Algebraic de Rham Cohomology

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1 Hypercohomology

Let X be a complex manifold. We still have the de Rham complex $\Omega^\bullet_{X/\mathbb{C}}$ given by

$$0 \to \mathcal{O}_X \to \Omega^1_{X/\mathbb{C}} \to \cdots$$

and naively, we could define

$$H^n_{dR}(X/\mathbb{C}) = H^n(\Gamma(X, \Omega^\bullet_{X/\mathbb{C}})).$$

Over the analytification, the Poincaré Lemma gives us that $\Omega_{X/\mathbb{C}}$ is an acyclic resolution of the constant sheaf $\underline{\mathbb{C}}$, and so we have

$$H^n_{dR}(X^{an}, \underline{\mathbb{C}}) = H^n(\Gamma(X^{an}, \Omega^\bullet_{X/\mathbb{C}})),$$

meaning we can use the de Rham complex to compute the cohomology. However, this is not the case for the algebraic de Rham complex. Instead, we will have to use hypercohomology.

Definition 1. Let $C^\bullet$ be a cochain complex and let F be a functor, say the global sections functor $\Gamma(\cdot)$. An injetive resolution of $C^\bullet$ is a quasi-isomorphism of $C^\bullet$ with a complex of injective objects $I^\bullet$, so we get a sequence of morphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \end{array}$$

which induces an isomorphism of cohomology of these sequences.

The n-th right hyperderived functor R^nF is given by

$$R^nF(C^\bullet) = H^n(F(I^\bullet)).$$

So, hypercohomology replaces a complex with a suitable complex of injectives, then uses the functor applied to that sequence to compute cohomology. As in cohomology of sheaves, we can take $I^\bullet$ to be a resolution of acyclic objects.

Definition 2. The nth algebraic de Rham cohomology of a smooth variety X over a field K is

$$H^n_{dR}(X/K) \coloneqq \mathbb{H}^n(X, \Omega^\bullet_{X/K}) \coloneqq R^n\Gamma(X, \cdot)(\Omega^\bullet_{X/K}).$$

In general, if $\pi \colon X \to Y$ is a smooth morphism of sheaves, we can consider the nth relative de Rham cohomolgoy as

$$\mathscr{H}^n_{dR}(X/Y) \coloneqq R^n\pi_*(\Omega^\bullet_{X/Y}).$$

Relative de Rham cohomology looks like a family of de Rham cohomlogy for each fiber. Also, quasi-coherent sheave are acyclic over affine schemes and so if X is affine, then hypercohomology is just the usual sheaf cohomology.

2 Spectral Sequences

Spectral sequences are often used to calculate de Rham cohomology or hypercohomology in general. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a totally ordered affine open covering on X. For each finite subset $I \subset \Lambda$, let

$$U_I := \bigcap_{\lambda\in I} U_i$$

and for each quasicoherent sheaf $\mathcal{F}$ on X, set

$$\mathcal{C}^n(\mathcal{F}) = \prod_{|I|=n+1} \mathcal{F}(U_I).$$

We have coboundary maps $d^n\colon \mathcal{C}^n(\mathcal{F}) \to \mathcal{C}^{n+1}(\mathcal{F})$ by mapping

$$d^n(x_I)_{I'} = \sum_{i=0}^{n+1}(-1)^j x_{I'\setminus\{i_j\}}, I' = i_1 < i_2 < \cdots < i_{n+1}.$$

Given a complex $\mathcal{F}^\bullet$, we naturally get complexes $\mathcal{C}^n(\mathcal{F}^\bullet)$. And, we get a spectral sequence

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{C}^1(\mathcal{F}^0) & \longrightarrow & \mathcal{C}^1(\mathcal{F}^1) & \longrightarrow & \mathcal{C}^1(\mathcal{F}^2) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathcal{C}^0(\mathcal{F}^0) & \longrightarrow & \mathcal{C}^0(\mathcal{F}^1) & \longrightarrow & \mathcal{C}^0(\mathcal{F}^2) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{F}^0 & \longrightarrow & \mathcal{F}^1 & \longrightarrow & \mathcal{F}^2 & \longrightarrow & \cdots
\end{array}$$

From this sequence, we can define the total complex $T^\bullet$ by

$$T^n := \bigoplus_{i+j=n} \mathcal{C}^i(\mathcal{F}^j),$$

and the coboundary maps by

$$d_T^n := \sum_{i+j=n} d_{hor}^{i,j} + (-1)^i d_{ver}^{i,j}.$$

Then, T^n is an acyclic resolution of $\mathcal{F}^\bullet$ and therefore

$$\mathbb{H}^n(\mathcal{F}^\bullet) = H^n(\Gamma(T^\bullet)).$$

Example 3. When $X = E$ is an elliptic curve, we can compute $H^n_{dR}(E/K)$ from this spectral sequence. Let

$$E\colon Y^2Z = X(X-Z)(X-\lambda Z).$$

Then, we can take the open covering $U = D_Z, V = D_Y$. Then, we have the spectral sequence

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_E(U\cap V) & \longrightarrow & \Omega^1_E(U\cap V) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_E(U)\times\mathcal{O}_E(V) & \longrightarrow & \Omega^1_E(U)\Omega^1_E(V) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_E(E) & \longrightarrow & \Omega^1_E(E) & \longrightarrow & 0
\end{array}$$

Then, we can calculate

$$H^1_{dR}(E/K) = \frac{\{(f,\alpha,\beta)\in\mathcal{O}_E(U\cap V)\times\Omega^1_E(U)\times\Omega^1(V) : df-\alpha-\beta=0\}}{\{(g-h,dg,dh)\}}.$$

Then, Cech cohomology gives the short exact sequence

$$0\to H^0(E,\Omega^1_E)\to H^1_{dR}(E/K)\to H^1(E,\mathcal{O}_E)\to 0$$

which is the Hodge filtration of E. A basis is given by $(0, dx/y, dx/y)$ and $(2y/x, xdx/y, xdx/y)$.

Spectral sequences allow us to calculate hypercohomology or these sequences.

Definition 4. A spectral sequence is a pair of pages a connecting morphisms (E_r, d_r) so that $E_r = \bigoplus_{i,j\geq 0} E_r^{i,j}$ is a lattice of objects and the connecting morphisms d_r is a set of maps $d_r^{i,j}\colon E_r^{i,j}\to E_r^{i+r,j-r+1}$ and $d_r\circ d_r = 0$. Moreover, the relation between the pages is that

$$E^{i,j}_{r+1} = \ker(d_r^{i,j})/im(d_r^{i-r,j+r-1}) = H^{i,j}(E_r).$$

The most common example of spectral sequences is by taking filtrations.

Theorem 5. *Let* $\mathrm{Fil}^\bullet$ *be a bounded decreasing filtration on a complex* $C^\bullet$*, so we have* $C^\bullet = \mathrm{Fil}^0 C^\bullet\supset\mathrm{Fil}^1 C^\bullet\supset\cdots\supset\mathrm{Fil}^{n+1}C^\bullet = 0$. *Define a spectral sequence* (E_r, d_r) *with* 0*th page given by*

$$E_0^{i,j} = \mathrm{Gr}^i C^{i+j} = \mathrm{Fil}^i C^{i+j}/\mathrm{Fil}^{i+1}C^{i+j}.$$

The 1*st page is given by*

$$E_1^{i,j} = H^{i+j}(\mathrm{Gr}^i C^\bullet).$$

Then, the spectral sequence converges to

$$E_\infty^{i,j} = \mathrm{Gr}^i H^{i,j}(C^\bullet),$$

where $\mathrm{Fil}^i H^n(C^\bullet) = Im(H^n(\mathrm{Fil}^i C^\bullet)\to H^n(C^\bullet)$.

Example 6. We saw that $T^\bullet$ gives an acyclic resolution of $\mathcal{F}^\bullet$. Taking $\mathcal{F}^\bullet$ to be the de Rham complex $\Omega_X^\bullet$, we can define a filtration on T^n by

$$\operatorname{Fil}^i T^n := \bigoplus_{i+j=n i\geq k} \mathcal{C}^i(\Omega_X^j).$$

The filtration on $H^n(\Gamma(T^\bullet)) = H^n_{dR}(X/K)$ is the Hodge filtration. The corresponding spectral sequence is called the Hodge–de Rham spectral sequence

$$E_1^{i,j} = R^j\Gamma(\Omega_X^i).$$

3 Gauss–Manin Connection

Definition 7. Recall that a connection on a quasicoherent sheaf $\mathcal{E}$ on a smooth scheme S is a homomorphism

$$\nabla\colon \mathcal{E} \to \Omega^1_{S/K} \otimes \mathcal{E}$$

such that $\nabla(f\sigma) = f\nabla(\sigma) + df \otimes \sigma$.

We extend this map to

$$\nabla_i\colon \Omega_S^i \otimes \mathcal{E} \to \Omega_S^{i+1} \otimes \mathcal{E}$$

by $\nabla_i(\omega \otimes e) = d\omega \otimes e + (-1)^i \omega \wedge \nabla(e)$.

A connection is integrable or flat if $\nabla_1 \circ \nabla\colon \mathcal{E} \to \Omega_S^2 \otimes \mathcal{E}$ is 0. A section of $\mathcal{E}$ is a horizontal section if $\nabla(e) = 0$.

Note that if we have a flat connection ∇ on $\mathcal{E}$, then we get a complex

$$0 \to \mathcal{E} \to \Omega_S^1 \otimes \mathcal{E} \to \Omega_S^2 \otimes \mathcal{E} \to \cdots$$

Now for a smooth morphism of smooth schemes $\pi\colon X \to Y$, the Gauss–Manin connection is a canonical integrable connection on $\mathscr{H}^n_{dR}(X/Y)$. Since π is smooth, we have the exact sequence

$$0 \to \pi^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0.$$

On this sequence, we have the Koszul filtration from last week

$$\operatorname{Fil}^i \Omega^j_{X/S} = \pi^*(\Omega^i_{Y/S}) \otimes \Omega^{j-i}_{X/S} to \Omega^j_{X/S}.$$

Then

$$\operatorname{Gr}^i \Omega^j_{X/S} \cong \pi^*(\Omega^i_{Y/S}) \otimes \Omega^{j-i}_{X/Y}.$$

Choosing a filtered injective (acyclic) resolution $I^\bullet$ of $\Omega^\bullet_{X/S}$ that is compatible with this Koszul filtration, we can consider the relative hypercohomology of $\operatorname{Fil}^\bullet I^\bullet$. Then, we have that

$$E_1^{i,j} = R^{i+j}\pi_*(\operatorname{Gr}^i \Omega^\bullet_{X/S}) \cong R^{i+j}\pi_*(\pi^*(\Omega^i_{Y/S}) \otimes \Omega^{\bullet-i}_{X/Y}) \cong \Omega^i_{Y/S} \otimes R^j\pi_*(\Omega^\bullet_{X/Y}) = \Omega^i_{Y/S} \otimes \mathscr{H}^i_{dR}(X/Y).$$

The Gauss–Manin connection is the composition

$$\nabla\colon \mathscr{H}^n_{dR}(X/Y) \cong E_1^{0,n} \to E_1^{1,n} \cong \Omega^1_{Y/S} \otimes \mathscr{H}^n_{dR}(X/Y).$$

The maps $d_1^{i,j}$ are simply the connecting homomorphisms of the long exact sequence gotten by applying $R^j\pi_*$ to

$$0 \to \operatorname{Gr}^{i+1} \Omega^\bullet_{X/S} \to \operatorname{Fil}^i \Omega^\bullet_{X/S} / \operatorname{Fil}^{i+2} \Omega^\bullet_{X/S} \to \operatorname{Gr}^i \Omega^\bullet_{X/S} \to 0.$$