BEM 103

10-23 Class 8:
The Portfolio approach to risk

- **More than one security out there and returns not perfectly correlated;**
- **Portfolios have better mean return profiles than individual stocks;**
- **Efficient frontier and the Sharpe value;**
- **Basic portfolio separation;**
- **Why not insurance contracts?**

Statistical Risk (correction and clarification)

- Take any security observe its prices for T periods
- Use whatever data X you can get your hands on to forecast its price at time T+1
- Produce a prediction P=F(X)+ε
- Risk involve the notion that F(.) is correct and thus the measure of risk is the distribution of the ε (mean zero)
- In this view uncertainty has to with the possibility (unmeasured) that F(.) as revealed by your investigation is wrong.
  - So your prediction P=F(X) is wrong not because there is error but because there has been a shift in fundamentals
- PB oil spill, this is a rare event and difficult to quantify. Occurs less than once ever 40,000 exploration days. So it is a big surprise, but it is not <u>unexpected.</u> That is risk.
- Alternative, the consequence of catastrophic failure in deep ocean oil drilling were not understood, so the likelihood of a 30 billion dollar loss-event was unknown and systematically mis-measured. That is uncertainty. We have to move to a totally different prediction p=G(x)

Portfolios

- A portfolio is simply a collection.
  - E.g. Past artistic achievement
  - Responsibilities (minister without portfolio!)
- A finance portfolio is thus a collection of assets.
  - These could be long positions—you own these assets and will enjoy the cash flow
  - They could be short—claims you promise to pay in the furture
- The portfolio approach to finance simply the realization that collections of assets may have better properties (lower variance, conditional on mean) than single assets because their variations offset.

Portfolios

- In Finance this idea is very old, (at least 1000 years old)
  - There is evidence that farmers have been pursuing portfolios of land and crops for far longer than that.
  - Shows up in shipping (where ventures are divided and individuals invest in ship shares for more than one venture).
  - Its fundamental in insurance contracts (that is why insurance companies can seem risk neutral)
- For finance there are two issues.
  - (1) Today
  - conditional on distributions (and taking prices as given) what are optimal portfolios?
  - (2) Monday
  - What is the impact on price of people choosing optimal portfolios? Or how do we get market equilibrium?

Mean and variance of a portfolio

- Let w_i be the weight of asset x_i in the portfolio (its share of total value)
- The mean return of the portfolio is the weighted average of the individual returns

$$E(w_1x_1 + w_2x_2) = w_1E(x_1) + w_2E(w_2)$$

- The variance of a portfolio is the weighted sum of the variances and covariances.

$$Var\,(w_1x_1 + w_2x_2) = \sum_{1}^{2}\sum_{1}^{2} w_iw_j\sigma_{ij}$$

- For two assets this is simply .

$$Var\,(w_1x_1 + w_2x_2) = w_1^2\sigma_{11} + 2w_1w_2\sigma_{12} + w_2^2\sigma_{22}$$

- So if covariance is low (let alone negative) your portfolio will have lower variance than either assets. Because the weights are less than 1, so their squares are small.

Optimal portfolio

- If you have a choice of only two assets all you need to decide are the weights
  - (e.g potatoes and rye or Amazon vs Google, 3 year T-Bill vs Wells Fargo Stock).
- Problem chose w_1 and w_2 to minimize .

$$Var\,(w_1x_1 + w_2x_2) = w_1^2\sigma_{11} + 2w_1w_2\sigma_{12} + w_2^2\sigma_{22}$$

- Subject to two constraints
  - Meet the return target $w_1r_1 + w_2r_2 \geq r$
  - and budget balance $w_1 + w_2 = 1$

Solution

- Substitution
- Start with the budget balance
- => $w_2 = (1- w_1)$ so replace in other equations
- Now the Return target ($w_1r_1 + w_2r_2 \geq r$)
  - $\Rightarrow w_1r_1 + w_2r_2 = w_1r_1 + (1- w_1)\, r_2 \geq r$
  - $\Rightarrow w_1 (r_1 - r_2) \geq r - r_2 \Rightarrow w_1 = (r-r_2)/(r_1 - r_2)$

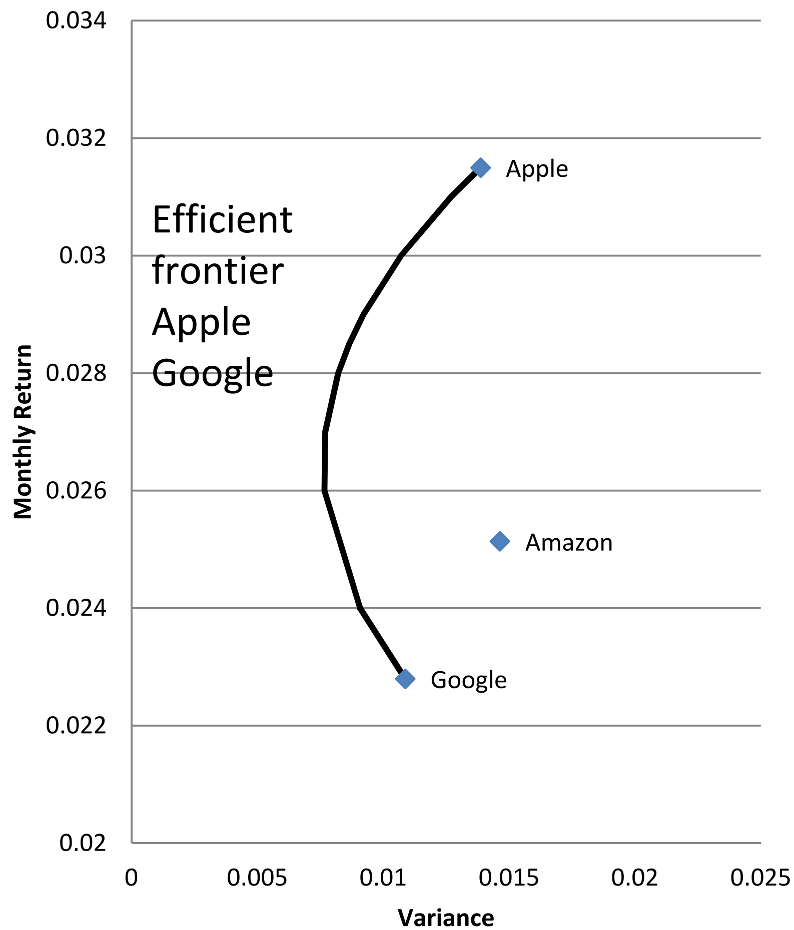
So in fact you do not have to maximize. Given the parameters the constraints always bind.

Now lets compute the variance the portolio

$$V(w_1r_1 + (1 - w_1r_2) = w_1^2\sigma_{11} + 2w_1(1 - w_1)\sigma_{12} + (1 - w_1)^2\,\sigma_{22}$$

Notice it's a quadratic function w_1 and thus of r

Mean Variance of Portfolios No Short Sales 2-assets

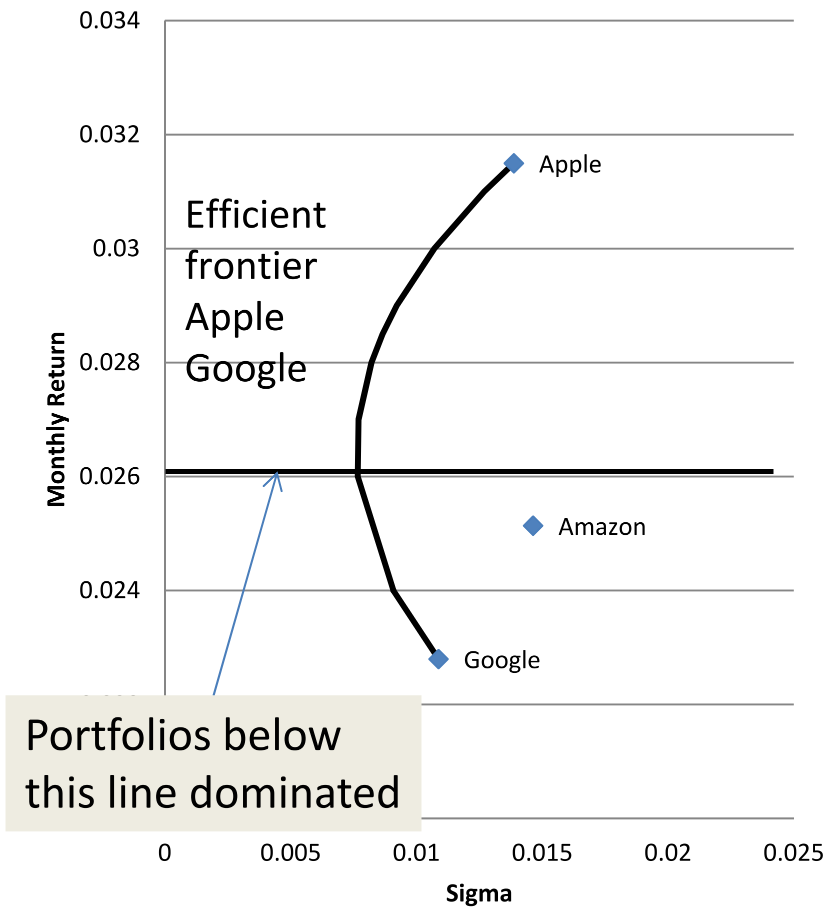


	Apple	Google
R	3.15	2.28
	Apple	Google
Apple	1.39	0.62
Google	0.62	1.08

Notice here you can't get a return higher than Apple (returns are weighted average) but you can get a better return than Google with a lower variance

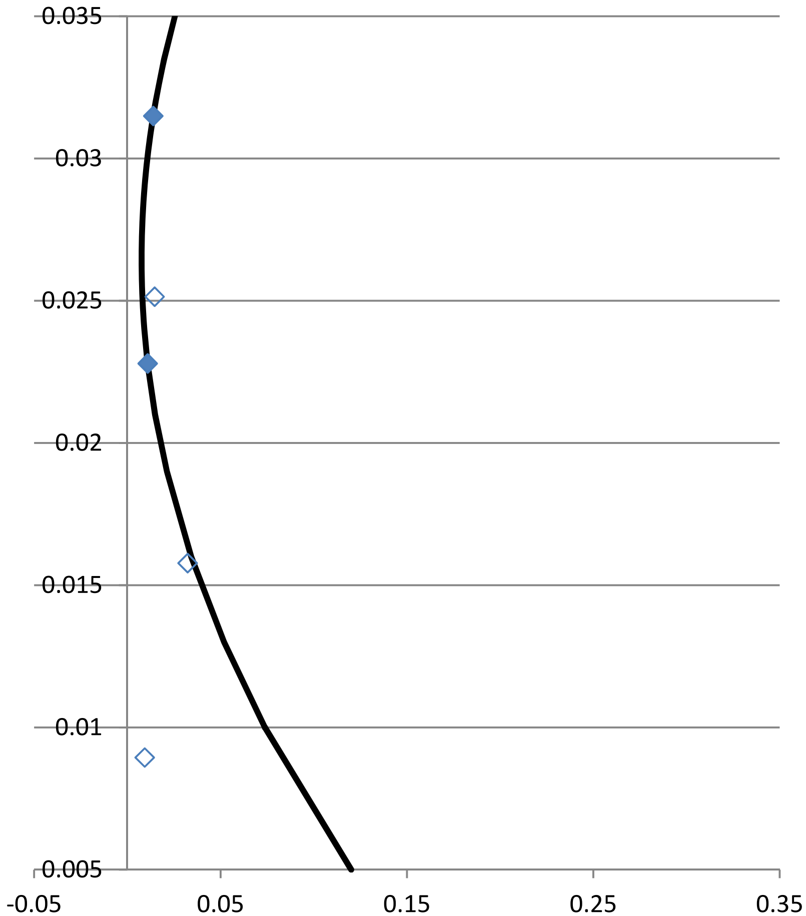
Efficient frontier is the whole choice set.

Sharpe Ratio



- So how to pick a point
- Sharpe Ratio
- $S(x)=(r_x-r_f)/(Stdev(x))$
- If agent is mean variance tradeoff type with parameter b
- Wants to chose X to set $S(x) = b^{½}$

With short sales



- With short sales you can drive your return down below what Google producing by Short selling Apple or up above what Apple returns by short selling Google
- Both increase variance (because the exposure now is greater than 1)

Beyond 2 stocks

The data
Note prices do not matter because you are figuring out proportions of your portfolio

x_1	x_2	…	x_i	…	x_n
r_1	r_2	…	r_i	…	r_n
σ_{11}	σ_{12}		σ_{1i}		σ_{1n}
σ_{21}	σ_{22}		σ_{2i}		σ_{2n}
….	….				
σ_{i1}	σ_{i2}		σ_{ii}		σ_{in}
…	…				
σ_{n1}	σ_{n2}		σ_{ni}		σ_{nn}

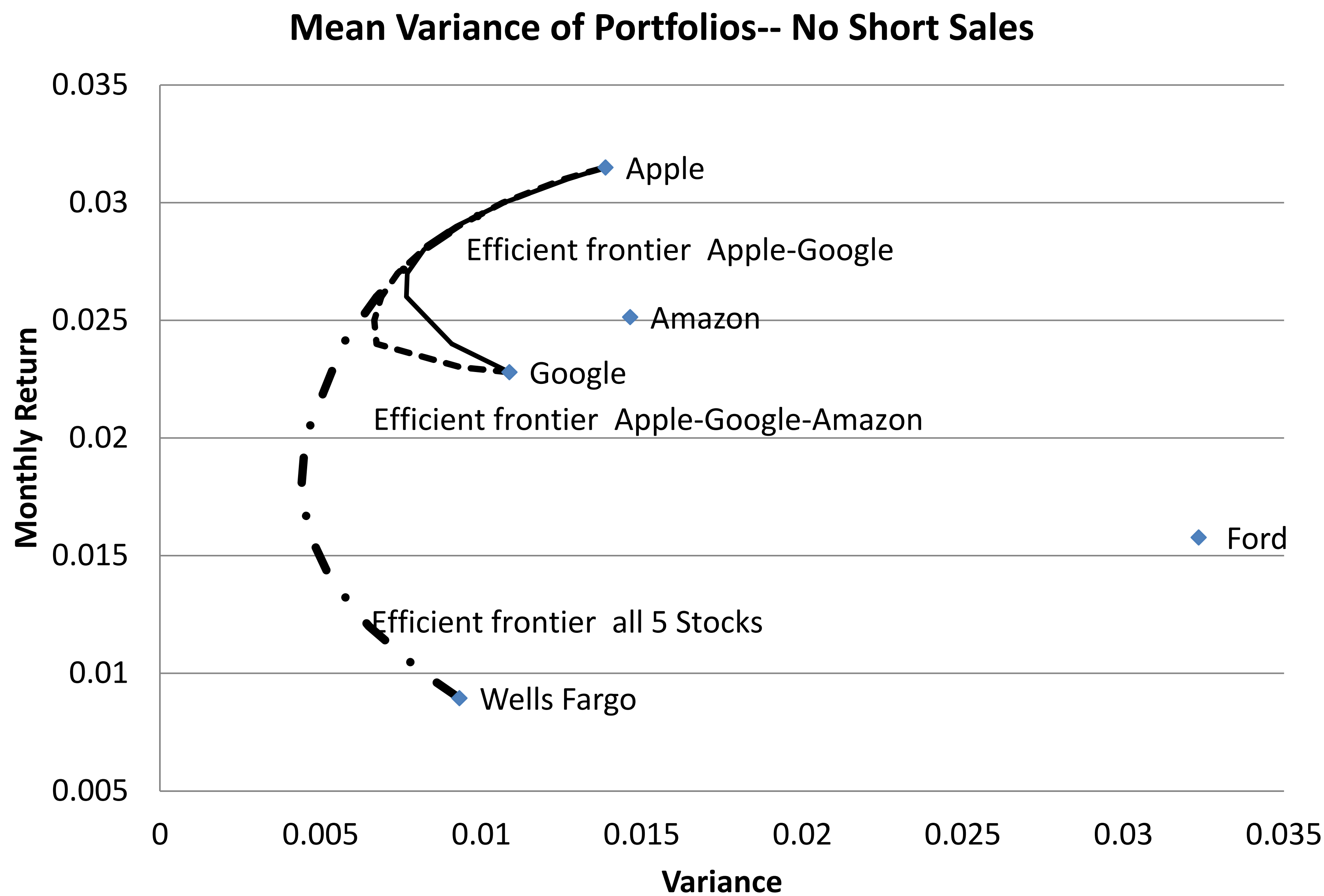
- The problem
- Find $W=\{w_1, w_2, \ldots w_i, \ldots w_n\}$
- That solve min Var(W) sbjt $r_W \geq r$
- So lets set this problem up

Beyond two stocks

- chose $w_1 \ldots w_n$ to minimize $$Var\left(\sum_{1}^{n} w_i x_i\right) = \sum_{i=1}^{n}\sum_{1}^{n} w_i w_j \sigma_{ij}$$
- Subject to two constraints
  - Meet the return target $$\sum_{i=1}^{n} w_i r_i \geq r$$
  - and budget balance $$\sum_{i=1}^{n} w_1 = 1$$
- Set up as a Lagrangean optimization $$L = \sum_{i=1}^{n}\sum_{1}^{n} w_i w_j \sigma_{ij} + \lambda_1 \left(r - \sum_{i=1}^{n} w_i r_i\right) + \lambda_2 \left(1 - \sum_{i=1}^{n} w_1\right)$$
- There are now n+2 unknowns $$(\mathrm{w}_1 \ldots \mathrm{w}_\mathrm{n}, \lambda_1, \lambda_2)$$
- So n+2 first order conditions. Because the Lagrangean is a polynomial of order 2, its FOC are n+2 linear equations with n+2 unknowns that can be solved uniquely.

The augmented data

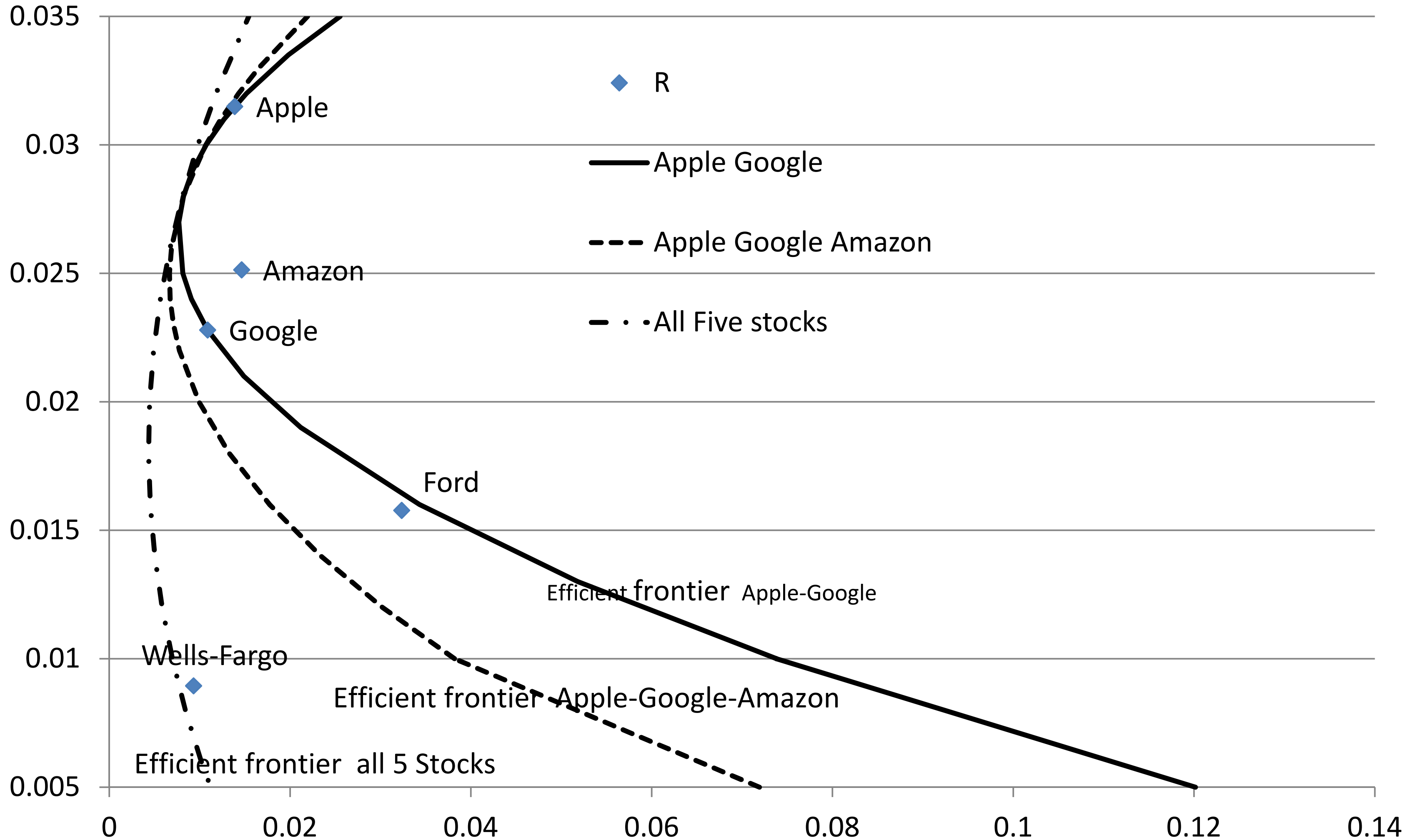
	Apple	Google	Amazon	Ford	WellsFargo	S&P500
Sigma2	1.39	1.09	1.46	3.23	0.93	0.20
R	3.15	2.28	2.51	1.58	0.89	0.67
	Apple	Google	Amazon	Ford	WellsFargo	
Apple	1.39	0.62	0.49	0.42	0.13	
Google	0.62	1.09	0.11	0.37	0.13	
Amazon	0.49	0.11	1.46	0.43	0.07	
Ford	0.42	0.37	0.43	3.23	0.80	
WellsFargo	0.13	0.13	0.07	0.80	0.93	


Mean Variance of Portfolios-- No Short Sales
0.035
0.03
0.025
0.02
0.015
0.01
0.005
Monthly Return
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
Variance
Apple
Efficient frontier Apple-Google
Amazon
Google
Efficient frontier Apple-Google-Amazon
Ford
Efficient frontier all 5 Stocks
Wells Fargo

No Short Sales

Target	WEIGHTS					Resulting
r	Apple	Google	Amazon	Ford	Wells Fargo	Sigma
0.90	0.00	0.00	0.00	0.00	1.00	0.009
1.00	0.00	0.07	0.01	0.00	0.92	0.008
1.20	0.00	0.14	0.07	0.00	0.79	0.006
1.40	0.00	0.21	0.14	0.00	0.66	0.005
1.60	0.00	0.27	0.20	0.00	0.52	0.005
1.80	0.05	0.30	0.24	0.00	0.42	0.004
2.00	0.13	0.29	0.25	0.00	0.33	0.005
2.40	0.29	0.28	0.28	0.00	0.14	0.006
2.60	0.37	0.28	0.29	0.00	0.05	0.007
2.80	0.52	0.20	0.28	0.00	0.00	0.008
2.90	0.65	0.11	0.25	0.00	0.00	0.009
3.00	0.77	0.01	0.22	0.00	0.00	0.011
3.10	0.92	0.00	0.08	0.00	0.00	0.013
3.04	0.95	0.00	0.01	0.00	0.00	0.013
3.15	1.00	0.00	0.00	0.00	0.00	0.014

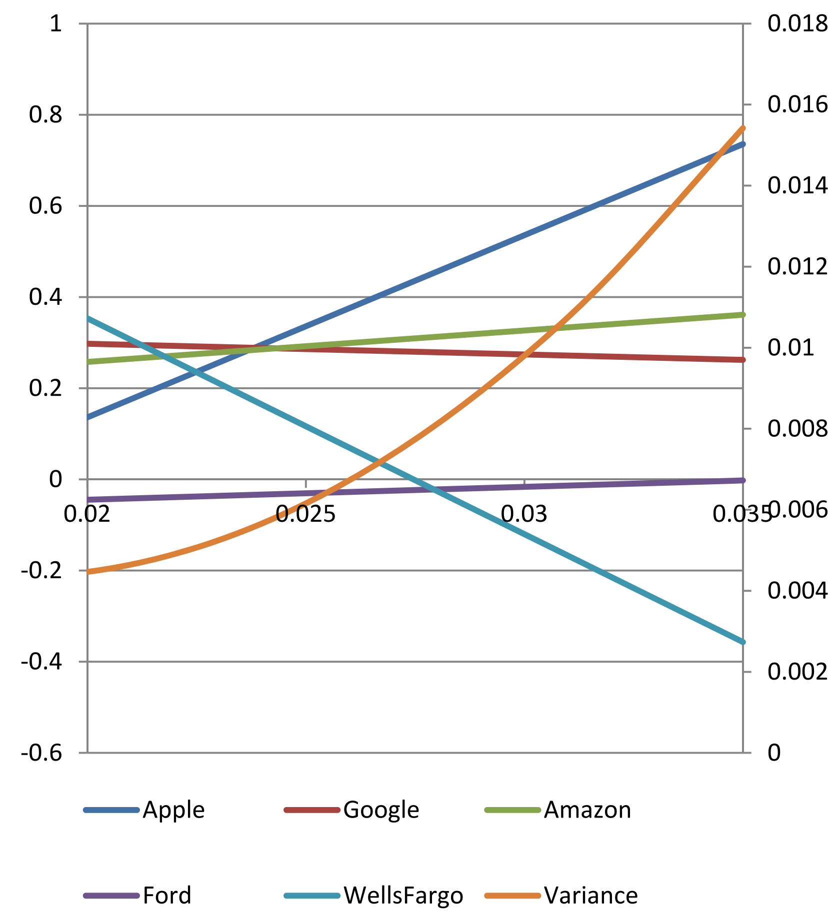
Short sales



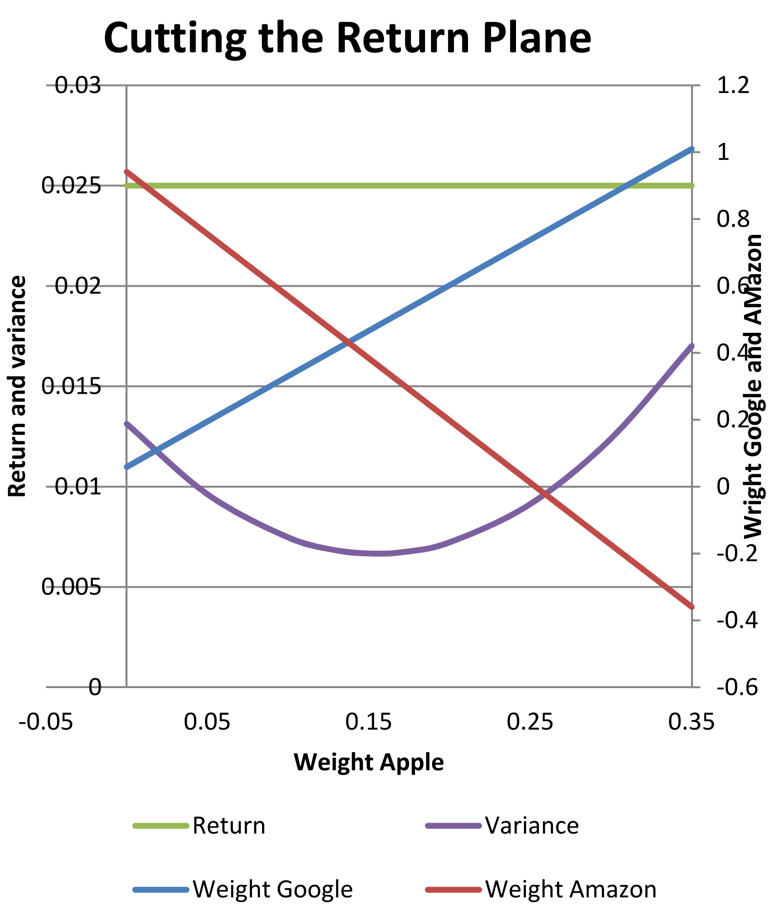
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
R
Apple Google
Apple Google Amazon
All Five stocks
Apple
Amazon
Google
Ford
Wells-Fargo
Efficient frontier Apple-Google
Efficient frontier Apple-Google-Amazon
Efficient frontier all 5 Stocks

Short Sales

- You short sell Ford
- But you can beat the no short sale portfolio at the top by short selling WellsFargo (low return low variance) overweighting the higher return stock



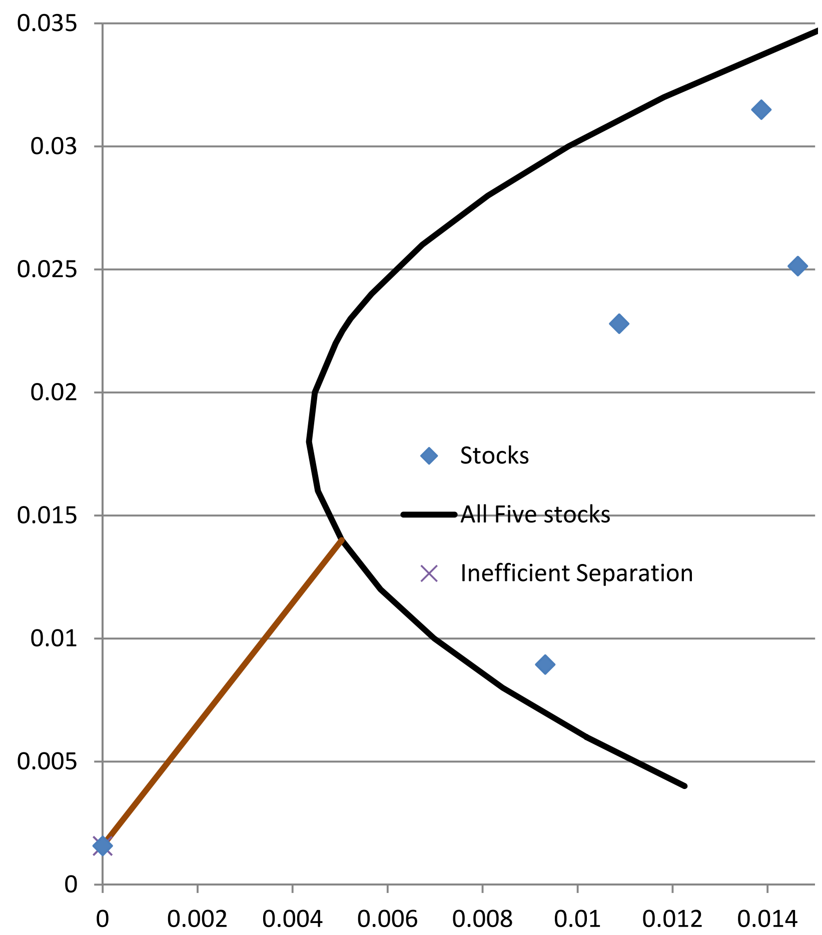
Why there is a return frontier



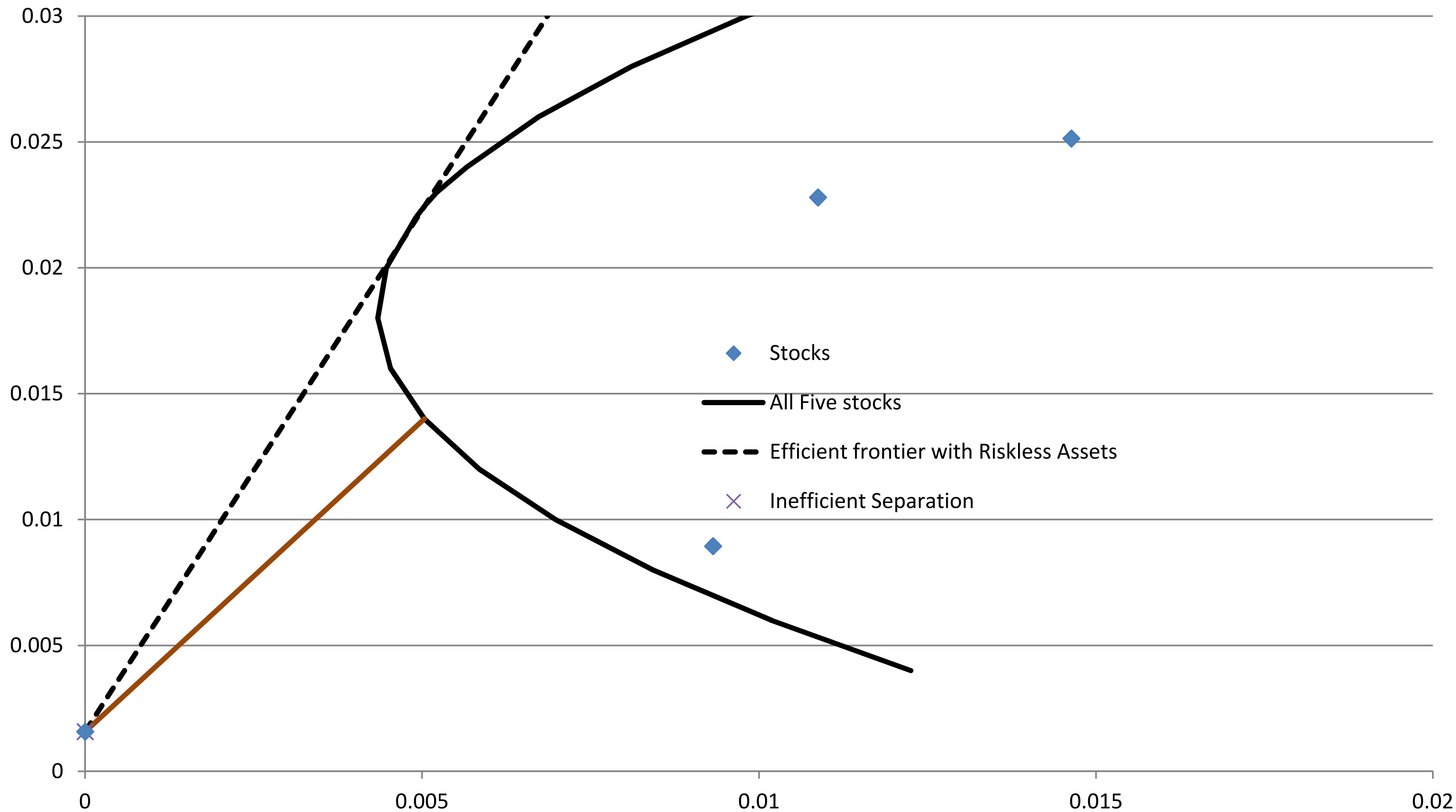
- 3 assets
- Fix the return
- Then conditional on a weight on Apple there is fixed proportion of Amazon and Google that give you that return.
- Over those variation the variance is a hyperbola with a unique min. that is the pt on the return frontier.

Risk free asset

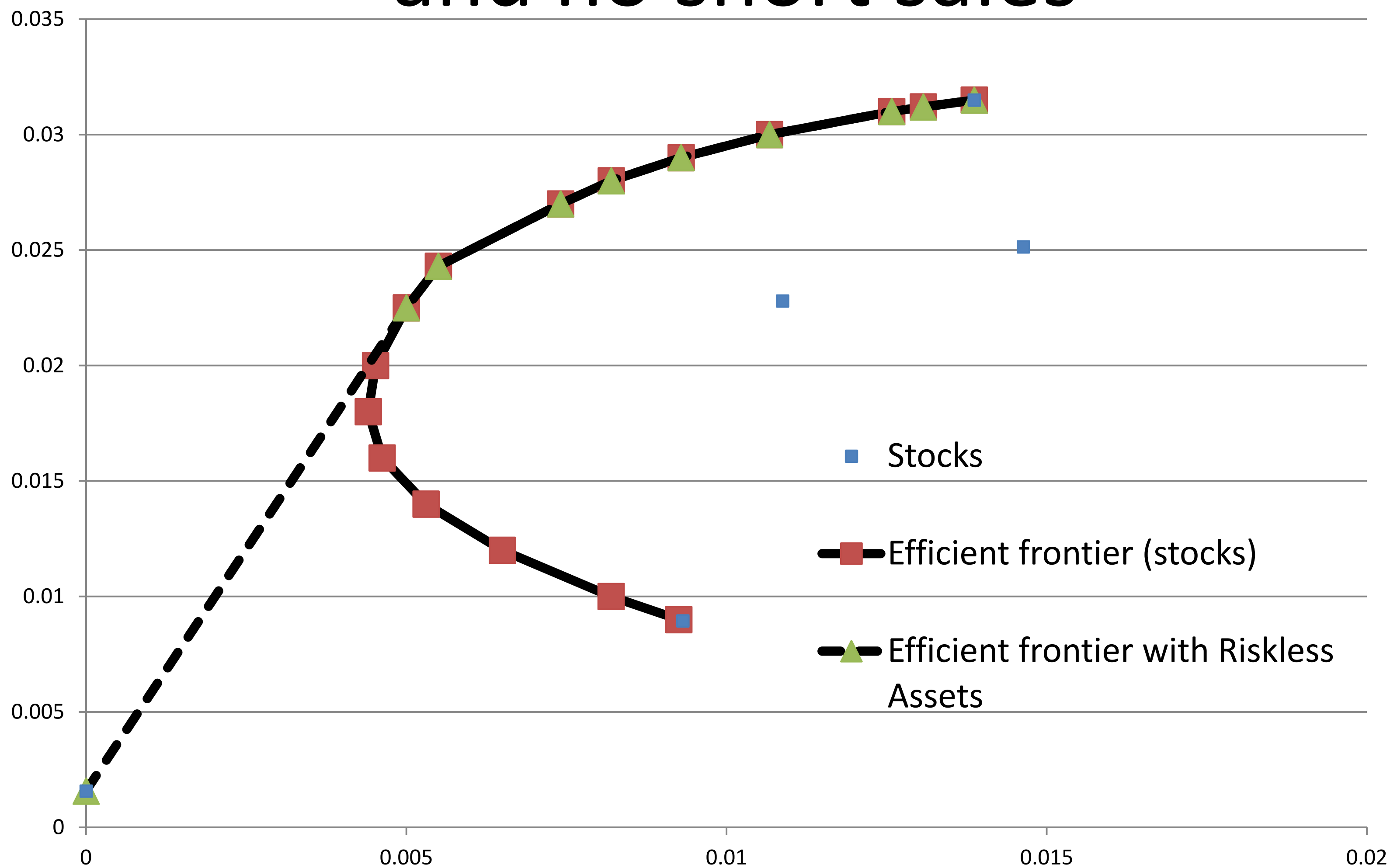
- You can construct a two part portfolio
- One that involves a mix of a portfolio on the efficiency frontier (there is no portfolio with the same mean and lower variance) and of the risky asset.
- But not so efficient



Efficient frontier with a riskless Asset (and short sales)



Efficient frontier with riskless asset and no short sales



Lessons from optimal portfolios

- Assets that are poorly correlated with a current portfolio have value
- There is an efficient frontier (where variance is minimized subject to return)
- Sharpe value connect portfolio choice with willingness to bear risk
- Short sales extend the range of the efficient frontier
- Existence of riskless asset implies portfolio separation into two parts a weight on riskless assets and a weight on the portfolio (strict with short sales)

Why not insurance contracts?

- Recall from last class, individuals are risk averse.   So they are willing to
  - 1. sell risky cash flows for less than their expected value
  - 2. buy insurance
- Indeed one could simply buy insurance,
  - But the portfolio approach says first find an efficient portfolio (because you get that insurance for free)
- Next step diversifiable vs undiversifiable risk

Next time

- 10-23 Class 8: The Portfolio approach to risk
- **More than one security out there and returns not perfectly correlated;**
- **Portfolios have better mean return profiles than individual stocks;**
- **Efficient frontier and the Sharpe value;**
- **Basic portfolio separation;**
- **Why not insurance contracts?**