Gauss–Manin Connection and Complex Period Morphism

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1 Riemann–Hilbert Correspondence

We begin by describing local systems, monodromy, and connections. Let B be a topological space which is path-connected so it has a universal cover.

Definition 1. A complex local system on B is a sheaf of complex vector spaces on B.

Remark. Given a local system $\mathbb{V}$ on B and if B is a complex manifold, then $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_B$ is a vector bundle on B. Local systems are also known as locally constant sheafs because locally they are the constant functions. The setting we care about is with respect to differential equations. If we have a homogenous system of n linear first order differential equations, then the sheaf of holomorphic global solutions form a local system.

Proposition 2. *If B is simply connected, then any complex local system on B is globally constant.*

Proof. Fix a point $b_0 \in B$ and for any other point $b \in B$, there is a path $\gamma\colon [0,1] \to B$ such that $\gamma(0) = b_0, \gamma(1) = b$. Then since $[0,1]$ is compact, every element of the fiber can be uniquely extended to a global section on $[0,1]$ which gives an isomorphism between $\mathbb{V}_{b_0}$ and $\mathbb{V}_b$. Note that we are using that on a constant sheaf $\mathbb{W}$, there are canonical isomorphisms of global sections $\Gamma(\mathbb{W})$ with $\mathbb{W}_x$ any fiber. For line bundles, given a path, we don't get a canonical isomorphism of fibers.

Given two homotopic paths γ_1, γ_2 from b_0 to b, we can cover the square $[0,1]^2$ deforming the paths and see that there is a unique way to extend fibers to global sections. This shows that the isomorphism $\mathbb{V}_{b_0} \to \mathbb{V}_b$ of fibers depends on the path from b_0 to b up to homotopy. Since B is simply connected, there is a single isomorphism from and therefore global sections are globally constant. $\square$

Remark. Note that this is different from line bundles. For instance, the space $\mathbb{P}^n$ is simply connected topologically but there are many nontrivial line bundles. Also, given any line bundle or vector bundle, we can take the constant functions to recover a locally constant sheaf. However, this depends on the particular trivialization and transition maps of the vector bundle and hence is not canonical.

Given a basepoint $b \in B$, we know that any loop around b that is homotopic to the constant map is constant and any two homotopic maps give the same endpoints. Thus, we get a well defined map

$$\pi_1(B, b) \to \mathrm{GL}(\mathbb{V}_b)$$

known as the monodromy representation.

Theorem 3. *The functor from local systems on B to representations of the fundamental group $\pi_1(B, b)$ given by the monodromy representation is an equivalence of categories.*

Proof. We gave the map from local systems to monodromy representations. For the reverse map, if we have a representation (ρ, V) of $\pi_1(B, b_0)$, then set

$$\mathbb{V}(U) = \left\{ \begin{array}{c|l} \text{functions} & \text{for any path } \gamma\colon [0,1] \to U, \text{ we have} \\ k\colon U \to V & k(\gamma(1)) = [\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)}] \cdot k(\gamma(0)) \end{array} \right\},$$

where α_b is a fixed path from b_0 to b. Whenever U is simply connected, then $\mathbb{V}(U)$ is constant and isomorphic to V. The map is given by fixing a point $b \in U$ and mapping to the fibers. It is also straightforward to check that this is the inverse to the local system to monodromy map from above. □

Now we suppose B is actually a complex manifold and so we can talk about its sheaf of germs of functions $\mathcal{O}_B$.

Definition 4. A vector bundle with connection (E, ∇) on B is a holomorphic vector bundle $E \to B$ on B with a $\mathbb{C}$-linear map

$$\nabla\colon E \to E \otimes_{\mathcal{O}_B} \Omega^1_{B/\mathbb{C}}$$

that satisfies the Leibniz rule

$$\nabla(f\sigma) = f\nabla(\sigma) + \sigma \otimes df$$

for $f \in \mathcal{O}_B(U)$ and $\sigma \in E(U)$.

The curvature is defined as $\nabla^2\colon E \to E\otimes\Omega^1_B\otimes\Omega^1_B \to E\otimes\Omega^2_B$, where $\Omega^2_B \coloneqq \wedge^2\Omega^1_B$. A connection is flat or integrable if it has curvature 0.

If D is a derivation (section of the tangent bundle $T_B \coloneqq \Omega^{1,\vee}_B$), then we can define $\nabla_D \coloneqq E \to E$ by $D \circ \nabla$. Then equivalently, a connection is flat if

$$\nabla_{D_1D_2 - D_2D_1} = \nabla_{D_1} \circ \nabla_{D_2} - \nabla_{D_2} \circ \nabla_{D_1}.$$

Example 5. Taking $E = \mathcal{O}_B$ and $\nabla \coloneqq d\colon \mathcal{O}_B \to \Omega^1_B$ is a vector bundle with connection. It is flat because $d^2 = 0$.

Another example is if B is an abelian variety of dimension g. Then $\Omega^1_B \cong \mathcal{O}^g_B$ is free of rank g with generators dz_i. Let $M \in \mathrm{End}(\Gamma(\Omega^1_B)) \cong M_n(\mathbb{C})$ be a matrix and then we can define a connection on $\mathcal{O}^n_B$ by

$$\nabla(f_1, \dots, f_n) = (df_1, \dots, df_n) + M \cdot f.$$

Then, the curvature is $\wedge^2 M$ a $n \times n$ matrix with ijth term $\sum_k m_i k \wedge m_{kj}$. So in general, it is not integrable.

Theorem 6 (Riemann–Hilbert)**.** *There is a bijection between local systems and vector bundles equipped with a flat connection.*

Proof. Given a local system $\mathbb{V}$, we have the vector bundle $\mathbb{V} \otimes \mathcal{O}_B$ and we can define a connection on it by acting as the identity on $\mathbb{V}$ and the derivation map $d\colon \mathcal{O}_B \to \Omega^1_B$. This is flat.

For the reverse, we map $(\mathbb{V}, \nabla)$ to the sheaf of flat sections, those sections v such that $\nabla v = 0$. Roughly, a connection gives you a way to compare the fiber of E at two different points depending on the path taken. You do this by integration. The fact that the connection is flat or integrable means that this isomorphism of fibers is invariant under homotopy of the path. This gives you a locally constant sheaf. □

Remark. These connections are actually linear differential equations. Given two connections ∇_1, ∇_2, then they differ by a function linear map because one can compute

$$(\nabla_1 - \nabla_2)(fe) = f(\nabla_1 - \nabla_2)(e).$$

So if $E = \mathcal{O}_B^n$ in which case we can take $\nabla = d$, any other connection ∇ on E can be written as

$$\nabla = d + M, M \in M_n(\Omega_B^1).$$

Setting $B = \operatorname{Spec} C(x)$, we have that

$$\nabla_{d/dx} = \frac{d}{dx} + M, M \in M_n(\mathbb{C}).$$

If we have $e \in E$ so that $e, \nabla_{d/dx}(e), \ldots, \nabla_{d/dx}^{n-1}(e)$ is a basis for E, then

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{pmatrix}$$

Now if A is a $\mathbb{C}(x)$ algebra so that the derivation $\frac{d}{dx}$ extends to it, then elements of $f \in A$ satisfying

$$\frac{d^n f}{dx^n} + b_{n-1}\frac{d^{n-1} f}{dx^{n-1}} + \cdots + b_0 f = 0$$

are in bijection with $\mathbb{C}(x)$-morphisms $E \to A$ that are compatible with the connection $\nabla_{d/dx}$ on E and $\frac{d}{dx}$ on A.

2 Gauss–Manin Connection

Let $f\colon X \to S$ be a smooth proper map of manifolds. Then, we have the sheaf of relative 1 forms

$$\Omega_{X/S}^1 := \Omega_X^1 / f^* \Omega_S^1.$$

It is a locally free sheaf of dimension the dimension of the fibers of X/S.

Definition 7. The deRham complex of X over S is the complex $(\Omega_{X/S}^\bullet, d^\bullet)$ given by

$$0 \to \underline{\mathbb{R}} \to \mathcal{O}_X \xrightarrow{d} \Omega_{X/S}^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{X/S}^n.$$

When $S = \operatorname{Spec}(\mathbb{R})$ and X a smooth real manifold, the Poincaré lemma says that this gives a resolution of the constant sheaf $\underline{\mathbb{R}}$. This means that the Betti cohomology $H^*(X, \mathbb{R})$ can be identified with the cohomology of this complex, and with the algebraic cohomology of the complex of Zariski sheaves of differential forms, which we will talk about next week. The same holds for complex cohomology when we replace $\underline{\mathbb{R}}$ with $\underline{\mathbb{C}}$. For complex manifolds, there is also a Hodge filtration.

When S is not a point, $\Omega^\bullet_{X/S}$ is a resultion of $f^{-1}\mathcal{O}_S$ and so it follows that fiberwise relative de Rham cohomology is

$$R^\bullet f_*(\Omega^\bullet_{X/S}) = R^\bullet f_*(\underline{\mathbb{C}}) \otimes \mathcal{O}_S.$$

One can use topology (Ehresmann's theorem and proper base change theorem) to show that the sheaf $R^\bullet f_*\underline{\mathbb{C}}$ is a local system on S. The expression on the right has a flat connection given by $1 \otimes d$. This connection ∇ on $H^\bullet_{dR}(X/S)$ is the Gauss–Manin connection.

We also have the Hodge filtration on $\Omega^\bullet_X$ given by

$$\operatorname{Fil}^p \Omega^q_X := Im(f^*\Omega^p_S) \otimes \Omega^{q-p}_X \to \Omega^q_X.$$

Then from the short exact sequence

$$0 \to \operatorname{Fil}^1 / \operatorname{Fil}^2 \to \operatorname{Fil}^0 / \operatorname{Fil}^2 \to \operatorname{Fil}^0 / \operatorname{Fil}^1 \to 0,$$

we get the short exact sequence

$$0 \to (\Omega^{i-1}_{X/S}) \otimes \Omega^1_S \to \Omega^i_X / \operatorname{Fil}^2 \Omega^i_X \to \Omega^i_{X/S} \to 0.$$

It turns out that the boundary map after taking cohomology gives the Gauss–Manin connection as well

$$\nabla \colon R^\bullet f_* \Omega^\bullet_{X/S} \to R^\bullet f_* \Omega^\bullet_{X/S} \otimes \Omega^1_S.$$

This will be how we define it algebraically without having to deal with topology.

3 Hodge Filtration

First we talk about Hodge theory, which was originally about finding explicit cohomology classes on Riemannian manifolds using the Dolbeault or de Rham complex.

Definition 8. Suppose X is a compact complex manifold with Hermitian metric h. It is Kähler if the real 2-form $\operatorname{Imag}(h)$ is closed, and its Kähler cohomomology class is the class in $H^2(X, \mathbb{R})$ associated to $\operatorname{Imag}(h)$.

Example 9. The Fubini–Study metric on projective space is given by

$$ds^2 = \sum_{i=1}^{n} \frac{dz_i d\overline{z}_i}{1 + |z|^2} - \sum_{1 \le i,j, \le n} \frac{z_i \overline{z_j} dz_j d\overline{z}_i}{(1 + |z|^2)^2}.$$

Its associated Kähler form is $\frac{i}{2}\partial\overline{\partial}\log|z|^2$.

Theorem 10. *Suppose X is a compact complex manifold admitting a Kähler matric. Then $\mathbb{C}^\times$ acts naturally on the cohomology on X and induces a splitting*

$$H^n(X, \mathbb{C}) \cong \oplus_{p+q=n} H^{p,q}(X),$$

where the action of z on $H^{p,q}$ is given by multiplication by $z^{-p}\overline{z}^{-q}$. Furthermore, the action respects the natural $\mathbb{R}$-structure so that $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

There is also a natural identification $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$*.*

We can also define a decreasing filtration

$$\operatorname{Fil}^p H^n(X, \mathbb{C}) := \oplus_{r \geq p} H^{r,n-r}(X),$$

and doing so, we get that $\operatorname{Fil}^p H^n(X, \mathbb{C})$ *can be identified with the Hodge filtration from above*

$$H^n(X, \operatorname{Fil}^p \Omega_{X/\mathbb{C}}) \to H^n(X, \Omega_X) \cong H^n(X, \mathbb{C}).$$

Now if we have a family $f \colon X \to S$ with X a Kähler complex manifold and assume that S is connected. Fix an n and $p \leq n$. For a point $s \in S$, let $s^{p,n} := \dim_{\mathbb{C}} \operatorname{Fil}^p H^n(X_s)$. It turns out that these Hodge numbers are constant in S if S is small enough.

Definition 11. The period morphism is

$$p^{p,n} \colon S \to Gr(s^{p,n}, H^n(X_s)).$$

The idea is that the Gauss–Manin connection allows us to identify the fibers $H^n(X_s)$ for different points s and once we fix an identification, different points correpond to different filtrations on $H^n(X_s)$ for some basepoint s. So, the period mappings encode the variations of such flags. In general, it cannot be any filtration. It must satisfy Griffiths transversality.

Proposition 12. *Let* ∇ *be the Gauss–Manin connection on* $\mathcal{H}_{dR}(X)$*. Then*

$$\nabla(\operatorname{Fil}^p(\mathcal{H}^\bullet_{dR}(X))) \subset \operatorname{Fil}^{p-1}(\mathcal{H}^\bullet_{dR}(X)) \otimes \Omega^1_S.$$

In terms of the period morphism, the differential of the period map of $\operatorname{Fil}^p H^n_{dR}(X)$ *in* $H^n_{dR}(X_s)$ *lands inside of*

$$\operatorname{Hom}(\operatorname{Fil}^p, \operatorname{Fil}^{p-1} / \operatorname{Fil}^p) \subset \operatorname{Hom}(\operatorname{Fil}^p, H^n(X_s)/ \operatorname{Fil}^p).$$