H w 9 Solutions

① $\frac{1}{a\sqrt{\pi}}\int_{-\infty}^{\infty} x e^{-x^2/a^2}\,dx \Rightarrow 0$ b/c x is odd and e^{-x^2} is even.

② $\frac{1}{2a\sqrt{\pi}}\int_{-\infty}^{\infty} x \cdot 4\frac{x^2}{a^2} e^{-x^2/a^2}\,dx \Rightarrow 0$ same reason.

③ $\int_{-\infty}^{\infty} \psi_0 \psi_1 x \cos(\omega t)\,dx = \cos(\omega t)\int_{-\infty}^{\infty} \psi_0\psi_1 x\,dx$

$$= \boxed{A\cos(\omega t)}$$ //

6-9.

begin with

/10

$$\left[q - \frac{d}{dq}\right]\left[q + \frac{d}{dq}\right]\psi_n(q) = (2\epsilon_n - 1)\psi_n(q)$$

hit it with $\left[q + \frac{d}{dq}\right]$

$$\left[q + \frac{d}{dq}\right]\left[q - \frac{d}{dq}\right]\underbrace{\left[q + \frac{d}{dq}\right]\psi_n(q)} = (2\epsilon_n - 1)\underbrace{\left[q + \frac{d}{dq}\right]\psi_n(q)}$$

group these into new function $\psi_n'(q)$

$$\left[q + \frac{d}{dq}\right]\left[q - \frac{d}{dq}\right]\psi_n'(q) = (2\epsilon_n - 1)\psi_n'(x)$$

$$= (2(\epsilon_n - 1) + 1)\psi_n'(x)$$

So $\psi_n'(q)$ is an eigenfunction with eigenvalue $\epsilon_n - 1$

So it must be the $\psi_{n-1}(q)$ state with energy ϵ_{n-1} since the states are quantized with energy $E = \epsilon \hbar\omega$

7.1 S.E becomes

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\,\psi(x)$$

Let $\psi(x)$ satesfy this equation

i) $[\hat{x},\hat{p}]\psi = \hat{x}\hat{p}\psi - \hat{p}\hat{x}\psi$

$$= x(-i\hbar)\frac{d}{dx}\psi(x) + i\hbar\frac{d}{dx}(x\,\psi(x))$$

$$= i\hbar\left[-x\,\psi'(x) + \psi(x) + x\,\psi'(x)\right]$$

$$= i\hbar\,\psi(x)$$

2

$\therefore\ [\hat{x},\hat{p}] = i\hbar$ //

i) $[\hat{x},\hat{H}]\psi = \hat{x}\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x)$

$$-\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)(x\,\psi(x))$$

3

$$-\frac{x\hbar^2}{2m}\psi''(x) + \cancel{xV(x)\psi(x)} + \frac{\hbar^2}{2m}\frac{d^2}{dx^2}(x\psi(x)) - \cancel{V(x)x\psi(x)}$$

$$= -\frac{x\hbar^2}{2m}\psi''(x) + \frac{\hbar^2}{2m}\frac{d}{dx}\left(\psi(x) + x\psi'(x)\right)$$

$$= -\frac{x\hbar^2}{2m}\psi''(x) + \frac{\hbar^2}{2m}\left[\psi'(x) + x\psi''(x) + \psi'(x)\right]$$

$$= \frac{\hbar^2}{2m}\left[\psi'(x) + \psi'(x)\right] = \boxed{\underbrace{\frac{\hbar^2}{m}\frac{d}{dx}}_{[\hat{x},\hat{H}]}\psi(x)} \; //.$$

(iii)

$$[\hat{p},\hat{H}]\psi(x) = \hat{p}\hat{H}\psi(x) - \hat{H}\hat{p}\psi(x)$$

remember apply operator from right to left

$$= \hat{p}\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)\right] - \hat{H}\left[-i\hbar\frac{d}{dx}\psi(x)\right]$$

$$= \hat{p}\left[\frac{-\hbar^2}{2m}\psi''(x) + V(x)\psi(x)\right] - \hat{H}\left[-i\hbar\psi'(x)\right]$$

$$= -i\hbar\left[\frac{-\hbar^2}{2m}\psi'''(x) + V'(x)\psi(x) + V(x)\psi'(x)\right]$$ cancel

3 cancel

$$+ i\hbar\left[\frac{-\hbar^2}{2m}\psi'''(x) + V(x)\psi'(x)\right]$$

$$= \left[+i\hbar V'(x)\right]\psi(x) \quad \text{so} \quad \boxed{[\hat{p},\hat{H}] = +i\hbar V'(x)} \; //$$

1/2

This means that states of <u>definite</u>

(i) position + momentum

(ii) position + energy

(iii) momentum + energy

cannot exist simultaneously.

Ch. 7:
Problem 8.

(a)
$$[\hat{r}\cdot\hat{p}, \hat{T}] = \left((\hat{r}\cdot\hat{p})\hat{T} - \hat{T}(\hat{r}\cdot\hat{p})\right) \quad \left(\text{note: } \hat{r} = r,\ \hat{p} = \frac{\hbar}{i}\nabla,\ \hat{T} = -\frac{\hbar^2}{2m}\nabla^2\right)$$

$$= \left[\left(\frac{\hbar}{i} r\cdot\nabla\right)\left(\frac{-\hbar^2}{2m}\nabla^2\right) - \left(\frac{-\hbar^2}{2m}\nabla^2\right)\left(\frac{\hbar}{i} r\cdot\nabla\right)\right]$$

$$= \frac{-\hbar^3}{2mi}\left[(r\cdot\nabla)\nabla^2 - \nabla^2(r\cdot\nabla)\right] \quad (\mathrm{I})$$

$$\nabla^2(r\cdot\nabla) = \nabla\cdot[\nabla(r\cdot\nabla)] = \nabla\cdot(\nabla + r\nabla^2)$$

$$= \nabla^2 + \nabla^2 + (r\cdot\nabla)\nabla^2 \quad (\mathrm{II})$$

(I) & (II) $[\hat{r}\cdot\hat{p}, \hat{T}] = \frac{-\hbar^3}{2mi}\left[-2\nabla^2\right]$

$$= \frac{\hbar^3}{mi}\nabla^2$$

$$= \frac{i\hbar}{m}\frac{\hbar^2}{i^2}\nabla^2 \qquad \text{note: } \hat{p}^2 = \frac{\hbar^2}{i^2}\nabla^2$$

1/2

$$= \frac{i\hbar}{m}\hat{p}^2$$

$[\hat{r}\cdot\hat{p}, \hat{V}]f = [(\hat{r}\cdot\hat{p})\hat{V} - \hat{V}(\hat{r}\cdot\hat{p})]f$ (note: f is our test function)

$$= \left[\left(r\cdot\frac{\hbar}{i}\nabla\right)\hat{V} - V\left(r\cdot\frac{\hbar}{i}\nabla\right)\right]f$$

$$= \frac{\hbar}{i}\left[(r\cdot\nabla)Vf - V(r\cdot\nabla)f\right] = \frac{\hbar}{i}\left[f(r\cdot\nabla V) + V(r\cdot\nabla f) - V(r\cdot\nabla f)\right]$$

1/2

$$\Rightarrow [\hat{r}\cdot\hat{p}, \hat{V}] = -\hbar i(r\cdot\nabla V) = -\hbar i r\frac{dV}{dr}$$

7-8

(b)

$$\int \psi_E^* [\hat{r}\cdot\hat{p}, \hat{H}] \psi_E \, d^3r = \int \psi_E^* (\hat{r}\cdot\hat{p}\hat{H} - \hat{H}\hat{r}\cdot\hat{p}) \psi_E \, d^3r$$

note that $\hat{H}$ is Hermitian, therefore:

$$\int \psi_E^* [\hat{r}\cdot\hat{p}, \hat{H}] \psi_E \, d^3r = \int \psi_E^* \hat{r}\cdot\hat{p}\hat{H}\psi_E \, d^3r - \int (\hat{H}\psi_E)^* \hat{r}\cdot\hat{p}\psi_E \, d^3r \quad \text{(III)}$$

note that ψ_E is an eigenfunction of $\hat{H}$ w/ a real eigenvalue E:

$$\hat{H}\psi_E = E\psi_E \Rightarrow (\hat{H}\psi_E)^* = E\psi_E^* \quad \text{(IV)}$$

1/2

(III) & (IV):

$$\int \psi_E^* [\hat{r}\cdot\hat{p}, \hat{H}] \psi_E \, d^3r = E\int \psi_E^* \hat{r}\cdot\hat{p}\psi_E \, d^3r - E\int \psi_E^* \hat{r}\cdot\hat{p}\psi_E \, d^3r$$

$$= 0$$

$$\int \psi_E^* [\hat{r}\cdot\hat{p}, \hat{H}] \psi_E \, d^3r = \int \psi_E^* [\hat{r}\cdot\hat{p}, \hat{T}+\hat{V}] \psi_E \, d^3r$$

$$= \int \psi_E^* [\hat{r}\cdot\hat{p}, \hat{T}] \psi_E \, d^3r + \int \psi_E^* [\hat{r}\cdot\hat{p}, \hat{V}] \psi_E \, d^3r$$

$$= \int \psi_E^* \frac{i\hbar}{m} \hat{p}^2 \psi_E \, d^3r + \int \psi_E^* \left(-i\hbar r \frac{dV}{dr}\right) \psi_E \, d^3r$$

$$= 0$$

(note: $\hat{T} = \frac{\hat{p}^2}{2m} \Rightarrow \hat{p}^2 = 2m\hat{T}$)

Ch 7.

Problem 8.

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(b)

$$\int \psi_E^* \, i\hbar \, 2\hat{T} \psi_E \, d^3r = \int \psi_E^* \, i\hbar \, r \frac{dV}{dr} \psi_E \, d^3r$$

$$\Rightarrow \boxed{2\int \psi_E^* \hat{T} \psi_E \, d^3r = \int \psi_E^* \, r \frac{dV}{dr} \psi_E \, d^3r}$$

(c)

$$\langle T \rangle = \int \psi_E^* \hat{T} \psi_E \, d^3r = \frac{1}{2}\int \psi_E^* \, r \frac{dV}{dr} \psi_E \, d^3r$$

(note: $\frac{dV}{dr} = m\omega^2 r$)

$$= \int \psi_E^* \frac{r}{2} (m\omega^2 r) \psi_E \, d^3r$$

$$= \int \psi_E^* \left(\tfrac{1}{2} m\omega^2 r^2\right) \psi_E \, d^3r$$

$$\langle V \rangle = \int \psi_E^* \hat{V} \psi_E \, d^3r = \int \psi_E^* \left(\tfrac{1}{2} m\omega^2 r^2\right) \psi_E \, d^3r$$

$\Rightarrow$

(note $\hat{V} = V(r)$)

1/2

$$\langle T \rangle = \langle V \rangle$$

7-8

(c)

$$\langle T \rangle = \int \psi_E^* \hat{T} \psi_E \, d^3r = \frac{1}{2} \int \psi_E^* \, r \frac{dV}{dr} \psi_E \, d^3r$$

$$\frac{dV}{dr} = \frac{e^2}{4\pi\varepsilon_0 r^2} \qquad = \frac{1}{2} \int \psi_E^* \frac{e^2}{4\pi\varepsilon_0 r} \psi_E \, d^3r$$

$$\langle V \rangle = \int \psi_E^* \hat{V} \psi_E \, d^3r = \int \psi_E^* \left(\frac{-e^2}{4\pi\varepsilon_0 r} \right) \psi_E \, d^3r$$

$\Rightarrow$

1/2

$$2\langle T \rangle = -\langle V \rangle$$

QP5.

$$\psi(x) = A\left(\frac{x}{x_0}\right)^b \exp(-x/x_0)$$

a) $$1 = \int_0^\infty |\psi(x)|^2 dx = |A|^2 \int_0^\infty \left(\frac{x}{x_0}\right)^{2b} \exp\left(\frac{-2x}{x_0}\right) dx \ominus \qquad u = \frac{x}{x_0} \quad du = \frac{dx}{x_0}$$

$$dx = du \cdot x_0$$

$$\ominus\ |A|^2 \int_0^\infty u^{2b} \exp(-2u)\, x_0\, du = |A|^2 x_0 \int_0^\infty u^{2b} \exp(-2u)\, du \ominus$$

$$\ominus\ |A|^2 x_0 \frac{(2b)!}{2^{2b+1}} = 1 \Rightarrow |A|^2 = \frac{2^{2b+1}}{(2b)!} \cdot \frac{1}{x_0}$$

given integral
$\int_0^\infty x^n \exp(-ax)dx = \frac{n!}{a^{n+1}}$

$$\Rightarrow \boxed{A = \sqrt{\frac{2^{2b+1}}{(2b)!} \cdot \frac{1}{x_0}} \quad \text{assuming } A \text{ is real}}$$

b) $$\langle x^m \rangle = \int_0^\infty \psi^* x^m \psi\, dx = \int_0^\infty |\psi|^2 x^m dx = |A|^2 \int_0^\infty \left(\frac{x}{x_0}\right)^{2b} x^m \exp\left(\frac{-2x}{x_0}\right) dx$$

$$u = \frac{x}{x_0} \quad du = \frac{dx}{x_0} \quad dx = du \cdot x_0 \quad \ominus\ |A|^2 \int_0^\infty u^{2b} x^m \exp(-2u)\, x_0\, du \quad \ominus$$

$$x^m = (x_0 u)^m \qquad \ominus\ |A|^2 \int_0^\infty u^{2b} (x_0 u)^m x_0 \exp(-2u)\, du = |A|^2 x_0^{m+1} \int_0^\infty u^{2b+m} \exp(-2u) du$$

$$\ominus\ |A|^2 x_0^{m+1} \frac{(2b+m)!}{2^{2b+m+1}} = \frac{2^{2b+1}}{(2b)!} \cdot \frac{1}{x_0} \cdot x_0^{m+1} \frac{(2b+m)!}{2^{2b+m+1}} = \boxed{\left(\frac{x_0}{2}\right)^m \cdot \frac{(2b+m)!}{(2b)!} = \langle x^m \rangle}$$

using given integral

substitute for A

c) $$\Delta x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\langle x^2 \rangle = \left(\frac{x_0}{2}\right)^2 \cdot \frac{(2b+2)!}{(2b)!} = \left(\frac{x_0}{2}\right)^2 \cdot \frac{(2b+2)(2b+1)\cancel{(2b)!}}{\cancel{(2b)!}} = \left(\frac{x_0}{2}\right)^2 (2b+2)(2b+1)$$

$$\Rightarrow \langle x \rangle = \frac{x_0}{2}(2b+1) \Rightarrow \langle x \rangle^2 = \left(\frac{x_0}{2}\right)^2 (2b+1)^2$$

$$\langle x^2 \rangle - \langle x \rangle^2 = \left(\frac{x_0}{2}\right)^2 (2b+2)(2b+1) - \left(\frac{x_0}{2}\right)(2b+1)^2$$

$$\ominus \left(\frac{x_0}{2}\right)^2\left(\cancel{4b^2}+2b+\cancel{4b}+2-\cancel{4b^2}-\cancel{2b}-\cancel{3b}-1\right)=\left(\frac{x_0}{2}\right)^2(2b+1)$$

$$\Rightarrow \Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\boxed{\frac{x_0}{2}\sqrt{2b+1}=\Delta x}$$

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d) $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)+V(x)\psi(x)=E\psi(x)$$

10/4

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}A\left(\frac{x}{x_0}\right)^b\exp(-x/x_0)=-\frac{\hbar^2}{2m}A\frac{d}{dx}\left(\frac{b}{x_0^b}x^{b-1}\exp(-x/x_0)+\left(\frac{x}{x_0}\right)^b\left(\frac{-1}{x_0}\right)\exp(-x/x_0)\right)$$

$$\ominus -\frac{\hbar^2}{2m}A\left[\frac{b(b-1)}{x_0^b}x^{b-2}\exp(-x/x_0)+\frac{b}{x_0^b}x^{b-1}\left(\frac{-1}{x_0}\right)\exp(-x/x_0)+\left(\frac{-1}{x_0}\right)\frac{b}{x_0^b}x^{b-1}\exp(-x/x_0)+\left(\frac{x}{x_0}\right)^b\frac{1}{x_0^2}\exp(-x/x_0)\right]$$

$$\ominus -\frac{\hbar^2}{2m}A\left[\frac{b(b-1)}{x_0^b}x^{b-2}\exp(-x/x_0)+\left(\frac{x}{x_0}\right)^b\frac{1}{x_0^2}\exp(-x/x_0)+\frac{-2b}{x_0^{b+1}}x^{b-1}\exp(-x/x_0)\right]$$

$$V(x)=\frac{E\psi(x)+\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}}{\psi(x)}$$

we can divide by $\psi(x)$ of x because we only consider $x>0$

$$V(x)=E+\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\cdot\frac{1}{\psi(x)}=E+\frac{\hbar^2}{2m}\frac{\cancel{A}\left[\frac{b(b-1)}{x_0^b}x^{b-2}\cancel{\exp(-x/x_0)}+\left(\frac{x}{x_0}\right)^b\frac{1}{x_0^2}\cancel{\exp(-x/x_0)}-\frac{2b}{x_0^{b+1}}x^{b-1}\cancel{\exp(-x/x_0)}\right]}{\cancel{A}\left(\frac{x}{x_0}\right)^b\cancel{\exp(-x/x_0)}}$$

$$=\underline{E+\frac{\hbar^2}{2m}\left[\frac{b(b-1)}{x^2}+\frac{1}{x_0^2}-\frac{2b}{x_0x}\right]=V(x)}$$

$$\lim_{x\to\infty}V(x)=E+\frac{\hbar^2}{2mx_0^2}=0\Rightarrow \underline{E=\frac{-\hbar^2}{2mx_0^2}}$$

$$\Rightarrow V(x)=\frac{-\hbar^2}{2mx_0^2}+\frac{\hbar^2}{2m}\left[\frac{b(b-1)}{x^2}+\frac{1}{x_0^2}-\frac{2b}{x_0x}\right]=\boxed{\frac{\hbar^2}{2m}\left[\frac{b(b-1)}{x^2}-\frac{2b}{x_0x}\right]=V(x)}$$