Zany strings and finicky banjo bridges

(with minimal math)[†]

David Politzer[*]

California Institute of Technology

(Dated: July 9, 2014)

Basic physics offers some clues about why a given banjo bridge's sound depends strongly on its tiniest and most subtle details. The strings present the bridge with sets of independent vibrations, with members of each set having nearly the same frequency. When there are at least two separate vibrations with the same frequency, any interaction between the two can have a substantial impact, no matter how weak that interaction. The bridge provides that interaction, allowing strings to talk to each other (and themselves), swapping energy back and forth during the time of a single pluck, all having a discernible impact on tone. This analysis does not predict anything about real bridges, nor give hints how to make "better" ones. However, it does offer some perspective on real banjo sound and how strings actually wiggle.

[*]politzer@caltech.edu; `http://www.its.caltech.edu/~politzer`; 452-48 Caltech, Pasadena CA 91125;

[†] No equations; those are in Ref. 1, which uses complex numbers and matrices at the level of college physics textbook solutions of damped and coupled oscillators to derive a compete, analytic solution.

I.   INTRODUCTION

Swapping bridges is one of the easiest (and potentially least expensive) hardware changes that can alter a banjo's sound. These days, there are a great number of artisanal or couture bridges available as well as high-end and low-end models. And simple ones are easy to make oneself. However, unlike many other possible choices that can be made in banjo set-up, there is little consensus on bridges.[2] With other items, even where people don't agree on what's desirable, they do mostly agree on the consequence of a given choice. Not so with bridges. And people who are very discerning about their sound often comment that the performance of a bridge depends on which banjo it's on.

The bridge builder actually has a huge number of design decisions to make all within that small object, and builders know that each choice matters. A physicist, trying to understand what the bridge does and how it works, might begin picturing it as a structureless, perfectly rigid blob that is the link between the strings' vibrations and the motion of the head.[4] The next level of approximation might involve locating the bridge at its rightful place, rather than at the center of the head and treating it as an extended object that can rock as well as go up and down. Now the math, even if computer aided, is formidable. And this is before getting to the internal details of the bridge itself.

In the following, I will try to explain why going further with the physics is pretty hopeless. **Accumulating wisdom from trial-and-error is really the only way to design a bridge.**

Part of this story relates to phenomena that are quite visible and pretty easily heard. For example, sometimes a plucked string will vibrate harder, then softer, then harder, then softer, then harder, before finally damping out. There's a warble in the sound. But some strings won't do that. And sometimes there's a similar throbbing in the resonance between a plucked string and some of the undamped ones. And in other cases not.

II.   RESONANCE & BEATS

The key idea is that, whenever there are two things that can oscillate at the same frequency, any small interaction between them, no matter how small, can eventually have a huge effect. Often, this is referred to as resonance.

As a first example, consider a kid on a swing. If you set him in motion, the period of

that motion is very nearly the same whether the amplitude is tiny or substantial.[3] If things start out small but you give a gentle push at just the right time in each cycle, the amplitude will grow steadily until it's pretty large. In the physicist's world of astute approximations, no matter how gentle the push, the large amplitude comes eventually.

But, if the pushes aren't timed right, it doesn't work. Even if the pushes are exactly periodic (i.e., spaced equally in time), the amplitude will not continue to build up. Furthermore, if the push timing is periodic and *almost* right, the swing amplitude will begin to grow but after a while the pushes will get out of synch and tend to damp the swing back down.

If you continue pushing at *exactly* a steady but not quite right rate, a while later you'll be getting back in synch and the swing amplitude will grow again. This is the phenomenon known as beats. And we hear it in music when tuning strings to each other. When the frequencies (or harmonics) aren't exactly the same, we hear louder-softer-louder. The louder-softer rate is precisely the difference in the rates of the two oscillations.

III. ONE STRING AS TWO OSCILLATORS

Imagine a string, as ideal as possible, stretched taut between two fixed points. The simplest vibration it can perform, once it's initially disturbed somehow, is to move back and forth with a frequency determined only by the length, tension, and the density of the string. For this simplest imagined motion, all parts of the string have the same frequency, and they move together, with the amplitude of motion largest at the center. (The shape along the string and the time dependence of this simplest motion are both precisely sine functions, but that detail is irrelevant for the present discussion.) The frequency of motion is the pitch that we hear when the string is plucked, e.g., some number of cycles per second.

That entire motion is to be thought of as one oscillator. But the string has a second oscillator with the same frequency. If the first motion is up and down with respect to the banjo head or sound board, the second motion is the possible side-to-side motion. There is only one string, and it can only do one thing at a time, but we consider the two possible motions as two separate oscillators in the sense that the string can do both at the same time. Furthermore, it can do any combination of the two, and that's the meaning of the idea that there are are two, separate oscillators present. For example, if there are equal amounts of the two that start at the same time, the motion of the string as viewed end-on

is at 45º to the head. If we started with more of one than the other, that angle is somewhat different. And, if they start at different times, the end-on view of the motion would trace out an ellipse.

Admitedly, this two-oscillator picture is very abstract and genuinely conceptually challenging.

IV.   ONE STRING AS MANY OSCILLATORS

In physics and engineering, the potentially complicated motion of a string stretched taut between fixed ends, even if confined to the vertical plane, is often described as a combination of simpler possible motions. Each of these motions has its own frequency. A thin string, not plucked too hard, is well approximated by the simplest mathematical model in which the frequencies of these simple motions are all integer multiples of the lowest one. These are the "harmonics." Typically, only the lowest couple dozen have anything to do with what you hear from a stringed instrument.

Because the string can vibrate in any direction perpendicular to its length, each of these harmonic motions actually counts as two oscillators with the same frequency.

V.   MANY STRINGS AS MANY COUPLED OSCILLATORS

Essentially all stringed instruments are tuned so that the ratios of frequencies of the pitches are ratios of small integers. For example, the ratio of frequencies of a C to the next higher G is 2:3. When you finger a chord, again the ratios of frequencies are ratios of small numbers.

For reference, an octave is 2:1, and a fifth is 3:2. In 4:3, the 4 is the octave of 2. So, in terms of note names, 4:3 is the same as 2:3, which is 3:2 backwards. "Backwards" makes it an interval called a fourth. For example, if 1 is a C, 2 is C up one octave; 3 is a G in that second octave; and 4 is the next C. 5 is the next E, i.e., 5:4 is a major third. 6 is the next G; 7 is a B♭; and 8 is the next C. This works great for C major chords in the key of C, but problems and conflicting viewpoints arise as we go further. For example, controversy surrounds the exact "right" frequency choice for the 7th or B♭. And, as you try to fill in other notes that you'd want for other chords, the harmonies eventually end up sounding

awful — because some frequencies have already been set by their ratio to something else. Actually, there's no single scheme that "works." This is the subject of "temperament," and there's no end of things that could be said.[5]

The import of the ratio thing is that, while each characteristic frequency of a given string is shared by two of its own motions, some of those frequencies also appear on other strings, which have their own oscillators in pairs. So, for any given string frequency, there are often four and sometimes six oscillators with that same frequency in the system with several strings.

The math is easy to work out for the case of two oscillators, but that is enough to give a good picture of what can happen with more.

VI. THE LESSON FOR BRIDGE PERFORMANCE

That's the point. The strings of a banjo are actually many (many dozen) oscillators. They come in sets containing at least two but often four or six members with the same or nearly the same frequency. When a pluck delivers energy to one of these, it starts talking to the others. How that shakes down over the duration of the pluck sound depends sensitively on the exact values of the tiny forces that connect those oscillators.

The only way the vertical motion of a string talks to the horizontal motion is *via* the bridge. The only way one string talks to another is *via* the bridge. And the overwhelmingly dominant way a string's vibration loses energy is *via* the bridge — and most of that energy becomes sound.

Yes, the bridge acts as a filter, selectively transmitting some frequencies better than others. And there are certainly other aspects of bridge performance besides how these equal-frequency oscillators couple, but these couplings are an audible component of the total sound. People find designs and materials that tend to deliver a sound they like. Some of these choices may have simple, physical bases. Total bridge mass is an example. Heavier means less initial volume, greater sustain, and less high frequency power.[6] However, at the level of the fine distinctions some people make, performance outliers within a set of bridges

made using the same design are to be expected because of uncontrollable slight variations in the forces that connect the "oscillators."

All stringed instruments manifest the coupling of vertical to horizontal string motions. Piano is a dramatic example. Most of its notes correspond to three adjacent strings, tuned in unison, and struck simultaneously by a single hammer. Those strings talk to each other in the same way, as described years ago.[7] But no common Western instrument rivals the banjo in the strength of the coupling of different strings' different harmonics. That is a consequence of the banjo's unique bridge and head design.

What follows is a description of the overly simplified simple case of a single string.

VII.   ADD DAMPING

Add damping, and it gets really interesting.

In a good, plucked stringed instrument, once a string is plucked, most of its energy is soon converted into sound. Relative to the string itself, this is termed "damping" — because the string is losing its energy.

VIII.   HISTORICAL INTERLUDE

The simplest possible example of the free, "transient" decay of coupled, damped oscillators is assiduously avoided in all physics courses and textbooks (at least all I've ever heard of). One reason might be because the math gets really hard almost instantly — unless you navigate very carefully and cleverly.

Elementary physics has long been part of the basic education of people going into physical sciences, engineering, and all manner of technological work. With the coupled, damped oscillator decay story absent from that preparation, people have had to figure it out on their own if they encountered a situation where it was relevant. But it's not totally trivial. If they succeeded, if felt like a discovery — even though the necessary concepts and mathematics are well over three hundred years old. But often they failed or didn't even know to try to think about it clearly. Had this material been a part of the elementary syllabus, progress in several fields of science and technology might have come faster.[1] It is possible that this

situation has begun to change in the past ten or fifteen years, but only because of advances and efforts in fluid mechanics and applied mathematics[8] that make essential use of related equations. For the present, physics teachers and their students largely remain ignorant of these matters.

IX.   DAMPED, COUPLED OSCILLATORS

For problems such as these, some people like ball and spring analogs. The minimal number of oscillators that can be coupled is two. There are several different ways that two oscillators could be damped. However, the goal of describing the plucked string suggests one in particular. Furthermore, that is the absolute simplest form that exhibits the generic complications of such systems. Referring back to the string whose damping is actually the production of sound, it is typically the motion perpendicular to the head or soundboard that takes energy out of the string. Call that direction "vertical." So it's the vertical motion that is damped. Horizontal motions lose energy much more slowly, which we'll approximate as no horizontal damping at all. This leaves us with the ball and spring model pictured below in FIG. 1. The motion of the directly damped left mass m represents the vertical motion of the string, and the right mass motion represents the horizontal oscillation of the string. If
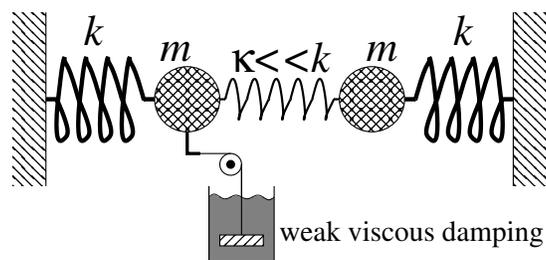

FIG. 1. two oscillator mechanical analog

it weren't for the weak coupling spring, labeled κ, and the weak damper (on just one of the masses), there would be two oscillators with equal frequencies.

The weak coupling spring allows the two equal frequency oscillators to work on each other, much like what's described in Section II. If you start one off with the other at rest, the moving one pushes on the other and gets it going. However, that first one is losing energy to the second. In fact, in the situation as described, the first one will eventually transfer all its energy to the second. Now, it's at rest and the second one is oscillating. But

that's just like the starting point, except that the roles have been reversed. So the second one starts pumping its energy into the first. If it weren't for damping, this back-and-forth dance would continue for ever. The specific damping chosen sucks energy out of one of the oscillators (i.e., corresponding to the vertical motion) and not the other. So string energy will be turned into sound only in as much as the string is oscillating vertically. Its horizontal component, which comes and goes, doesn't produce sound. Hence, the beats.

The rate of the beating depends on the strength of the coupling.

X. MORE DETAILS OF THE EXACT SOLUTION

The weak coupling and damping can be characterized by two numbers. For the coupling, in the absence of damping, we can ask how many times does the first oscillator go through its cycle before all its energy is transferred to the second one. That's a big number, and its inverse characterizes the strength of the coupling. For the damping, in the absence of coupling, we can ask how many cycles does it take for the damped oscillator to lose half of its energy. That's a big number, and its inverse characterizes the strength of the damping.[9]

A simple paper-and-pencil solution of this problem requires that the initial frequencies be almost the same and that the coupling and damping be weak. And when you work it out, there are three qualitatively different kinds of total motion, depending on the ratio of the two small parameters (coupling to damping). If, in this sense, the coupling is stronger than the damping, then the solution is just as described at the end of Section IX. Sound is produced in spurts of decreasing maxima. Note that the direction of the initial pluck is important, too. If the string is plucked vertically, the sound starts right away. If the initial pluck is horizontal, time has to elapse before that horizontal motion pumps up the vertical one.

FIG. 2 shows a graph of the calculated "sound" or energy dissipation rate for a horizontal pluck *versus* time. The sound volume is measured in dB, which is a logarithmic scale. The coupling and damping parameters were chosen to have values that could plausibly occur for a stringed instrument (but without reference to any particular piece of hardware). Those are numbers around 1/100. However, their ratio could be 1/4, 1, or 4 — or something else. In FIG. 2 the ratio of coupling to damping parameters is 2.

Surprisingly (at least it was to me), if the damping is stronger than the coupling (but both

still very weak), there is no pulsing. However, the produced sound volume falls according to what's known as a double exponential. Mathematically, it is the sum of two exponentials, but in practice that means it dies off quickly and then dies off more slowly.
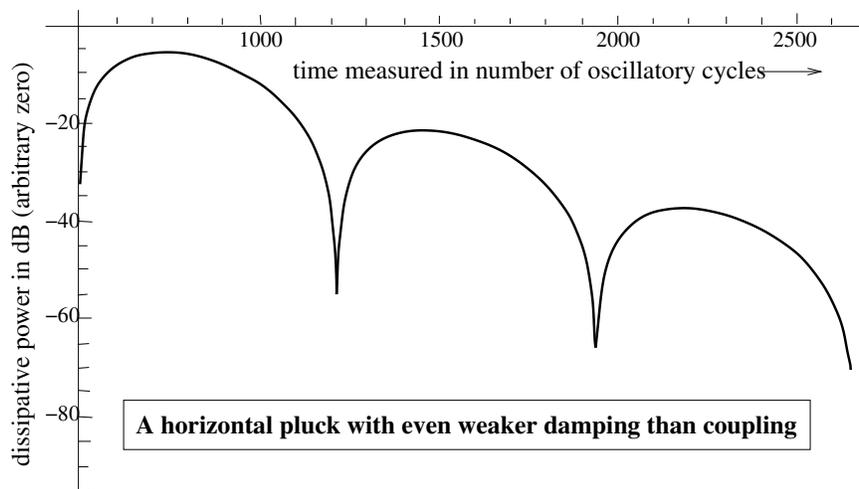

FIG. 2. The calculated "sound" of a particular horizontal pluck.

When the coupling and damping parameters are about equal, the produced sound follows a single exponential: no pulsing and only one exponential decay rate.

You can usually see some of this just by looking at the strings of an instrument. The coupling and damping parameters are likely a bit different for each string and even different for each frequency mode of a given string. With some strings after a pluck, the most visible amplitude of vibrations clearly gets little and big and little and big before damping out. Other strings have motion that decays steadily, while yet others have a rapid decay phase followed by a distinctly slower one.

Exponential decay looks like a straight downward pointing line on a logarithmic plot.

Thom Moore is a professor at Rollins College, Winter Park, Florida, who works with undergraduates to do acoustical science experiments. He and a student, Laurie Stephey are the authors of one of the only papers published in a refereed journal reporting careful banjo sound measurements.[10] They plucked the first string with the others damped, in the vertical and the horizontal direction. And they plucked the second string likewise. They then extracted the lowest three dominant frequency components in each case. Each such component is the sound made by a system of two weakly coupled, damped oscillators with equal initial frequencies. Their results are displayed below in FIG. 3, a figure copied from

their paper.[10]

The possible characteristic behaviors predicted by the analysis discussed above are a straight line (sloping down) with a kink or sharp bend — when there's more damping than coupling; a single line — when the damping and coupling are comparable; and a throbbing whose average follows a single line — when the coupling is more effective than the damping. There are two kinds of such curves in this last category: either the damping starts right away and the sound volume falls or the sound volume increases at first before it soon begins to decrease.

Each of the possible behaviors is represented in the measurements somewhere, even though there's only one bridge. Even more noteworthy is the fact that a single string, crossing the bridge at a given point, can have a variety of the tiny relevant parameters that describe the string-bridge-string interaction — different values and ratios for different harmonics.

It is hard to imagine that there are any bridge design and construction details that do not effect these parameters. And it is even harder to imagine how one might deduce the parameters from design and construction details. Also, the parameters are so small in an absolute, dimensionless sense (i.e., they are pure numbers, the same using centimeters or inches) that they are likely to be effected by tiny variations from bridge to bridge of the same design.

---

[1] David Politzer, "The plucked string: an example of non-normal dynamics," http://arxiv.org/abs/1406.6939; for a slightly better, revised version, see http://www.its.caltech/~politzer.

[2] Browse an on-line banjo forum: http://www.banjohangout.org; in the upper right hand corner, do: SEARCH, choose: Forum, enter: bridge.

[3] Galileo is famous for noting this property of swinging ceiling fixtures when he was in church, not paying attention to the service. Actually, large amplitude swing cycles do take longer; it's noticeable when the angle from the vertical is more than 45º, and the swings take longer and longer as that angle approaches 180º.

[4] Joe Dickey, "The Banjo: the Model Instrument," http://www.joedickey.com/assets/acousticstoday-

manuscript-copy.pdf; and "The structural dynamics of the American five-string banjo," J. Acoust. Soc. Am., **114** (5) 2958 (2003).

[5] Aside for math geeks: the twelfth root of two is irrational, which implies that none of the ratios mentioned (except 2:1) agree exactly with an equal-temperament tuned piano or a normally laid out fretboard. On the other hand, the integers sound better to many people in many circumstances, but you have to make compromises eventually, and you can't "play" in different keys if you've fixed the first few frequencies as small integer ratios.

[6] Those features all follow from a simple differential equation analysis or even a qualitative argument based on $F = ma$. Mike's Banjo Mute$^{\circledR}$ and all other inertial mutes work that way. Mike's mute increases the bridge mass by a factor of 20, which swamps any other conceivable effect; http://www.mikesbanjomute.com.

[7] Gabriel Weinreich, "Coupled piano strings,"J. Acoust. Soc. Am. **62** (6) 1474–1484 (1977).

[8] See Ref. 1 for mention and references to the fluid mechanics, applied math, and other applications.

[9] The complete solution to the (single) damped harmonic oscillator time evolution is reviewed by S. Lewicki, D. Politzer, and D. Priest (1986) here: http://www.its.caltech.edu/~politzer/SHO.mp3; for undamped coupled oscillators, perhaps the most intriguing example where the frequencies are nearly equal is the Wilberforce pendulum. Try Google for videos.

[10] Laurie A. Stephey and Thomas R. Moore, "Experimental investigation of an American five-string banjo," J. Acoust. Soc. Am. **124** (5) 3276 (2008).
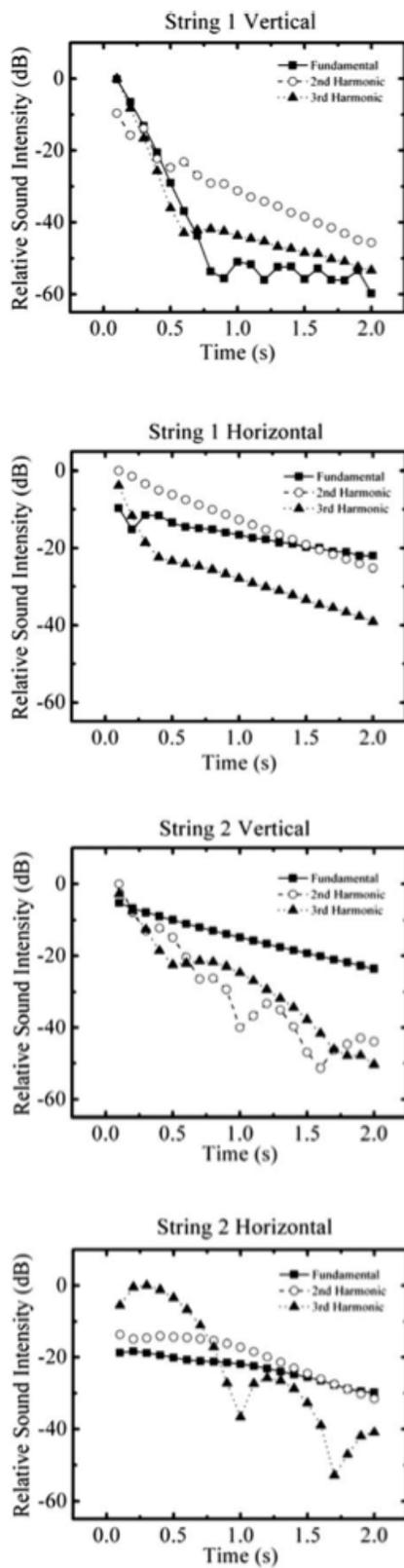
FIG. 3. Measured loudness in dB (log scale) *vs.* time for string harmonics, copied from Ref. 10; twelve curves displaying (roughly) four characteristic, possible behaviors.