CRYSTALLINE SITE AND CRYSTALLINE COHOMOLOGY

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1. ring of Witt vector

Lemma 1.1 (addition and multiplication rule)**.** *Let $X_0, X_1, \dots$ be an infinite sequence of variables, and p a prime number. For each $n \in \mathbb{Z}_{\geq 1}$, let $W_n(X_0, \dots, X_n) := X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n$. Then, there exists polynomials $S_0, S_1, \dots; P_0, P_1, \dots \in \mathbb{Z}[X_0, X_1, \dots; Y_0, Y_1, \dots]$ such that*

$$W_n(S_0, S_1, \dots, S_n) = W_n(X_0, X_1, \dots, X_n) + W_n(Y_0, Y_1, \dots, Y_n)$$
$$W_n(P_0, P_1, \dots, P_n) = W_n(X_0, X_1, \dots, X_n) \cdot W_n(Y_0, Y_1, \dots, Y_n)$$

Remark 1.2. We can treat $\mathbb{Z}[X_0, X_1, \cdots,]$ as a "ring scheme", such that $\Delta(X_i) = S_i, m(X_i) = P_i$. It is an ring object in the category of scheme, that is, the functor takes value in rings.

Definition 1.3 (Witt ring attached to A)**.** *Let A be a commutative ring. Then we define $W(A) := \prod_{n\geq 0} A$, such that the mutiplication rule and addition rule is defined as follows: Let $\underline{a} = (a_0, a_1, \dots), \underline{b} = (b_0, b_1, \dots) \in W(A)$. Then*

$$\underline{a} + \underline{b} := (S_0(a, b), S_1(a, b), \dots) \text{ and } \underline{a} \cdot \underline{b} := (P_0(a, b), P_1(a, b), \dots).$$

Then $W(A)$ has the $1 := (1, 0, \dots)$ being the multiplicative identity, and $p := 1 + 1 + \cdots + 1$ is the element $(0, 1, 0, \dots)$ in $W(A)$.

We also define the n^{th} truncated Witt vector being $W_n(A) = W(A)/p^n W(A)$.

Remark 1.4. Notice that $W(A) = \mathrm{Hom}(\mathbb{Z}[X_0, \cdots], A)$ and $W_n(A) = \mathrm{Hom}(\mathbb{Z}[X_0, \cdots, X_n], A)$, together with the ring structure introduced by m and Δ.

Now assume that $A = K$ where K is a perfect field of characteristic p. Then:

(1) $W(K)$ is a complete discrete valuation ring, with residue field K and maximal ideal $pW(K)$. $W(K)$ can be viewed as a thickening of K with ideal (p).
(2) $W(K)$ is endowed with the mappings $V, F : W(K) \to W(K)$,
$$V(\underline{a}) := (0, a_0, a_1, \dots) \text{ and } F(\underline{a}) := (a_0^p, a_1^p, \dots).$$
and $V \circ F(\underline{a}) = F \circ V(\underline{a}) = p \cdot \underline{a} = (0, a_0^p, a_1^p, \dots)$.

Example 1.5. The most important example should be when $K = \mathbb{F}_{p^n}$ and $K = \overline{\mathbb{F}}_p$. When $K = \mathbb{F}_{p^n}$, then $W(K) = \prod \mathbb{F}_{p^n} \cong \mathbb{Z}_{p^n}$, the integral closure of $\mathbb{Z}_p$ in the unique degree n unramified extension of $\mathbb{Q}_p$. But what is the isomorphism? Notice that we have the Techimuller lift $[] : \mathbb{F}_{p^n}^\times \to \mathbb{Z}_{p^n}^\times$ such that for $x \in \mathbb{F}_{p^n}$, $[x]$ is the unique element in $\mathbb{Z}_{p^n}$ satisfying $[x]^{p^n-1} - 1 = 0$ and $[x] \equiv x \mod (p)$. Then, the isomorphism $f : W(K) \cong \mathbb{Z}_p^{un}$ is given by $f(\underline{a}) = \sum [a_i] p^i$. Similarly, $W(\overline{\mathbb{F}}_p) \cong \mathbb{Z}_p^{un}$,

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the integral closure of $\mathbb{Z}_p$ in the maximal unramified extension of $\mathbb{Z}_p$. F on $W(K)$ is the Frobenius we know for $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_p^{un}$.

2. Divided power ring

In this section, we define the divided power structure on a pair (A, I). The prototype should be $\mathbb{Z} < x >$, the ring generated by the symbol $\frac{x^n}{n!}$. We want to define the element $\frac{x^n}{n!}$ in a ring A where "inverting n" only makes sense for $x \in I$ (Not on all A).

Definition 2.1. *Given a ring A and an ideal $I \subset A$, an* ***divided power structure*** *on (A, I) is a collection of maps $\{\gamma_n\}_{n\geq 0}$ where $\gamma_n : I \to A$ satisfying the following properties for all $x, y \in I, \lambda \in A$:*

1. $\gamma_0(x) = 1, \gamma_1(x) = x, \gamma_n(x) \in I$ *for all* $n \geq 1$.
2. $\gamma_n(x + y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$.
3. $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$
4. $\gamma_i(x)\gamma_j(x) = \frac{(i+j)!}{i!j!}\gamma_{i+j}(x)$
5. $\gamma_p(\gamma_q(x)) = \frac{(pq)!}{p!(q!)^p}\gamma_{pq}(x)$.

The motive of Divided power structure is to make sense of the elements $\frac{x^n}{n!}$. Indeed, as a consequence of these axioms, we can deduce two more properties:

- *From (3), we may deduce that $\gamma_n(0) = 0$ for all $n > 0$*
- *From $(1), (4)$ we may deduce that $n!\gamma_n(x) = x^n$.*

We say that (A, I, γ) is a ***divided power ring*** *or P.D. ring if $\gamma = \{\gamma_n\}_{n\geq 0}$ is a divided power structure on A, I.*

There are some accessible examples of divided power ring:

Example 2.2. If A is a $\mathbb{Q}$-algebra, then take any ideal I, (A, I) has a unique divided power structure given by

$$\gamma_n(x) = \frac{x^n}{n!}$$

So this notion is really interesting when there are some $n \in A$ that is not invertible.

Remark 2.3. When A is torsion free as an additive group (So in particular characteristic 0 ones), then for any $I \subset A$, the divided power structure is unique if it exists. (Basically it will be $\gamma_n(x) = \frac{x^n}{n!}$)

Example 2.4. Let R be a DVR with residue characteristic p. Let $\pi \in R$ be a uniformizer, and define e by $(p) = (\pi)^e$ (The ramification index). Then, $R, (\pi)$ has a divided power structure (And it is necessarily unique) if and only if $e \leq p - 1$.

Indeed, (π) has a divided power structure if and only if $x \in (\pi) \implies \frac{x^n}{n!} \in (\pi)$, which holds exactly when $\mathrm{ord}_\pi(\frac{\pi^n}{n!}) > 0$ for all $n \geq 1$. But if we write $n = \sum_i a_i p^i$, we have

$$\mathrm{ord}_\pi \frac{\pi^n}{n!} = n - e\mathrm{ord}_p(n!) = n - \frac{e}{p-1}(n - \sum_i a_i) = \frac{p-1-e}{p-1}n + \frac{e}{p-1}\sum_i a_i$$

which is positive for all $n \geq 1$ if and only if $e \leq p - 1$.

Example 2.5. As a more explicit example, notice that $W(K)$ for K a perfect field of char p has a unique $P.D.$ structure. Indeed, Let $I = (p)$, then $e = 1$ and $p \geq 2$.

Hence, $\gamma_n : (p) \to W(K) : \gamma_n(x) = \frac{x^n}{n!}$ is well defined and is the divided power structure on $W(K)$.

Similarly, for the truncated Witt ring W_n, it has a unique P.D.structure $(W_n, (p), \gamma_m(x)$ where $\gamma_m(x) = \frac{x^m}{m!}$ if $m < n$ and $\gamma_m(x) = 0$ for $m \geq n$.

Example 2.6. Let A be a ring. One can define the following A-algebra with divided power structure:

$$A\langle x_1, \cdots, x_r\rangle = \oplus_{n\geq 0}\Gamma^n$$

Where Γ^n is an A module generated by symbols $x_1^{[k_1]} \cdots x_r^{[k_r]}$, where $k_1+\cdots+k_r = n$. The algebra structure is given by $x_m^{[k_m]} x_n^{[k_n]} = \binom{m+n}{n} x_{m+n}^{[k_m+k_n]}$. The ideal $I = \oplus_{n\geq 1}\Gamma^n$ posses a unqiue P.D. structure $\gamma_n(x_i) = x_i^{[n]}$.

Remark 2.7. If A is canceled by an integer $n > 2$ and if the ideal $I \subset A$ has a $P.D.$structure, the I is a nilpotent ideal, since $x^n = n!\gamma_n(x) = 0$ for all $x \in I$. In particular Spec(A) and Spec(A/I) has the same topological space.

Definition 2.8. *Let (A, I, γ) and (B, J, δ) be divided power rings. A divided power morphism $f : (A, I, \gamma) \to (B, J, \delta)$ is a ring map $f : A \to B$ such that $f(I) \subseteq J$ and $\delta_n \circ f = f \circ \gamma_n$, for all $n \geq 0$.*

We give a criteria which divided powers may be extended from one ring to another.

Proposition 2.9. *Let (A, I, δ) be a P.D. ring and B be an A-algebra, and suppose that $\mathrm{Tor}_A(A/I, B) = 0$(i.e. $I \otimes_A B \cong IB$. Then, the divided powers (A, I, γ) extend uniquely to (B, IB, δ). In particular, when B is a flat A algebra, then we can apply this propositition.*

3. Divided power envelope

Lemma 3.1. *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. There exists a divided power ring $(D, \overline{J}, \overline{\gamma})$ and a homomorphism of divided power rings $(A, I, \gamma) \to (D, \overline{J}, \overline{\gamma})$ such that*

$$\mathrm{Hom}_{(A,I,\gamma)}((D, \overline{J}, \overline{\gamma}), (C, K, \delta)) = \mathrm{Hom}_{(A,I)}((B, J), (C, K))$$

$$Id_D \iff f_D$$

functorially in the divided power algebra (C, K, δ) over (A, I, γ). Here the LHS is morphisms of divided power rings over (A, I, γ) and the RHS is morphisms of (ring, ideal) pairs over (A, I)

Definition 3.2 (divided power envelope)**.** *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. The divided power algebra $(D, \overline{J}, \overline{\gamma})$ as above is called the* ***divided power envelope*** *of J in B relative to (A, I, γ) and is denoted $D_B(J)$ or $D_{B,\gamma}(J)$.*

Let $(A, I, \gamma) \to (C, K, \delta)$ be a homomorphism of divided power rings. The universal property of $D_{B,\gamma}(J) = (D, \overline{J}, \overline{\gamma})$ is

$$\text{ring maps } B \to C \text{ which map } J \to K \iff \text{ divided power homomorphisms } (D, \overline{J}, \overline{\gamma}) \to (C, K, \delta)$$

$$g \circ f_D : B \to D \to C \iff g : D \to C$$

So, how to construct this P.D.envolope $(D, \overline{J}, \overline{\gamma})$? By the universal property, the surjection $B \to B/J$ factors through D

$$B \to D \to B/J$$

The first arrow maps J into $\overline{J}$ and $\overline{J}$ is the kernel of the second arrow. The ideal $\overline{J}$ is generated by $\overline{\gamma}_n(x)$, where $n > 0$ and x is an element in the image of $IB \to D$. These $\overline{\gamma}_n(x)$ also generate D as a B-algebra.

Example 3.3. (1) (A, I, γ) is its own $P.D.$ envolpe of the identity $A \to A$.

(2) Given (A, I, γ) and consider $\varphi : A \to A/I$, $B = A/I$ and $J = (0)$. Then $D_{B,\gamma}((0)) = (A, I, \gamma)$. This can be seen by either showing that (A, I, γ) satisfies the required universal property.

(3) In the case where $\varphi : B \to B$ sending I to J, we have the following lemma for $D_{A,\gamma}(J)$:

Lemma 3.4. *Let (B, I, γ) be a divided power algebra. Let $I \subset J \subset B$ be an ideal. Let $(D, \overline{J}, \overline{\gamma})$ be the divided power envelope of J relative to γ. Choose elements $f_t \in J, t \in T$ such that $J = I + (f_t)$. Then there exists a surjection*

$$\Psi : B\langle x_t \rangle \to D$$

of divided power rings mapping x_t to the image of f_t in D. The kernel of Ψ is generated by the elements $x_t - f_t$ and all $\gamma_n(\sum r_t x_t - r_0)$, wherever we have a relation $\sum r_t f_t - r_0 = 0$ in B for some $r_t \in B, r_0 \in I$.

(4) Consider A and $I = (0), \gamma$ be the trivial divided power structure. Let $B = A[t]$ with $J = (t)$. Then $D_{B,\gamma}(J) = A\langle t \rangle$, with $\gamma_n(t) = t^{[n]}$. More generally, for (A, I, γ) and $B = A[t_1, \cdots, t_n]$ and $J = IB + (t_1, \cdots, t_n)$, we have $D_{B,\gamma}(J) = A\langle t_1, \cdots, t_n \rangle$, $\overline{J} = J = IA\langle x1, \ldots, x_t \rangle + A\langle x1, \ldots, x_t \rangle_+$, and the divided power structure δ is that $\delta_n(x_i) = x_i^{[n]}$.

It has the following universal property: $(A, I, \gamma) \to (A\langle x_1, \cdots, x_t \rangle, J, \delta)$ is a homomorphism of divided power rings. Moreover, A homomorphism of divided power rings $\varphi : (A\langle x_1, \cdots, x_t \rangle, J, \delta) \to (C, K, \epsilon)$ is the same thing as a homomorphism of divided power rings $A \to C$ and elements $k_1, \cdots, k_t \in K$

(5) We have a relative version of Divided power polynomial algebra: Let (A, I, γ) be a divided power ring. Let B be an A-algebra and $IB \subset J \subset B$ an ideal. Let x_i be a set of variables. Then

$$D_{B[x_i],\gamma}(JB[x_i] + (xi)) = D_{B,\gamma}(J)\langle x_i \rangle$$

The construction of divided power envelope is functorial in both variable B and in the base ring A:

Lemma 3.5. *Let (A, I, γ) be a divided power ring, B an A-algebra, $J \subset B$, then:*

- *If B' is a flat B algebra $\implies D_{B',\gamma}(JB') = D_{B,\gamma}(J) \otimes_B B'$.*
- *Let $(A, I, \gamma) \to (A', I', \gamma')$ be a P.D. morphism. Then $D_{B,\gamma}(J) \otimes_A A' = D_{B\otimes_A A',\gamma'}(J \otimes_A A')$*

The notion of divided powers may be globalized as follows. We replace A by a scheme S and I by a quasi-coherent ideal $\mathcal{I}$ of $\mathcal{O}_S$.

Definition 3.6. *An divided power structure on* $(\mathcal{O}_S, \mathcal{I})$ *is* $\{\gamma_n\}_{n\geq 1}, \gamma_n : \mathcal{I} \to \mathcal{O}_S$*, such that for each open set* U *of* S*,* $\gamma_n : \mathcal{I}(U) \to \mathcal{O}_S(U)$ *is a divided power structure defined as before, and the restriction maps are divided power morphisms.*

A P.D. morphism $f : (S, I, \gamma) \to (S', \mathcal{I}', \gamma')$ *is a morphism* $f : S \to S'$ *such that* $f^{-1}\mathcal{I}'\mathcal{O}_S \subseteq \mathcal{I}$ *, and for all* $U' \subseteq S'$ *open, the map* $f : (\mathcal{O}_{S'}(U'), \mathcal{I}'(U'), \gamma') \to (\mathcal{O}_S(f^{-1}(U')), \mathcal{I}'(f^{-1}(U')), \gamma)$ *is a P.D. morphism.*

Since localizations are flat, we have that for any ring A and any ideal $I \subset A$, P.D. structures on A and I correspond to P.D. structures on Spec$A$ and $\tilde{I}$. Moreover, P.D. morphisms $(A, I, \gamma) \to (B, J, \delta)$ correspond to P.D. morphisms (Spec$B, \tilde{J}, \delta) \to$ (Spec$A, \tilde{I}, \gamma$).

Since quasi-coherent ideals $\mathcal{I}$ of S correspond to closed immersions $U \to S$, in the following, we will write a P.D. scheme $(S, \mathcal{I}, \gamma)$ as $(U \hookrightarrow S, \gamma)$, where $U \hookrightarrow S$ is the closed immersion corresponding to $\mathcal{I}$. In what follows, we will primarily be interested in divided power thickenings, which are divided powers schemes $(U \hookrightarrow S, \gamma)$ for which $\mathcal{I}$ is nilpotent ideal (Equivalent to $U \hookrightarrow S$ being a homeomorphism).

As an application of *P.D.* envelope in scheme, we have the following construction:

Proposition 3.7 (P.D. thickening)**.** *Let* K *be a perfect field of characteristic* p *and* X/K *a* K *scheme. Let* $i : X \hookrightarrow Z$ *be a closed immersion of* X *into a smooth* W_n *scheme* Z*. Then there exists a unique scheme* $\varphi : \tilde{Z} \to Z$*, called the P.D. envolope of* i*, such that* $i = \varphi \circ i'$*. Moreover, for any other P.D scheme* T, δ *such that* $T \to X$ *is a morphism and* T *is over* Z*, then this morphism factor through* $T \to \tilde{Z} \to X$*.*

Proof. We look at the ring theoretic picture. Since W_n has a P.D. structure and Z is a smooth W_n scheme, then Z has a unique extended P.D. structure. $X \to Z$ corresponds to $A \to B = A/I$. Then $\tilde{Z} = D_{B,\gamma}((0))$ is a P.D. envolope of $(A, I, \gamma), A \to A/I$. ☐

Example 3.8. (1) If $Z \otimes_{W_n} k \cong X$, then $\tilde{Z} = Z$, as is the affine case.

(2) If $X = \text{Spec}k$, $Z = \text{Spec}W_n[t]$, then $\tilde{Z} = \text{Spec}W_n\langle t\rangle$. Indeed, if we look at the affine picture, let $B = W_n, J = (p)$, then $\tilde{Z}$ corresponds to $D_{W_n(t),\gamma}(\mathcal{I})$, where $\mathcal{I}$ is the ideal of Z cutting out X, so it is $(p) + (t)$. By the (5) of the example, it is equal to $D_{B,\gamma}((p))\langle t\rangle = W_n\langle t\rangle$.

4. Crystalline site and crystals

Remark 4.1. A site is a category $\mathcal{C}$ along with coverings for each element in $\mathcal{C}$: For each $X \in \mathcal{C}$, $cov(X) = \{X_i \to X\}_{i\in I}$ such that $cov(X)$ contains all isomorphisms, and is closed under base change and composite. The Crystalline site is a site under the covering specified by the Crystalline topology.

Let K be a perfect field of characteristic $p > 0$. Let $W = W(k)$ be the ring of Witt vector of K, $W_n = W/p^n$ (So $W_1 = K.W_n$ should be considered as nilpotent thickenings of K) Then $(W_n, (p))$ has a unique divided power structure. Let X be a K-scheme

Definition 4.2 (The Crystalline site)**.** *The Crystalline site of* X *over* W_n*, denoted as* Crys(X/W_n)*, is the category together with coverings, where:*

- *The objects are commutative diagram:* $\begin{array}{ccc} U & \hookrightarrow & T \\ \downarrow & & \downarrow \\ Spec(k) & \hookrightarrow & W_n \end{array}$ *, where* $U \subset X$ *is a zariski open, and* $U \hookrightarrow T$ *are divided power thickening of* U*. That is,* $U \hookrightarrow T$ *is a closed immersion of* W_n *scheme, defined by an ideal* $\mathcal{I}$*, such that* $\mathcal{I} \supset (p)\mathcal{O}_T, (\mathcal{O}_T, \mathcal{I})$ *has a divided power structure* δ *satisfying* $\delta(pa) = \gamma_n(p)a^n, pa \in \mathcal{I}$*. We denote such object as* (U, T, δ)*.*
- *The morphisms* $f : (U, T, \delta) \to (U', T', \delta')$ *is* $f : T \to T'$ *which is a P.D. morphism and* $f\mid_U$ *is an open immersion.*
- *For* (U, T, δ)*, the family of covers are* $\{(U_i, T_i, \delta_i)\}_{i \in I}$ *such that* $T_i \hookrightarrow T$ *is an open immersion and* $T = \cup_i T_i$*.*

Definition 4.3 (The Sheaf on $\text{Crys}(X/W_n)$)**.** *A sheaf* $\mathcal{F}$ *on* $\text{Crys}(X/W_n)$ *is equivalent to the following data: For every element* $(U \hookrightarrow T, \gamma)$ *of* $\text{Crys}(X/W_n)$*, a Zariski sheaf* $\mathcal{F}_{(U \hookrightarrow T, \gamma)}$ *on* T*, and for every morphism* $u : (U \hookrightarrow T, \gamma) \to (U' \hookrightarrow T', \gamma')$ *in* $\text{Crys}(X/W_n)$*, a map*

$$\rho_u : u^{-1}\mathcal{F}_{(U' \hookrightarrow T', \gamma')} \to \mathcal{F}_{(U \hookrightarrow T, \gamma)}$$

of sheaves on T *satisfying the following properties:*

(1) If $v : (U' \hookrightarrow T', \gamma') \to (U'' \hookrightarrow T'', \gamma'')$ *is another morphism in* $\text{Crys}(X/W_n)$

$$\begin{array}{ccc} u^{-1}v^{-1}\mathcal{F}_{(U'' \hookrightarrow T'', \gamma'')} & \xrightarrow{u^{-1}\rho_u} & u^{-1}\mathcal{F}_{(U' \hookrightarrow T', \gamma')} \\ \downarrow & & \downarrow \rho_u \\ (v \circ u)^{-1}\mathcal{F}_{(U'' \hookrightarrow T'', \gamma'')} & \xrightarrow{\rho_{v \circ u}} & \mathcal{F}_{(U \hookrightarrow T, \gamma)} \end{array}.$$

That is, $\rho_u \circ u^{-1}\rho_v = \rho_{v \circ u}$*.*

(2) If $u : T \to T'$ *is an open immersion, then* ρ_u *is an isomorphism of sheaves on* T*.*

Example 4.4. A particularly important example of Sheaf on $\text{Crys}(X/W_n)$ is given by the structure sheaf $\mathcal{O}_{\text{Crys}(X/W_n)}$. It is a sheaf valued in rings given by $(U \hookrightarrow T, \gamma) \to \mathcal{O}_T$. We define a sheaf of modules on $\text{Crys}(X/W_n)$ to be a sheaf of $\mathcal{O}_{\text{Crys}(X/W_n)}$ -modules. For a sheaf of modules $\mathcal{F}$, note that the maps ρ_u define maps $\rho_u : u^*\mathcal{F}_{(U' \hookrightarrow T', \gamma')} \to \mathcal{F}_{(U \hookrightarrow T, \gamma)}$ of $\mathcal{O}_T$ -modules.

Definition 4.5. *A sheaf* $\mathcal{F}$ *of modules on* $\text{Crys}(X/W_n)$ *is a* ***crystal*** *if each* ρ_u *is an isomorphism. Note that* $\mathcal{O}_{\text{Crys}(X/W_n)}$ *is itself a crystal, since* $\mathcal{O}_T$ *is a quasi-coherent sheaf, and so we do have* $\rho_u : u^*\mathcal{O}_{T'} \cong \mathcal{O}_T$*.*

We say that $\mathcal{F}$ *is a crystal of quasi-coherent modules if each* $\mathcal{F}_{(U \hookrightarrow T, \gamma)}$ *is a quasi-coherent module on* T*.*

Example 4.6. Let us consider crystals of quasi-coherent modules on $\text{Crys}(\text{Spec}\mathbb{F}_p)$. Since $\text{Spec}\mathbb{F}_p$ is affine and has only one nonempty open subset, we may identify objects in $\text{Spec}\mathbb{F}_p$ with P.D. rings (A, I, γ) where I is a nil ideal and $A/I = \mathbb{F}_p$. Since A is a $W(\mathbb{F}_p) = \mathbb{Z}_p$-algebra. We may therefore define a crystal of quasi-coherent modules on $\text{Crys}(\text{Spec}\mathbb{F}_p/W)$ by fixing a $\mathbb{Z}_p$-module M and setting $\mathcal{F}_{(A,I,\gamma)}; = A \otimes_{\mathbb{Z}_p} M$.

Now that we have the notion of Crystalline site, we can define the Crystalline cohomology as the cohomology of $\mathcal{O}_{\mathrm{Crys}(X/W_n)}$.

Definition 4.7. *With the assumption as above, we define*

$$H^i(X/W_n) = H^i((X/W_n)_{Crys}; \mathcal{O}_{X/W_n}), H^i(X/W) = \varprojlim_n H^i(X/W_n)$$

This definition is not computable. Below we show a comparison theorem that makes the crystalline cohomology computable.

5. comparison result of Crystalline and De Rham

Let $j : X \hookrightarrow Z$ a closed immersion of X in a smooth scheme Z over W_n. If we look at the structural sheaf, we get $\mathcal{O}_Z \to \mathcal{O}_X$. Then, since $\mathcal{O}_Z$ is a W_n module, we have a $P.D.W_n$ algebra $\mathcal{O}_{\tilde{Z}}$ with a W_n linear map $\iota : \mathcal{O}_Z \to \mathcal{O}_{\tilde{Z}}$, such that $\mathcal{O}_Z \to \mathcal{O}_X$ factors through $\mathcal{O}_Z \to \mathcal{O}_{\tilde{Z}} \to \mathcal{O}_X$, and $X \to \tilde{Z}$ is a divided power morphism over W_n. We call this $\tilde{Z}$ the divided power thickening of X in Z. As name suggested, $X \to \tilde{Z}$ is a nil-immersion, so they have the same topological space.

Let $\tilde{\mathcal{I}}$ be the ideal in $\tilde{Z}$ defining $\mathcal{O}_X$. Then $\mathcal{O}_X = \mathcal{O}_{\tilde{Z}}/\tilde{\mathcal{I}}$. There exists a unique integrable connection

$$d : \mathcal{O}_{\tilde{Z}} \to \mathcal{O}_{\tilde{Z}} \otimes_{\mathcal{O}_Z} \Omega^1_{Z/W_n}$$

such that $d\gamma_n(x) = \gamma_{n-1}(x) \otimes dx$, for all $x \in \tilde{\mathcal{I}}$. Thus, $\mathcal{O}_{\tilde{Z}} \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/W_n}$ is a complex of abelian sheaf on $\tilde{Z}$ that has the same underlying space as X.

Theorem 5.1. *There is a canonical isomorphism between crystalline cohomology and the Hypercohomology* $H^i(X/W_n) \cong \mathbb{H}^i(X, \mathcal{O}_{\tilde{Z}} \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/W_n})$

Corollary 5.1.1. *If Z/W_n is a smooth lifting of X/k, Then $\tilde{Z} = Z$, and*

$$H^i(X/W_n) \cong H^i_{dR}(Z/W_n)$$

5.1. **Properties of Crystalline cohomology.** A lot of the properties of Crystalline cohomology is true without assuming that X has a smooth lifting over W_n. But to see these properties, it is easier to assume that X/k is proper smooth and admits a lifing Z over W_n that's also smooth.

(1) $H^*(X/W_1) = H^*_{dR}(X/k)$
(2) $H^n_{\mathrm{Crys}}(X/W_m)$ is a contravariant functor in X. These groups are finitely generated W_m modules, and zero if $n < 0 orn > 2dim(X)$
(3) There is a cup-product structure

$$\cup_{i,j} : H^i_{\mathrm{Crys}}(X/W)/torsion \times H^j_{\mathrm{Crys}}(X/W)/torsion \to H^{i+j}_{\mathrm{Crys}}(X/W)$$

Moreover, $H^{2\dim(X)}_{\mathrm{Crys}}(X/W) \cong W$, and $\cup_{n,2dim(X)-n}$ induces a perfect pairing modulo torsion, called Poincare duality.
(4) $H^n_{\mathrm{Crys}}(X/W)$ defines an integral structure on $H^n_{dR}(X/K)$.
(5) If l is a prime different from p,

$$\dim_{\mathbb{Q}_l} H^n(X, \mathbb{Q}_l) = rank/W H^n_{\mathrm{Crys}}(X/W)$$

(6) We have the universal coefficient lemma

$$0 \to H^n_{\mathrm{Crys}}(X/W) \otimes_W k \to H^n_{dR}(X/k) \to Tor_1^W(H^{n+1}_{\mathrm{Crys}}(X/W), k) \to 0$$

This can be derived from UFC of De Rham cohomology if X has a lift, but is true even if X does not have a lift.

(7) the absolute Frobenius morphism $F : X \to X$ induces a σ-linear morphism $\varphi : H^n_{\mathrm{Crys}}(X/W) \to H^n_{\mathrm{Crys}}(X/W)$ of W-modules.

Example 5.2. Let X be a smooth and proper variety over a perfect field k of positive characteristic p, and assume that the Using only the properties of crystalline cohomology mentioned above, then the following are equivalent

- For all $n \geq 0$, the W-module $H^n_{\mathrm{Crys}}(X/W)$ is torsion-free.
- We have $\dim_{\mathbb{Q}_l} H^n(X, \mathbb{Q}_l) = \dim_k H^n_{dR}(X/k)$ for all $n \geq 0$ and all primes $l \neq p$.

Example 5.3. Let us give a two fundamental examples.

(1) Let A be an Abelian variety of dimension g. Then, all $H^n_{\mathrm{Crys}}(A/W)$ are torsion-free W-modules. More precisely, $H^1_{\mathrm{Crys}}(A/W)$ is free of rank $2g$ and for all $n \geq 2$ there are isomorphisms $H^n_{\mathrm{Crys}}(A/W) \cong \wedge^n H^1_{\mathrm{Crys}}(A/W)$. Also, $\mathbb{D}(A[p^\infty]) \cong H^1_{\mathrm{Crys}}(A/W)$, compatible with the Frobenius-actions on both sides.

(2) For a smooth and proper variety X, let $\alpha : X \to Alb(X)$ be its Albanese morphism. Then, α induces an isomorphism $H^1_{\mathrm{Crys}}(X/W) \cong H^1_{\mathrm{Crys}}(Alb(X)/W)$ In particular, $H^1_{\mathrm{Crys}}(X/W)$ is always torsion-free.