

The Information Geometry of the Multinomial Distribution

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1.

The Fisher metric on the n -simplex corresponding to the $n+1$ dimensional multinomial

$$p(x|\theta) = \frac{\left(\sum_{i=1}^{n+1} x_i\right)!}{x_1! x_2! \dots x_{n+1}!} \theta_1^{x_1} \theta_2^{x_2} \dots \theta_{n+1}^{x_{n+1}} \quad (1)$$

is

$$g_{ij} = \sum_{k=1}^{n+1} \frac{1}{\theta_k} (\partial_i \theta_k) (\partial_j \theta_k) . \quad (2)$$

Note that i and j are indices in some unspecified local coordinate system and have nothing to do with the other index k . Differentiation w.r.t to the corresponding coordinates is denoted ∂_i and ∂_j and is assumed to apply to the immediately following symbol only.

To derive (2), the key step is to rewrite the Fisher metric

$$g_{ij} = \int (\partial_i \log p(x|\theta)) (\partial_j \log p(x|\theta)) p(x|\theta) dx \quad (3)$$

as

$$g_{ij} = - \int (\partial_j \partial_i \log p(x|\theta)) p(x|\theta) dx = -\mathbb{E}[\partial_j \partial_i \log p(x|\theta)] \quad (4)$$

To see the equivalence of (3) and (4) note that the latter is equal to

$$- \int \left(\frac{\partial_j \partial_i p(x|\theta)}{p(x|\theta)} - \frac{\partial_j p(x|\theta) \partial_i p(x|\theta)}{(p(x|\theta))^2} \right) p(x|\theta) dx$$

where the first term vanishes because $\int p(x|\theta) dx = 1$ is a constant. The second term gives exactly (3) by $\partial_i \log p(x|\theta) = (\partial_i p(x|\theta))/p(x|\theta)$.

We may now compute

$$\begin{aligned} \partial_j \partial_i \log p(x|\theta) &= \sum_{k=1}^{n+1} \partial_j \left(\partial_i \theta_k \frac{\partial \log p(x|\theta)}{\partial \theta_k} \right) = \sum_{k=1}^{n+1} \partial_j \left(\partial_i \theta_k \frac{x_k}{\theta_k} \right) = \\ &= \sum_{k=1}^{n+1} \partial_j \partial_i \theta_k \frac{x_k}{\theta_k} - \sum_{k=1}^{n+1} \partial_i \theta_k \partial_j \theta_k \frac{x_k}{\theta_k^2} \end{aligned}$$

and plug it into (4), noting that $E[x_k] = N\theta_k$ where $N = \sum_{k=1}^{n+1} x_i$:

$$\begin{aligned} g_{ij} &= -E[\partial_j \partial_i \log p(x|\theta)] = -\sum_{k=1}^{n+1} \partial_j \partial_i E[x_k] + \sum_{k=1}^{n+1} \partial_i \theta_k \partial_j \theta_k \frac{E[x_k]}{\theta_k^2} = \\ &= -N \partial_i \partial_j \underbrace{\sum_{k=1}^{n+1} \theta_k}_1 + N \sum_{k=1}^{n+1} \partial_i \theta_k \partial_j \theta_k \frac{1}{\theta_k} = N \sum_{k=1}^{n+1} \frac{1}{\theta_k} \partial_i \theta_k \partial_j \theta_k . \end{aligned}$$

Up to the constant N , which can be ignored, this gives (2).

2.

The metric (2) on the n -simplex is the pull-back of the natural metric (inherited from the embedding space) on the positive quadrant of the n -sphere related to the simplex by $\theta \mapsto \omega$ by $\theta_i = \sqrt{\omega_i}$.

Let e_1, e_2, \dots, e_{n+1} be the extension of the basis of the local tangent space (the one that the indices in ∂_i and ∂_j refer to) to the embedding space. Any vector $z = \sum_{k=1}^{n+1} \alpha_k \omega_k$ can be expressed in this basis as $z = \sum_{i=1}^{n+1} \beta_i e_i$, with $\alpha_k = \sum_{i=1}^{n+1} \beta_i \partial_i \omega_k$. To recover the natural norm $\|z\|^2 = \sum_{k=1}^{n+1} \alpha_k^2$, we write

$$\|z\|^2 = \sum_{k=1}^{n+1} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \beta_i \beta_j \partial_i \omega_k \partial_j \omega_k = \mathbf{z}^T \mathbf{G} \mathbf{z}$$

where $\mathbf{z} = (\alpha_1, \alpha_2, \dots, \alpha_{n+1})$ is the vector form of z in our coordinate system and \mathbf{G} is the metric tensor in the same coordinate system with entries

$$G_{i,j} = \sum_{k=1}^{n+1} \partial_i \omega_k \partial_j \omega_k .$$

This (or rather its restriction to the tangent space) is the metric that the sphere inherits from the embedding space. It is now just a matter of applying the chain rule to show that the pull-back of this is the multinomial metric on the n -simplex:

$$\sum_{k=1}^{n+1} \partial_i \omega_k \partial_j \omega_k = \sum_{k=1}^{n+1} \left(\partial_i \theta_k \frac{\partial \omega_k}{\partial \theta_k} \right) \left(\partial_j \theta_k \frac{\partial \omega_k}{\partial \theta_k} \right) = \sum_{k=1}^{n+1} \frac{1}{\theta_k} \partial_i \theta_k \partial_j \theta_k .$$