Quiz 1 Solutions

/2

1a) From the setup to the problem, we know that we have 6 particles in cell 1 and we have 2 spin up (because $U_1 = 2$).

There are $\binom{6}{2} = \frac{6 \cdot 5}{2} = \boxed{15}$ ways to arrange such a system

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↑↓↑↓↓↓, ↓↑↓↑↓↓, ↓↓↑↓↑↓, ↓↓↓↑↓↑,
↑↓↓↑↓↓, ↓↑↓↓↑↓, ↓↓↑↓↓↑, ↑↓↓↓↑↓, ↓↑↓↓↓↑
↑↓↓↓↓↑

/2 b) Well, the multiplicity of the combined system is

$$g_{tot} = g_1(U_1) \cdot g_2(U_2) = \binom{6}{2} \cdot \binom{6}{4} = \frac{6 \cdot 5}{2} \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3}{4 \cdot 3 \cdot 2} = 15 \cdot 15 = \boxed{225}$$

$$\boxed{g_{tot} = 225} \qquad \boxed{\sigma_{tot} = \ln(g_{tot}) = \ln(225)}$$

/2 c)

Total energy is conserved So $U_1 + U_2 = 6$

U_1	U_2	g_1	g_2	Combined multiplicity
0	6	$\binom{6}{0} = 1$	$\binom{6}{6} = 1$	1
1	5	$\binom{6}{1} = 6$	$\binom{6}{5} = 6$	36
2	4	$\binom{6}{2} = 15$	$\binom{6}{4} = 15$	225
3	3	$\binom{6}{3} = 20$	$\binom{6}{3} = 20$	400
4	2	$\binom{6}{4} = 15$	$\binom{6}{2} = 15$	225
5	1	$\binom{6}{5} = 6$	$\binom{6}{1} = 6$	36
6	0	$\binom{6}{6} = 1$	$\binom{6}{0} = 1$	1

- The most probable configuration is the one w/ the most accessible states. This is apparently $\boxed{U_1 = 3,\ U_2 = 3}$
- This makes sense, since the systems are of the same size, evenly splitting the energy between them is the most likely configuration.

1/2 d) Fundamental entropy: $\boxed{\sigma_{tot} = \ln(g_{tot}) = \ln(400)}$

1/2 e) This part of the problem is a bit tricky. What it basically wants you to show is that:

$$\frac{\binom{600}{290}}{\binom{600}{300}} \overset{?}{\approx} \frac{\binom{6}{2}}{\binom{6}{3}}$$

- Let me explain what I mean by this. We are asked to show that fractional deviations from the most probable value $\sim \frac{1}{\sqrt{N}}$. This was shown in the textbook by approximating the spin system as a gaussian. Here, we do it a bit more explicitly.

- The right hand side looks at an $N=6$ system. It is comparing the probabilities of a fractional deviation of $1/6$ to the maximum value.
- The left hand side looks at a system 100 times larger, $N=600$. It is then comparing the probabilities of a fractional deviation of $10/600 = 1/60$ to the maximum value.
- If these two are approximately equal, we'll see that fractional deviations for a 100 times larger system $1/10$ as large. or fractional deviations $\sim 1/\sqrt{N}$

$$\frac{\frac{\cancel{600!}}{290!\,310!}}{\frac{\cancel{600!}}{300!\,300!}} \overset{?}{\approx} \frac{\frac{\cancel{6!}}{2!\,4!}}{\frac{\cancel{6!}}{3!\,3!}} \Rightarrow \frac{300!\,300!}{290!\,310!} \overset{?}{\approx} \frac{3!\,3!}{2!\,4!} = 0.75$$

$$\frac{300!\,300!}{290!\,310!} = e^{\ln\left(\frac{300!\,300!}{290!\,310!}\right)}$$

approximate $\ln\left(\frac{300!\,300!}{290!\,310!}\right)$ by stirling's approximation

$\Rightarrow \ln(300!) + \ln(300!) - \ln(290!) - \ln(310!)$

Stirling approximation: $\ln(N!) \approx N\ln(N) - N$ for large N

$\Rightarrow 2\cdot 300\ln(300) - \cancel{300\cdot 2} - 290\ln(290) + \cancel{290} - 310\ln(310) + \cancel{310}$

$\Rightarrow 2\cdot 300\ln(300) - 290\ln(290) - 310\ln(310) = -0.3334$

$$e^{-0.3334} = 0.7165 \approx \frac{300!\,300!}{290!\,310!}$$

$\Rightarrow$ $\boxed{0.7165 \overset{?}{\approx} 0.75 \quad \text{Yes, this seems reasonable}}$

2) I am going to employ the following notation: $\uparrow = \varphi_1$, $\downarrow = \psi_2$

/10

	Distinguishable	Bosons	Fermions
	$\uparrow\uparrow$	$\uparrow\uparrow$	
These are distinct states for distinguishable particles [	$\uparrow\downarrow$ $\downarrow\uparrow$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$
	$\downarrow\downarrow$	$\downarrow\downarrow$	
Probability of both particles Being in same state	$\frac{2}{4} = \frac{1}{2}$	$\frac{2}{3}$	$\frac{0}{1} = 0$

Bosons are more likely to be in the same state, than Distinguishable particles or fermions.

The key idea is, that when you consider indistinguishable Bosons you lose unique states from the distinguishable case (i.e. you combine them into one) but you keep the states where all particles are in the same state. Thus, overall, it becomes more likely to be in the same state.