Spring 2024 Caltech Number Theory Learning Seminar: Lawrence-Venkatesh Method

The goal is to better understand the new method Lawrence and Venkatesh used to prove Faltings' Theorem in [LV20]. This involves inputs from p-adic Hodge theory and variations of p-adic Galois representations, and can be applied in higher-dimensional situations. We will first cover the necessary background in p-adic hodge theory and crystalline cohomology and then prove the theorem while black-boxing Parshin's trick.

(1) **4/12 Strategy of Proof** I will recall the proof of Faltings' original proof of the Mordell conjecture following Section 1 of [BS]. Then, I will give an outline of Lawrence and Venkatesh's proof.

(2) **4/19 The Gauss-Manin Connection and Complex Period Morphism** Define local systems, monodromy, and connections, and state Riemann–Hilbert. Give the definition of de Rham cohomology and the Gauss–Manin connection over $\mathbb{C}$. Give the construction of the period map and period domain over $\mathbb{C}$ for K ahler Manifolds as well. This should follow [Con] and [Lit].

(3) **4/26 Algebraic de Rham cohomology** Define hypercohomology and algebraic de Rham cohomology. Show that it agrees with the classical de Rham cohomology. This is the first two pages of [Gro66]. Give the algebraic definition of the Gauss–Manin connection following the first section of [KO68].

(4) **5/3 Crystalline Cohomology** Define the crystalline site as well as crystals and crystalline cohomology. Follow the first two sections (I.1, I.2) of [CL98].

(5) **5/10 p-adic Hodge Theory** Review the theory of local fields and their ℓ-adic Galois representations. State the basic theorems with Fontaine's period rings and if time sketch the construction of the period rings. The

basic theorems are given in the first two sections of [Ill90] and Section 2 of [Ber04]

(6) **5/17 The p-adic Period Morphism** Focusing on the p-adic period mapping, cover Section 3 of [LV20] by going through all the proof. Assume and cite the results of Section 2 of [LV20] as necessary.

(7) **5/24 The S-unit Equation** Give the proof of the S-unit equation, that there are finitely many pairs $u, v \in \mathcal{O}_S^\times$ of S-integer units such that $u + v = 1$ following Section 4 of [LV20]. Prove Theorem 4.1 and Lemma 4.2 after stating but no need to prove Lemma 4.3 and 4.4.

(8) **5/31 Proof of Faltings' Theorem** Black box the Kodaira-Parshin family and use results about it (e.g. monodromy, simplicity) to prove Faltings' Theorem following Section 5 of [LV20]. State Proposition 5.3 and go through the proof of Theorem 5.4.

References

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