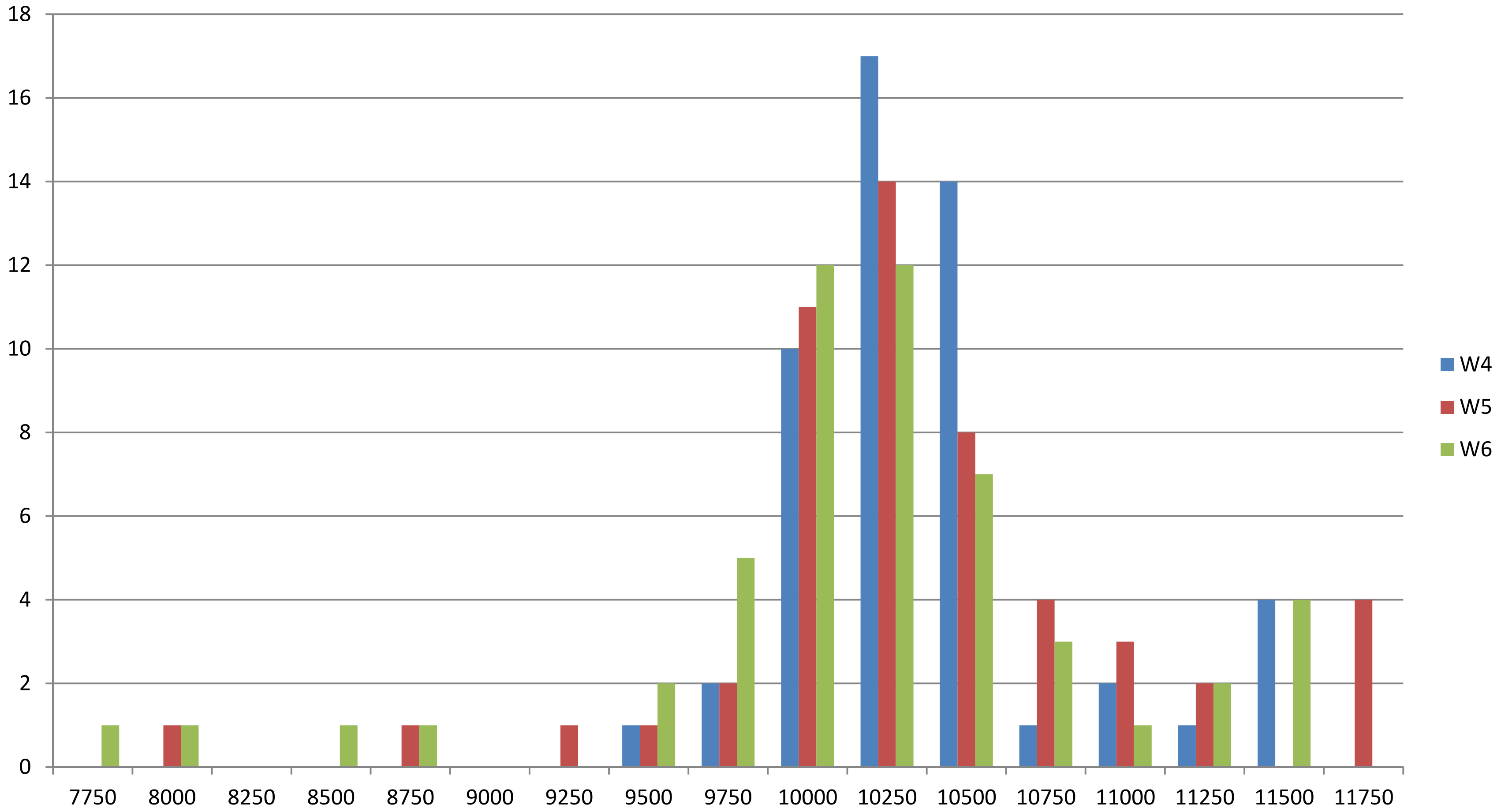
$$C(S,K,T) = SN\left(\frac{\ln\,(SK(1+r)^{-T}}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right) - \frac{K}{(1+r)^{T}} N\left(\frac{\ln\,(SK(1+r)^{-T}}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}\right)$$

- The first part is the stock side
- The second is the amount borrowed
- So in fact Black Scholes is a mathematical proof that the binomial approach works in a broad context

Options since 1973

- Theory
  - More and more version of Replicating portfolio with more and more variations
  - Each time you need to build the replicating portfolio
- Can input Black Scholes formula in programmable calculators…So can use on the trading floor.
- Become standard pricing mechanism

- Average number of stocks W2 1 (by fiat)
- Average number of stocks W3 5.1 (your choices)
- Average number of stocks W4 3.9 (your choices) and 0.77 shorts (55% of you did not short)
- Average number of stocks W5 1.5. some of you did not invest at all
- Average number of stocks W6 3.3 average leverage 18K. 40% did not borrow



18
16
14
12
10
8
6
4
2
0
7750
8000
8250
8500
8750
9000
9250
9500
9750
10000
10250
10500
10750
11000
11250
11500
11750
W4
W5
W6

Returns (w3)

- Correlation between the absolute value of change and the number of stocks is -0.41
- So undiversified portfolios are the big winners and Loosers (TESLA)
- Diversifed stocks portfolios more stable
- Correlation in returns net of S&P 500 returns week 3-4 -0.03, Week 4-5 0.06 and W5-6 -0.01
  - That efficient markets