Proofs of the Mordell Conjecture

Roy Zhao

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We will first give an overview of Faltings' proof of the Mordell Conjecture and then see which parts were replaced by the new methods of Lawrence and Venkatesh. First, we recall the statement of the Mordell Conjecture.

Theorem 1. *Suppose X/K is an algebraic curve of genus $g \geq 2$. Then, the set $X(K)$ is finite.*

1 Faltings' Proof

Faltings proved the Mordell Conjecture through a series of reductions, which we go through here. The first is the Shafarevich Conjecture for curves.

Theorem 2. *There exist only finitely many smooth, projective curves defined over K of genus g and with good reduction outside a finite set of places S.*

Proof. This reduction uses Parshin's trick, which will be relevant later. Given X/K a curve and $P \in X(K)$ a point, Parshin constructs a finite cover $X_P \to X$ defined over a finite extension K_1/K, depending only on X, that is ramified precisely at P. The genus and primes of bad reductions set S again depends only on X and is independent of the point P. This is done by fixing some mapping of $X \to Jac(X)$ and then taking X' to be $[2]^*X$, the pullback of X under the map $[2] : Jac(X) \to Jac(X)$. Choosing S to include the prime 2 and enlarging K_1 so that $Jac(X)[2]$ is defined over K_1, we get that X'/X is a Galois cover outside S defined over K_1. Let $D = [2]^{-1}(P)$ and let $P' \in D$ be some point. Now consider the generalized Jacobian of X' with respect to the divisor $D - P'$ on X_1. This is like the ray class field, we take Pic^0 over all line bundles that are larger than $D - P'$ (functions that vanish on $D - P'$). We can embed $X' - (D - P')$ into this generalized Jacobian $Jac(X', D - P')$ and again pull back by [2] to get our desired curve X_P, which is smooth over X_1 away from $D - P'$. Mapping down to our original curve X gives a curve that is branched only at P.

Thus, we get a map

$$X(K) \to M_{g'}(K', S')$$

to curves of bounded genus over some fixed number field K', with good reduction outside a finite set of places S' of K'. The latter set is bounded. Moreover, this map is finite to one due to De Franchis' theorem which states that if X, Y are curves and the genus of Y is ≥ 2, then $\mathrm{Hom}_K(X, Y)$ is finite. □

The second simple step is to pass to the weak Shafarevich conjecture over abelian varieties by taking Jacobians of the aforementioned curves.

Theorem 3. *There exist finitely many abelian varieties over K of dimension g, good reduction outside S.*

Proof. Torelli's theorem says that by taking Jacobians, the map from curves to abelian varieties with a fixed choice of principal polarization is injective. The map to just the set of abelian varieties is finite to one because a given abelian variety has only finitely many principal polarizations. □

The third step is to use the Faltings' height to reduce the question about isomorphism equivalence to isogeny equivalence.

Theorem 4. *There exist finitely many abelian varieties over K of dimension g, good reduction outside S, up to isogeny.*

Proof. This follows from showing that there are finitely many isomorphism classes of abelian varieties over K in a given K-isogeny class. This will follow from a Northcott property on heights and a comparison of how the height changes from an isogeny. He proves that for a fixed abelian variety A, there exists an $N \in \mathbb{N}$, depending only on the isogeny class of A, such that if $A \to B$ is a K-isogeny of degree coprime to N, then $h(A) = h(B)$. And, he proved that the Faltings height of isogenies dividing a power of N have bounded difference $h(B) - h(A)$. This shows that in an isogeny class of A, the Faltings height is bounded, and by the Northcott property, there are finitely many abelian varieties in an isogeny class. Note that the proof of these theorems rely on computations with group schemes and p-adic Hodge theory (Tate decomposition). □

The last reduction is from isogeny classes to ℓ-adic representations. Let

$$T_\ell(A) := \varprojlim A[\ell^n], V_\ell(A) := T_\ell(A) \otimes \mathbb{Q}_\ell$$

denote the Tate module. Then the Galois group $G_K := Gal(\overline{K}/K)$ acts on V_ℓ, a $\mathbb{Q}_\ell$-vector space of dimension $2g$. Moreover, by studying the natural map

$$\mathrm{Hom}_K(A, B) \otimes \mathbb{Q}_\ell \to \mathrm{Hom}_{G_K}(V_\ell(A), V_\ell(B)),$$

we see that if A, B are K-isogenous, then they give rise to isomorphic G_K-modules. Thus, we get a well defined map from abelian varieties up to isogeny to isomorphism classes of $2g$-dimensional ℓ-adic G_K-representations. The latter space is not finite, but Faltings shows that the image is finite and the map has finite fibers. The latter statement comes from the following theorem.

Theorem 5. *If A, B are abelian varieties defined over K, the natural map*

$$\mathrm{Hom}_K(A, B) \otimes \mathbb{Q}_\ell \to \mathrm{Hom}_{G_K}(V_\ell(A), V_\ell(B))$$

is an isomorphism.

This states that if two abelian varieties have isomorphic Tate modules, then they are isogenous, showing that the map from abelian varieties up to isogeny to Tate modules is not only finite to one, but an injection.

The statement about finiteness of Galois representations comes from two statements. The first is a semisimplicity argument.

Theorem 6. *If A is an abelian variety over K, then $V_\ell(A)$ is a semi-simple representation of G_K.*

Thus, it suffices to consider only semisimple representations. Faltings proves the follwing about those:

Theorem 7. *Let K be a number field and S a finite set of places. There are finitely many isomorphism classes of rational semisimple ℓ-adic representations of G_K of dimension d unramified outside S.*

Proof. This follows from a statement that two representations are isomorphic if and only if the traces of Frobenius agree at a finite set of places. The existence of this finite set of places is by Chebotarev density theorem (choosing primes so that Frob generates G_K) and then the traces are Weil q-integers, of which there are finitely many by Hermite-Minkowski. □

To summarize, Faltings performs the following reductions

$$\{X(K)\} \to \{C/K \text{ with good reduction outside } S\}$$

$$\to \{A \text{ of dimension } g \text{ with good reduction outside } S\}/\{\text{isom}\}$$

$$\to \{A \text{ of dimension } g \text{ with good reduction outside } S\}/\{\text{isog}\}$$

$$\to \{\text{Rational semisimple } \ell\text{-adic representations of dimension } 2g \text{ unramified oustide } S\}.$$

The last set Faltings proves is finite by Chebotarev density theorem and Hermite–Minkowski. This fact will be used also by Lawrence and Venkatesh.

2 Lawrence–Venkatesh's Proof

The proof of Lawrence and Venkatesh is similar to Faltings proof in that it still uses Parshin's trick to reduce to looking at curves with good reduction outside a fixed set of places and then using Faltings result on the finiteness of rational semisimple ℓ-adic representations unramified outside a fixed set of places. The general setting is a smooth projective family $X \to Y$, where Y is a smooth K-variety, and S a finite set of places outside of which Y has a smooth model, and $\mathcal{O}_S$ the ring of S-integers in K. Suppose that we have integral models $\mathcal{X} \to \mathcal{Y}$ over $\mathcal{O}_S$. For every point $y \in \mathcal{Y}(\mathcal{O}_S)$ and p unramified in K and not in S, we get a Galois representation ρ_y of G_K on $H^*_{\text{ét}}(X_{y,\overline{K}}, \mathbb{Q}_p)$.

In the case of Faltings theorem, Y is our curve of genus $g \geq 2$ and X is a family of abelian varieties over Y given by Parshin's trick and then taking the Jacobian. What Faltings shows is that the Galois representation on $H^*_{\text{ét}}(X_{y,\overline{K}}, \mathbb{Q}_p)$ is semisimple for all points y and the representation ρ_y determines X_y up to isogeny.

The method of Lawrence and Venkatesh is weaker in that sense. Instead of looking at ρ_y, they look at $\rho_{y,v}$, the restriction of the representation to G_{K_v} for some place v of K above p. They then show that the map

$$\{Y(K)\} \to \{\rho_{y,v}\}$$

has finite fibers, and $\rho_{y,v}$ is semisimple for all but finitely many $y \in Y(K)$.

For general $X \to Y$, they show that the map

$$\{Y(K)\} \to \{(\rho_y^{ss})_v\}$$

of taking the semisimplification of ρ_y, and then restricting to $G_{K_v} \subset G_K$, is not finite to one, but has fibers that are not Zariski dense (if Y is a curve, this means finite to one). This has applications in showing that $Y(K)$ is not Zariski dense in Y for some class of hypersurfaces.

The reason of restricting to G_{K_v} is that one can use p-adic Hodge theory to determine $\rho_{y,v}$. It says that every restricted representation corresponds to a triple

$$(H_{dR}(X_y/K_v), \mathrm{Fil}^\bullet, \phi),$$

where ϕ is a semilinear Frobenius map and $\mathrm{Fil}^\bullet$ is the Hodge filtration on H_{dR}. The Gauss–Manin connection identifies $H_{dR}(X_z/K_v) \cong H_{dR}(X_y/K_v)$ as $z \in Y(K_v)$ varies in a residue disk around y and under this identification, the only thing that varies is the Hodge filtration. Thus, we get a map

$$Y(K_v) \to Gr(H_{dR}(X_y, K_v), \mathrm{Fil}^\bullet)(K_v)$$

where Gr is a Grassmannian flag variety which identifies the filtration in the total space of $H_{dR}(X_y/K_v)$. This is the p-adic period map. This is an injective map but different filtrations can give rise to isomorphic ϕ-modules. An isomorphism is given by a linear endomorphism of $H_{dR}(X_y/K_v)$ that commutes with ϕ. Thus, to show that the map

$$Y(K) \to \{\rho_{y,v}\}$$

is finite to one, it suffices to show that the intersection of the image of the period map has finite intersection with the orbit of the centralizer $Z(\phi)$ of ϕ acting on $\rho_{y,v}$. This will be done by showing that the orbit of $\rho_{y,v}$ under $Z(\phi)$ is contained in a proper algebraic subvariety of Gr, and that the image of $Y(K_v)$ in the Grassmannian is Zariski dense. Then the intersection points are the zeroes of a nonvanishing K_v-analytic function on a residue disk, and therefore finite. This is the same approach used by Chabauty and Kim.

Thus, we need to show three things:

1. The image of $Y(K_v)$ under the period map is Zariski dense.
2. The orbit of $Z(\phi)$ is proper algebraic subvariety.
3. ρ_y is semisimple for all but finitely many points $Y(K)$.

The first point is done by comparing the p-adic period map to the complex period map. Once this is done, it suffices to show that the image of $Y(\mathbb{C})$ is Zariski dense under the period map, and this can be done by appealing to topology. The action of the fundamental group $\pi_1(Y)$ on the filtration is the monodromy representation. The fact that this monodromy representation is large was previously shown by others (Looijenga, Grunewald et al., Slater and Tshishiku).

For the second point, the Frobenius map ϕ is a semilinear operator on a vector space over an unramified extension L_w over $\mathbb{Q}_p$. If $[L_w : \mathbb{Q}_p]$ is very large, this semilinearity gives a strong bound on the size of the centralizer. In order to guarantee that the size of $L_w = K_v$ is large, remember that K was actually K' and not K. It is a finite extension of our base field determined by the field of definition of Parshin's trick where we pulled back by [2]. But, we can pull back by $[2^n]$ and as we set $n \to \infty$, the degree of K goes to infinity as well. Then, some argument shows that this makes L_w very large.

Finally the last step is very similar to the previous two steps. Suppose that the local representation ρ_y is not semisimple. Then, there is some subrepresentation W of ρ_y and under p-adic Hodge theory, this corresponds to a Frobenius-stable subspace $W_{dR} \subset H_{dR}(X_y/K_v)$. Moreover, by again comparing to the complex period map, the Hodge filtration of W_{dR} must have prescribed dimensions. So, we again get an unlikely intersection problem where the orbit of $Z(\phi)$ is small and so the orbit of the bad W_{dR} is very small and intersects with $Y(K_v)$ only finitely many times.

Outline:

1. **4/19 The Gauss-Manin Connection and Complex Period Morphism** Define local systems, monodromy, and connections, and state Riemann–Hilbert. Give the definition of de Rham cohomology and the Gauss–Manin connection over $\mathbb{C}$. Give the construction of the period map and period domain over $\mathbb{C}$ for K ahler Manifolds as well. This should follow [Con] and [Lit].

2. **4/26 Algebraic de Rham cohomology** Define hypercohomology and algebraic de Rham cohomology. Show that it agrees with the classical de Rham cohomology. This is the first two pages of [Gro66]. Give the algebraic definition of the Gauss–Manin connection following the first section of [KO68].

3. **5/3 Crystalline Cohomology** Define the crystalline site as well as crystals and crystalline cohomology. Follow the first two sections (I.1, I.2) of [CL98].

4. **5/10 p-adic Hodge Theory** Review the theory of local fields and their ℓ-adic Galois representations. State the basic theorems with Fontaine's period rings and if time sketch the construction of the period rings. The basic theorems are given in the first two sections of [Ill90] and Section 2 of [Ber04]

5. **5/17 The p-adic Period Morphism** Focusing on the p-adic period mapping, cover Section 3 of [LV20] by going through all the proof. Assume and cite the results of Section 2 of [LV20] as necessary.

6. **5/24 The S-unit Equation** Give the proof of the S-unit equation, that there are finitely many pairs $u, v \in \mathcal{O}_S^\times$ of S-integer units such that $u + v = 1$ following Section 4 of [LV20]. Prove Theorem 4.1 and Lemma 4.2 after stating but no need to prove Lemma 4.3 and 4.4.

7. **5/31 Proof of Faltings' Theorem** Black box the Kodaira-Parshin family and use results about it (e.g. monodromy, simplicity) to prove Faltings' Theorem following Section 5 of [LV20]. State Proposition 5.3 and go through the proof of Theorem 5.4.

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