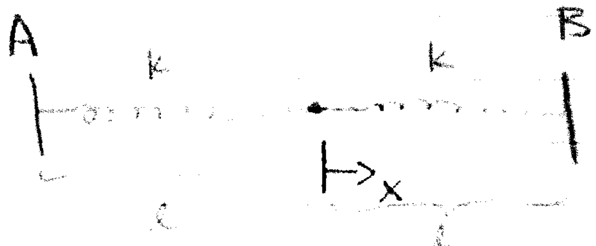
Ph2a - 2011 HW #1 solutions

2

3-19



(a)

for small oscilations in the x-direction we have

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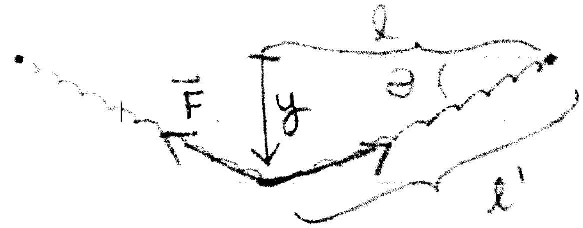
$$m\ddot{x} = -k(l+x-l_0) + k(l-x-l_0)$$
$$= -kl - kx + kl_0 - kl_0 - kx + kl$$
$$= \boxed{-2kx}$$

(b)



where $l^2 + y_1^2 = l'^2$

note $l'\sin(\theta) = y$

$l\tan(\theta) = y$

$\tan(\theta) \sim \sin(\theta)$

$\theta \ll 1 \quad l' \approx l$

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$$\vec{F} = F_x\hat{i} + F_y\hat{j}$$
$$F_x = F\cos(\theta)$$
$$F_y = F\sin(\theta)$$

$$m\ddot{y} = 2F_y = 2F\sin(\theta) = 2F\frac{y}{l}$$

and $F = -k(l' - l_0) \approx -k(l - l_0)$

since $l' \approx l$

$$\boxed{m\ddot{y} = -\frac{2ky(l-l_0)}{l}}$$

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(c) for ocsillation $\omega = \sqrt{\frac{k}{m}}$ where m is the mass and k is what appears in the SHO equation.

1/2

$$m\ddot{x} = -\underline{\underline{k}}x$$

$\therefore$ x: $\omega_x = \sqrt{\frac{2k}{m}}$ $\quad$ y: $\omega_y = \sqrt{\frac{2k(\ell-\ell_0)}{m\ell}}$

$$\boxed{\frac{\omega_x}{\omega_y} = \frac{T_y}{T_x} = \sqrt{\frac{\ell}{\ell-\ell_0}}}$$

(d) In general:

$$x(t) = A_x\sin(\omega_x t) + B_x\cos(\omega_x t)$$
$$y(t) = A_y\sin(\omega_y t) + B_y\cos(\omega_y t)$$

So

$$V_x(t) = A_x\omega_x\cos(\omega_x t) - B_x\omega_x\sin(\omega_x t)$$
$$V_y(t) = A_y\omega_y\cos(\omega_y t) - B_y\omega_y\sin(\omega_y t)$$

1/2

So $A_x = 0$ and $A_y = 0$ b/c @ rest; $t=0$
and since

$x(0) = A_0$ $\quad$ $B_x = A_0$
$y(0) = A_0$ $\quad$ $B_y = A_0$

$$\boxed{\begin{aligned} x(t) &= A_0\cos(\omega_x t) \\ y(t) &= A_0\cos(\omega_y t) \end{aligned}}$$

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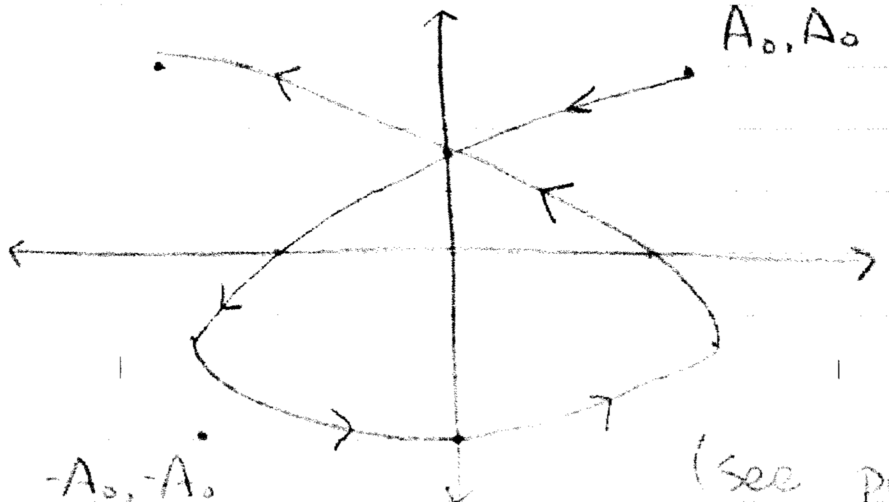
(e) if $\ell = \frac{9}{5}\ell_0$ then $\frac{\omega_x}{\omega_y} = \frac{3}{2}$

$\omega_y = \frac{2}{3}\omega_x$

$x(t) = A_0 \cos(\omega_x t)$

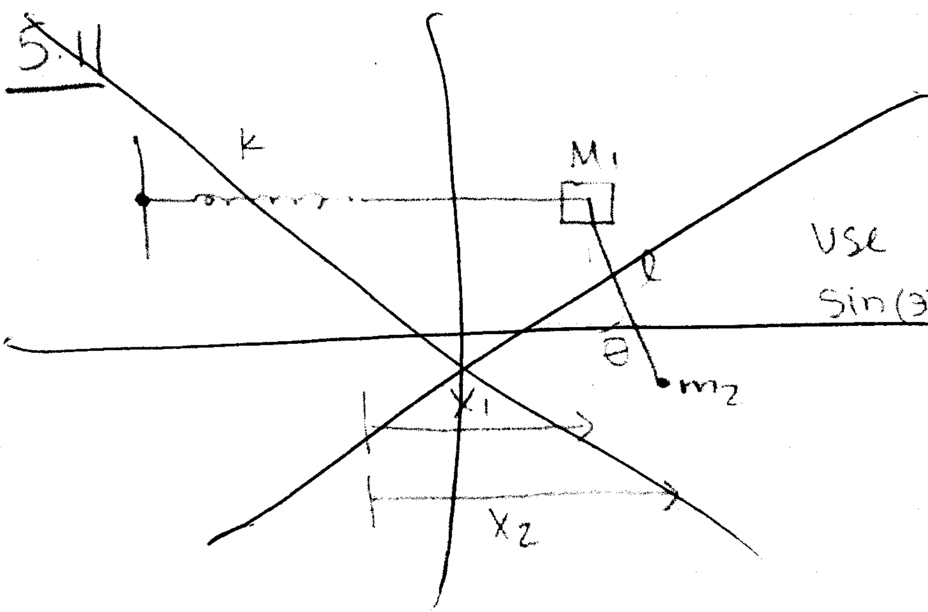
$y(t) = A_0 \cos(\frac{2}{3}\omega_x t)$

$\frac{}{2}$



(see pg 36)

5.11



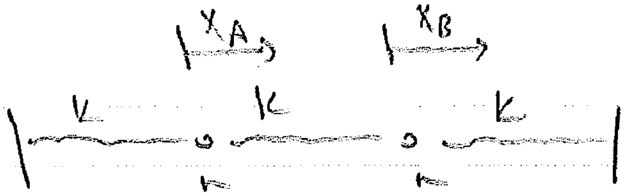
Use

$\sin(\theta) \sim \tan(\theta) \sim \frac{x_2 - x_1}{\ell}$

a) $\overline{4}$ b) $\overline{3}$

6

5-6(a)+(b)



1 fixed
$m\frac{\partial^2 x}{\partial t^2} = -2kx$

$\omega^2 = \frac{2k}{m}$ $\omega = \sqrt{\frac{2k}{m}}$

$T = \frac{2\pi}{\omega} = 3s$

$\omega = \frac{2\pi}{3} = \sqrt{\frac{2k}{m}}$

$\omega_0 = \frac{2\pi}{3\sqrt{2}}$

$$m\frac{d^2x_A}{dt^2} = -kx_A - k(x_A - x_B)$$

$$m\frac{d^2x_B}{dt^2} = -kx_B - k(x_B - x_A)$$

assume $x_A = A e^{i\omega t}$ $x_B = Be^{i\omega t}$

$$-A\omega^2 e^{i\omega t} = -\omega_0^2 Ae^{i\omega t} - \omega_0^2(A-B)e^{i\omega t}$$

$$-B\omega^2 e^{i\omega t} = -\omega_0^2 Be^{i\omega t} - \omega_0^2(B-A)e^{i\omega t}$$

$$-A\omega^2 + 2A\omega_0^2 - B\omega_0^2 = 0$$

$$-B\omega^2 + 2B\omega_0^2 - A\omega_0^2 = 0$$

or $\begin{bmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$

$$\det\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0 = (2\omega_0^2 - \omega^2)^2 - \omega_0^4$$

$$2\omega_0^2 - \omega^2 = \pm\omega_0^2$$

$$\underline{\underline{\omega^2 = 3\omega_0^2,\ \omega_0^2}}$$

$$\boxed{\begin{aligned} T_{\omega_0} &= 3\sqrt{2} \\ T_\omega &= \frac{3\sqrt{2}}{\sqrt{3}} \\ &= \underline{\underline{\sqrt{6}}} // \end{aligned}}$$

$\therefore$

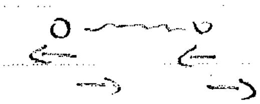
if $\omega^2 = \omega_0^2$ $-A\cdot\omega^2 + 2A\omega_0^2 - B\omega_0^2 = 0$

$-A + 2A - B = 0$; $A = B$

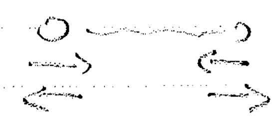
$\omega^2 = 3\omega_0^2$ $-3A\omega_0^2 + 2A\omega_0^2 - B\omega_0^2 = 0$

$-3A + 2A - B = 0$ $A = -B$

<u>eigen modes</u> : $\begin{bmatrix} x_A \\ x_B \end{bmatrix}_1 = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\omega_0 t}$



and $\begin{bmatrix} x_A \\ x_B \end{bmatrix}_2 = C_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\sqrt{3}\omega_0 t}$



5-6 (a) (b) cont.

$$\begin{bmatrix} x_A \\ x_B \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i(\omega_0 t + \alpha)} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i(\omega_0 t\sqrt{3} + \beta)}$$

@ $t=0$, $x_A = 0$, $x_B = 5$
$V_A = 0$, $V_B = 0$

$$x_A = C_1 \cos(\omega_0 t + \alpha) + C_2 \cos(\omega_0 \sqrt{3} t + \beta)$$
$$x_B = C_1 \cos(\omega_0 t + \alpha) - C_2 \cos(\omega_0 \sqrt{3} t + \beta)$$
$$V_A = -C_1 \omega_0 \sin(\omega_0 t + \alpha) - C_2 \omega_0 \sqrt{3} \sin(\omega_0 \sqrt{3} t + \beta)$$
$$V_B = -C_1 \omega_0 \sin(\omega_0 t + \alpha) + C_2 \omega_0 \sqrt{3} \sin(\omega_0 \sqrt{3} t + \beta)$$

α and $\beta = 0$ so $V_A, V_B = 0$ @ $t=0$

$C_1 = -C_2$ so $x_A = 0$ @ $t=0$

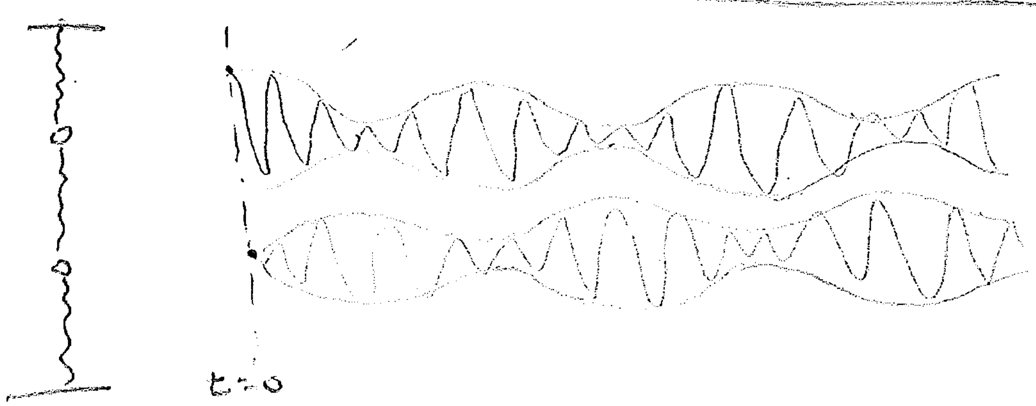
$\therefore$ $x_B = 5 = 2C_1$ $C_1 = 2.5$

$$x_A = 2.5 \cos(\omega_0 t) + 2.5 \cos(\sqrt{3}\,\omega_0 t)$$
$$x_B = 2.5 \cos(\omega_0 t) - 2.5 \cos(\sqrt{3}\,\omega_0 t)$$

or

$$\boxed{\begin{aligned} x_A &= 5 \cos\left(\frac{\sqrt{3}-1}{2}\omega_0 t\right)\cos\left(\frac{\sqrt{3}+1}{2}\omega_0 t\right) \\ x_B &= 5 \sin\left(\frac{\sqrt{3}-1}{2}\omega_0 t\right)\sin\left(\frac{\sqrt{3}+1}{2}\omega_0 t\right) \end{aligned}}$$ //

from trig identities



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S-6 (c)

$\frac{\sqrt{3}-1}{2}\omega_0 t = \pi$, time of 2nd peak (slow)

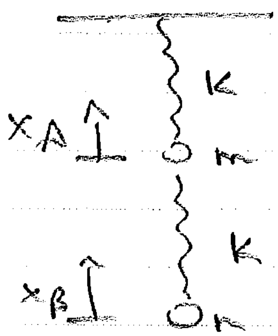
$\frac{\sqrt{3}+1}{2}\omega_0 t = \pi$ time of 2nd fast peak

$t_s = \frac{2\pi}{\sqrt{3}-1}\frac{1}{\omega_0} \rightarrow \boxed{t_s = 5.7955}$ but notice

$t_f = \frac{2\pi}{\sqrt{3}+1}\frac{1}{\omega_0}$ $\frac{t_s}{t_f} = \boxed{3.73}$, non integer.

but not exact state as before!

S-10 (a) Finding ratios of frequencies is 5 pts, ratio of amplitudes is 5 points



$$m\frac{d^2 x_A}{dt^2} = -kx_A - k(x_A - x_B)$$

$$m\frac{d^2 x_B}{dt^2} = -k(x_B - x_A)$$

Assume $\begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} e^{i\omega t}$

our D.E's becomes. A: $Ame^{i\omega t}(-\omega^2) = -2Ake^{i\omega t} + Bke^{i\omega t}$

B: $Bme^{i\omega t}(-\omega^2) = -Bke^{i\omega t} + Ake^{i\omega t}$

$2A\frac{k}{m} - B\frac{k}{m} - A\omega^2 = 0$ $B\frac{k}{m} - A\frac{k}{m} - B\omega^2 = 0$

5-10 (a) cont.

$$(2\omega_0^2-\omega^2)A - \omega_0^2 B = 0$$
$$(-\omega_0^2)A + (\omega_0^2-\omega^2)B = 0$$

$$\begin{bmatrix} 2\omega_0^2-\omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2-\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = 0$$

Find ω^2 by det $= 0$

$$(2\omega_0^2-\omega^2)(\omega_0^2-\omega^2) - \omega_0^4 = 0$$
$$2\omega_0^4 - 2\omega_0^2\omega^2 - \omega_0^2\omega^2 + \omega^4 - \omega_0^4 = 0$$

$$\omega^4 + (-3\omega_0^2)\omega^2 + \omega_0^4 = 0$$

quadratic eq.: $\dfrac{3\omega_0^2 \pm \sqrt{9\omega_0^4 - 4\omega_0^4}}{2} = \boxed{\dfrac{3\pm\sqrt{5}}{2}\omega_0^2 = \omega^2}$ //

aside $\left(\dfrac{\omega_1}{\omega_2}\right)^2 = \dfrac{3+\sqrt{5}}{3-\sqrt{5}} = \dfrac{(3+\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \dfrac{14+6\sqrt{5}}{4}$

$$= \left(\frac{6+2\sqrt{5}}{4}\right)^2 = \left(\frac{5+2\sqrt{5}+1}{5-1}\right)^2 = \left(\frac{(\sqrt{5}+1)(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}\right)^2$$
$$= \left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)^2 = \left(\frac{\omega_1}{\omega_2}\right)^2$$

aside over.

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To find ratio of amplitudes stick the two ω's back into DE's. i.e.

$$(2\omega_0^2-\omega^2)A - \omega_0^2 B = 0$$

over →

S-10 cont.

for $w^2 = \frac{3+\sqrt{5}}{2} w_0^2$

$$A\left(2w_0^2 - \frac{3+\sqrt{5}}{2} w_0^2\right) = B \cdot w_0^2$$

$$\frac{4-3-\sqrt{5}}{2} = \frac{B}{A} = \frac{1-\sqrt{5}}{2}$$ (Note take absolute value to make amplitude)

$$w^2 = \frac{3-\sqrt{5}}{2} w_0^2 \ ; \quad A\left(2w_0^2 - \frac{3-\sqrt{5}}{2} w_0^2\right) = B w_0^2$$

$$\frac{4-3+\sqrt{5}}{2} = \frac{B}{A} = \frac{1+\sqrt{5}}{2}$$

$\boxed{\text{one mode amplitude ratio} = \frac{\sqrt{5}-1}{2}}$

$\boxed{\text{other " " } = \frac{\sqrt{5}+1}{2}}$ //

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QP 1

(a) (i) $w = \sqrt{\frac{10}{5}} = \boxed{\sqrt{2}\ s^{-1}}$ //

(ii) $w = \sqrt{\frac{20}{5}} = \boxed{2\ s^{-1}}$ //

(iii)

$$m \frac{d^2 X_A}{dt^2} = -k_1 X_A - k_2 X_A$$

$$= -(k_1+k_2) X_A$$

$$w = \sqrt{\frac{k_1+k_2}{5}} = \boxed{\sqrt{6}\ s^{-1}}$$ //

- QP 1 Solution

a)

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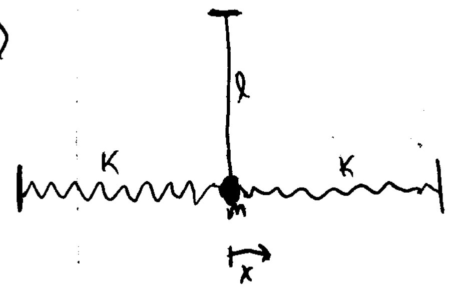

- Before we can write the differential equations, we need to know the forces involved.

$F_{springs} = -2Kx$ $\qquad F_{x,grav} = mg\sin\theta \underset{\uparrow}{\cong} -\frac{mgx}{l}$

small angle approximation

$\Rightarrow F_{x,total} = -2Kx - \frac{mgx}{l} = -(2K + \frac{mg}{l})x$

$\Rightarrow m\frac{d^2x}{dt^2} = -(2K + \frac{mg}{l})x \quad \Rightarrow \quad \boxed{\frac{d^2x}{dt^2} = -(\frac{2K}{m} + \frac{g}{l})x}$

b) The student's should be able to argue directly from this differential equation, what the frequency is.

/1

guess: $x = A\cos(\omega t + \phi)$ $\qquad \frac{d^2x}{dt^2} = -\omega^2 A\cdot\cos(\omega t + \phi) \ominus$ [illegible]

$\ominus \; -(\frac{2K}{m} + \frac{g}{l})A\cos(\omega t + \phi)$

$\Rightarrow \omega^2 = (\frac{2K}{m} + \frac{g}{l}) \quad \Rightarrow \quad \boxed{|\omega| = \sqrt{(\frac{2K}{m} + \frac{g}{l})}}$

c)

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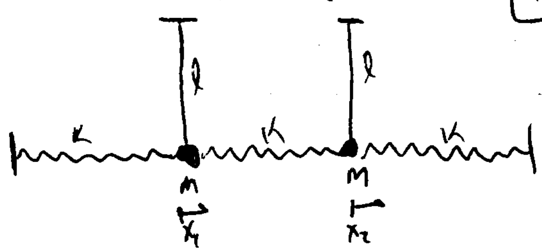

- Using same reasoning as before: $F_{1,grav} = \frac{-mgx_1}{l}$, $F_{2,grav} = \frac{-mgx_2}{l}$

$F_{1,spring} = -Kx_1 + K(x_2 - x_1) = -2Kx_1 + Kx_2 \qquad F_{2,spring} = Kx_1 - 2Kx_2$

now add all the ~~equations~~ Forces together and set equal to $ma = m\frac{d^2x}{dt^2}$

$$\Rightarrow m\frac{d^2x_1}{dt^2} = \frac{-mgx_1}{l} - 2Kx_1 + Kx_2 \qquad m\frac{d^2x_2}{dt^2} = \frac{-mgx_2}{l} - 2Kx_2 + Kx_1$$

$$\Rightarrow \frac{d^2x_1}{dt^2} = -\left(\frac{g}{l} + \frac{2K}{m}\right)x_1 + \frac{K}{m}x_2$$

$$\frac{d^2x_2}{dt^2} = \frac{K}{m}x_1 - \left(\frac{g}{l} + \frac{2K}{m}\right)x_2$$

$$\Rightarrow \begin{bmatrix} \frac{d^2x_1}{dt^2} \\ \frac{d^2x_2}{dt^2} \end{bmatrix} = \begin{bmatrix} -\left(\frac{g}{l}+\frac{2K}{m}\right) & K/m \\ K/m & -\left(\frac{g}{l}+\frac{2K}{m}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \ddot{\vec{X}} = \begin{bmatrix} -\left(g/l+\frac{2K}{m}\right) & K/m \\ K/m & -\left(\frac{g}{l}+\frac{2K}{m}\right) \end{bmatrix} \vec{X}$$ where "dot" notation represents time derivative.

12 d)

now we make the guess that $\vec{X} = \vec{V}e^{\alpha t}$. Let's plug this in.

$$\Rightarrow \alpha^2 \vec{V}\cancel{e^{\alpha t}} = \begin{bmatrix} -\left(g/l+\frac{2K}{m}\right) & K/m \\ \frac{K}{m} & -\left(\frac{g}{l}+\frac{2K}{m}\right) \end{bmatrix} \vec{V}\cancel{e^{\alpha t}} \Rightarrow \begin{bmatrix} -\left(g/l+\frac{2K}{m}\right) & K/m \\ K/m & -\left(\frac{g}{l}+\frac{2K}{m}\right) \end{bmatrix} \vec{V} = \alpha^2 \vec{V}$$

The normal mode frequencies, α, must be the eigenvalues of this matrix and the relative amplitudes of the normal modes, $\vec{V}$, must be the eigenvectors.

Find the eigenvalues: $\det\begin{bmatrix} -\left(g/l+\frac{2K}{m}\right)-\alpha^2 & K/m \\ K/m & -\left(\frac{g}{l}+\frac{2K}{m}\right)-\alpha^2 \end{bmatrix} = 0$

$$\Rightarrow \left(\frac{g}{l}+\frac{2K}{m}\right)^2 + (\alpha^2)^2 + 2\left(\frac{g}{l}+\frac{2K}{m}\right)\alpha^2 - \frac{K^2}{m^2} = 0$$

$$\Rightarrow \boxed{\alpha = \pm i\sqrt{\frac{g}{l}+\frac{K}{m}},\ \pm i\sqrt{\frac{g}{l}+\frac{3K}{m}}}$$

more explicitly, the eigen frequencies are $\boxed{\omega_1 = \sqrt{\frac{g}{\ell}+\frac{K}{m}},\ \omega_2 = \sqrt{\frac{g}{\ell}+\frac{3K}{m}}}$

– e)

now find the eigenvectors:

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$$\begin{bmatrix} -(g/\ell + \frac{2K}{m}) - \omega_1^2 & K/m \\ K/m & (-\frac{g}{\ell} + \frac{2K}{m}) - \omega_1^2 \end{bmatrix} \vec{V}_1 = \vec{0}$$

$$\Rightarrow \begin{bmatrix} -\frac{K}{m} & \frac{K}{m} \\ \frac{K}{m} & -\frac{K}{m} \end{bmatrix} \vec{V}_1 = \vec{0} \Rightarrow \underline{\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

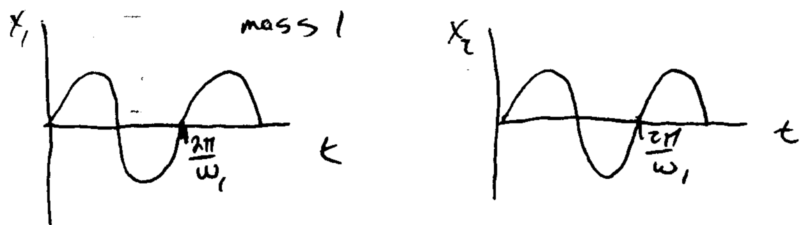
$$\begin{bmatrix} -(g/\ell + \frac{2K}{m}) - \omega_2^2 & K/m \\ K/m & (-\frac{g}{\ell} + \frac{2K}{m}) - \omega_2^2 \end{bmatrix} \vec{V}_2 = \vec{0}$$

$$\Rightarrow \begin{bmatrix} K/m & K/m \\ \frac{K}{m} & \frac{K}{m} \end{bmatrix} \vec{V}_2 = \vec{0} \Rightarrow \underline{\vec{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$\Rightarrow \text{eigenfunction 1: } (a e^{+i\omega_1 t} + b e^{-i\omega_1 t}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \overset{\text{equivalent}}{\cong} \boxed{A_1 \cos(\omega_1 t + \phi_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

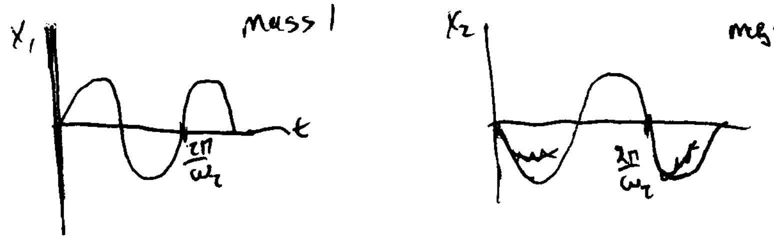
$$\text{eigenfunction 2: } (c e^{+i\omega_2 t} + d e^{-i\omega_2 t}) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cong \boxed{A_2 \cos(\omega_2 t + \phi_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Sketch eigenmode 1:



- in eigen mode 1 the masses move with the same amplitude, phase, and frequency.

Sketch eigenmode 2:



- In eigenmode two the masses move with the same amplitude, and frequency but with 180° phase difference.

The general solution is the sum of two normal modes.

$$\vec{X} = A_1 \cos(\omega_1 t + \phi_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A_2 \cos(\omega_2 t + \phi_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Let's look at the easier initial condition first: $\dot{x}_1(0) = \dot{x}_2(0) = 0$

$$\Rightarrow \dot{\vec{X}}(0) = 0 = -A_1 \omega_1 \sin(\phi_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - A_2 \omega_2 \sin(\phi_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \phi_1 = \phi_2 = 0$$

$$\vec{X} = A_1 \cos(\omega_1 t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A_2 \cos(\omega_2 t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{X}(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 + A_2 \\ A_1 - A_2 \end{bmatrix} \Rightarrow A_1 = A_2, \quad A_1 = A_2 = \frac{x_0}{2}$$

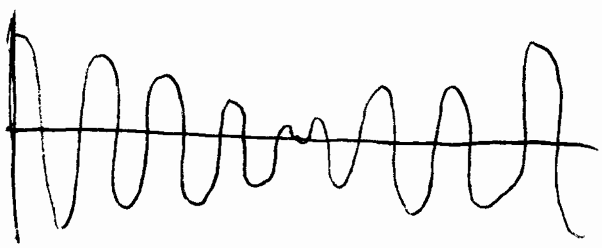
$$\Rightarrow \boxed{\vec{X} = \frac{x_0}{2} \cos(\omega_1 t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x_0}{2} \cos(\omega_2 t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

f) All the students now have to do is plot this function.

h) if $kl = mg/10$ then $\frac{\omega_2^2}{\omega_1^2} = \frac{13}{11} \Rightarrow \frac{\omega_2}{\omega_1} = \sqrt{\frac{13}{11}}$.

pick some value for ω_1 and do a plot. I'd be happy for something like this.

x_1



x_2