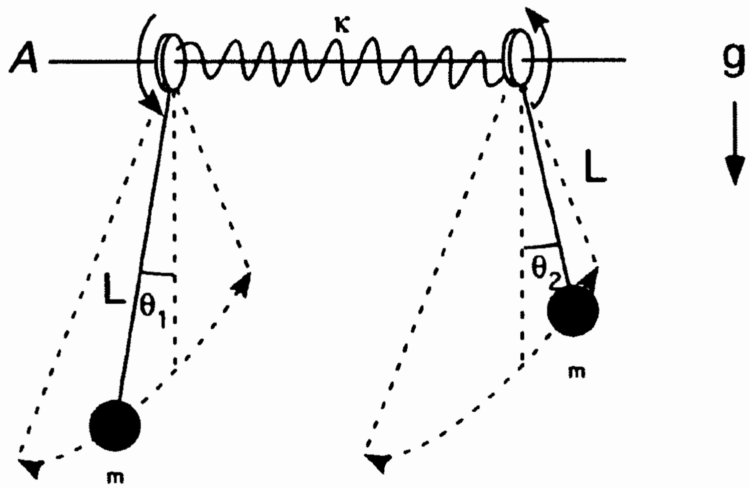
Final Solution

Problem 1



Coupled Oscillators: Two pendula pivot about an axis A and are constrained to remain in planes perpendicular to that axis. Furthermore, the pivot points remain fixed. Each has a length L and a mass m. The angles from the vertical are θ_1 and θ_2. [In the picture $\theta_1 < 0$ and $\theta_2 > 0$]. The two pendula are coupled by a weak torsional spring that obeys Hooke's Law with torsional spring constant κ.

$$\tau_2 = \kappa(\theta_1 - \theta_2)$$

where τ_2 is the torque on mass 2 and τ_1 is the torque on mass 1.

(a) [2.5 pts] Write down the linear coupled differential equations describing the motion of the pendula when the angles $\theta_i << 1$.

(b) [2.5 pts] Find the two normal-mode frequencies (eigenfrequencies).

(c) [2.5 pts] Find the two normal-mode functions (eigenfunctions). Each eigenfunction can be written as a vector amplitude times a phasor (complex exponential). The overall amplitude and phase (or complex amplitude) for each eigenfunction can remain undetermined.

(d) [2.5 pts] Determine the time evolution of each pendulum with the initial conditions $\theta_1(0) = 0$ and $\theta_2(0) = A$, with both starting at rest, and sketch the two functions for at least two beat cycles. You can assume that the spring constant $\kappa = \frac{1}{8}mgL$

~~Quiz 1 Solutions~~

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Note: The first thing we should do is take a look at the Hooke's Law equation that the torsional springs obey. The first thing we should note is that

$$[\tau_i] = \text{Force} \cdot \text{Length}\,,$$

so we know we are dealing with a situation unlike those seen in class and in sections (though an example was given during the quiz review). Therefore, because θ_1 and θ_2 are unitless, we should also take note that

$$[\kappa] = \text{Force} \cdot \text{Length}\,.$$

The only other physical constants given are g which has units of acceleration and the mass and length of the pendulum L which obviously have units of mass and length, respectively. The only way we can get a frequency out of these is by the combinations $\sqrt{g/L}$ and $\sqrt{\kappa/mL^2}$. The symmetric mode has both masses swinging in phase which means that the spring is doing nothing. Therefore we know that

$$\omega_{\text{symmetric}} = \sqrt{g/L}\,.$$

The antisymmetric mode is a little more complicated, but if you think carefully you will know that this problem is now completely analogous to two pendula coupled by a standard spring, a problem we have done before which by direct analogy $k/m \to \kappa/mL^2$ gives us the antisymmetric mode's frequency:

$$\omega_{\text{antisymmetric}} = \sqrt{\frac{g}{L} + 2\frac{\kappa}{I}}\,,$$

where $I = mL^2$ is the moment of inertia. Even without making this analogy we know that because the mode is antisymmetric for some distance θ traveled by one, the other travels an equal and opposite distance, meaning that the effect of the spring is doubled. The signs of the gravitational frequency and the spring frequency have to be the same since they are both restoring forces and therefore act in the same direction. So we could have come to the correct answer by just thinking physically about the problem.

(a) We may just write down the solution:

$$I\ddot{\theta}_1 = -(mgL+\kappa)\theta_1 + \kappa\theta_2$$

Because, the problem is symmetric about the center plane we know the equation of motion for the second mass is the same as for the first, with the indices switched:

$$I\ddot{\theta}_2 = -(mgL+\kappa)\theta_2 + \kappa\theta_1 \,.$$

(b) Normal modes are those which oscillate with the same frequency, and therefore have the form: $\theta_i = A_i \exp i\omega t$, with $i = 1,2$. Plugging these in to the equations of motion above and writing the resulting equations in matrix form yields:

$$\begin{pmatrix} -\omega^2 + (\frac{g}{L}+\frac{\kappa}{I}) & -\frac{\kappa}{I} \\ -\frac{\kappa}{I} & -\omega^2 + (\frac{g}{L}+\frac{\kappa}{I}) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0\,. \tag{1}$$

This equation has nontrivial solutions (something without $A_1 = A_2 = 0$) only when the matrix is singular. Using the fact that singular matrices have a zero determinant we may get the final relation to determine the frequencies of oscillation:

$$\begin{aligned}
0 &= \begin{vmatrix} -\omega^2 + (\frac{g}{L}+\frac{\kappa}{I}) & -\frac{\kappa}{I} \\ -\frac{\kappa}{I} & -\omega^2 + (\frac{g}{L}+\frac{\kappa}{I}) \end{vmatrix} && (2)\\
&= \left\{-\omega^2 + \left(\frac{g}{L}+\frac{\kappa}{I}\right)\right\}^2 - \left(\frac{\kappa}{I}\right)^2 && (3)\\
&= (\omega^2)^2 - 2\left(\frac{g}{L}+\frac{\kappa}{I}\right)\omega^2 + \left(\frac{g}{L}+\frac{\kappa}{I}\right)^2 - \left(\frac{\kappa}{I}\right)^2 && (4)\\
&= (\omega^2)^2 - 2\left(\frac{g}{L}+\frac{\kappa}{I}\right)\omega^2 + \frac{g}{L}\left(\frac{g}{L}+2\frac{\kappa}{I}\right) && (5)
\end{aligned}$$

If the student were able to deduce that the symmetric mode has a frequency of $\sqrt{g/L}$ then they could divide the RHS of the the final equation by $\omega^2 - g/L$ to get the final mode. Supposing that one did not do this we go on. Using the quadratic equation we get:

$$\begin{aligned}
\omega^2 &= \frac{2\left(\frac{g}{L}+\frac{\kappa}{I}\right) \pm \sqrt{4\left(\frac{g}{L}+\frac{\kappa}{I}\right)^2 - 4\frac{g}{L}\left(\frac{g}{L}+2\frac{\kappa}{I}\right)}}{2} && (6)\\
&= \left(\frac{g}{L}+\frac{\kappa}{I}\right) \pm \sqrt{\left(\frac{g}{L}+\frac{\kappa}{I}\right)^2 - \frac{g}{L}\left(\frac{g}{L}+2\frac{\kappa}{I}\right)} && (7)\\
&= \left(\frac{g}{L}+\frac{\kappa}{I}\right) \pm \frac{\kappa}{I} && (8)\\
&= \frac{g}{L} \text{ and } \frac{g}{L}+2\frac{\kappa}{I} && (9)
\end{aligned}$$

as expected.

(c) Plugging g/L in for ω^2 in equation 1 we find that $A_1 = A_2$ (as expected), and plugging in $g/L + 2\kappa/I$ for ω^2 yields $A_1 = -A2$ (also as expected). Therefore, given $\omega_S \equiv \sqrt{g/L}$ and $\omega_A \equiv \sqrt{\frac{g}{L} + 2\frac{\kappa}{I}}$ we have the two normal mode functions:

$$\vec{\theta}_S = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_S t} \quad \text{and} \quad \vec{\theta}_A = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_A t} \tag{10}$$

(d) So we need to find a linear combination of $\vec{\theta}_S$ and $\vec{\theta}_A$ such that at time $t = 0$ we get the vector $(0, A)$. That is,

$$\begin{pmatrix} 0 \\ A \end{pmatrix} = C_1\vec{\theta}_S + C_2\vec{\theta}_A \tag{11}$$

$$= C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \tag{12}$$

This gives the two equations

$$C_1 + C_2 = 0 \tag{13}$$

$$C_1 - C_2 = A\,, \tag{14}$$

which finally gives $C_1 = A/2$ and $C_2 = -A/2$. Writing out the full solution we have

$$\vec{\theta}(t) = \frac{A}{2}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_S t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{i\omega_A t} \right\} . \tag{15}$$

Not that it simplifies our solution in any way, but we can make the suggested assumption that $\kappa = \frac{1}{8}mgL$ to get $\omega_A = \sqrt{\frac{5}{4}\frac{g}{L}}$. We can simplify the final equation above by applying some simple exponential algebra and defining $\omega_- \equiv \frac{\omega_S - \omega_A}{2}$ and $\omega_+ \equiv \frac{\omega_S + \omega_A}{2}$:

$$\vec{\theta}(t) = \frac{A}{2} e^{i\omega_+ t} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_- t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{i\omega_- t} \right\} \tag{16}$$

$$= \frac{A}{2} e^{i\omega_+ t} \begin{pmatrix} 2i\sin(\omega_- t) \\ 2\cos(\omega_- t) \end{pmatrix} \tag{17}$$

$$= A \begin{pmatrix} -\sin(\omega_+ t)\sin(\omega_- t) \\ \cos(\omega_+ t)\cos(\omega_- t) \end{pmatrix} . \tag{18}$$

The envelope is defined by the trig function of ω_-, and the beats are separated by half the period: $t_{\text{beat}} = \frac{\pi}{\omega_-} = \frac{2\pi}{\omega_S - \omega_A}$. The motion will oscillate $\frac{1}{2}\frac{\omega_+}{\omega_-} = \frac{1}{2}\frac{\omega_S + \omega_A}{\omega_S - \omega_A} = \frac{1}{2}\frac{2+\sqrt{5}}{2-\sqrt{5}}$ many times during each beat. Note that this is independent of the relationship between g and L.

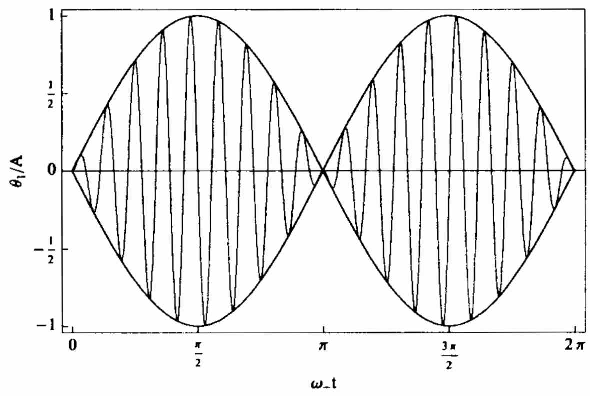


Figure 1: The motion of mass 1 as a function of $\omega_- t$.

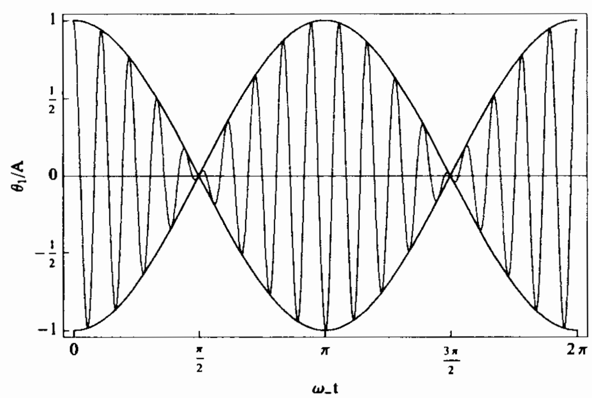


Figure 2: The motion of mass 2 as a function of $\omega_- t$.

The value of g/L determines t_{beat}, indicating that changing g/L only stretches or contracts the plot in the t direction. Therefore we plot the motion of θ_1 and θ_2 in dimensionless units of $\omega_- t$ and θ_i/A without needing to specify g/L. See figures 1 and 2 for the particle motion of masses 1 and 2 respectively.

Problem 2

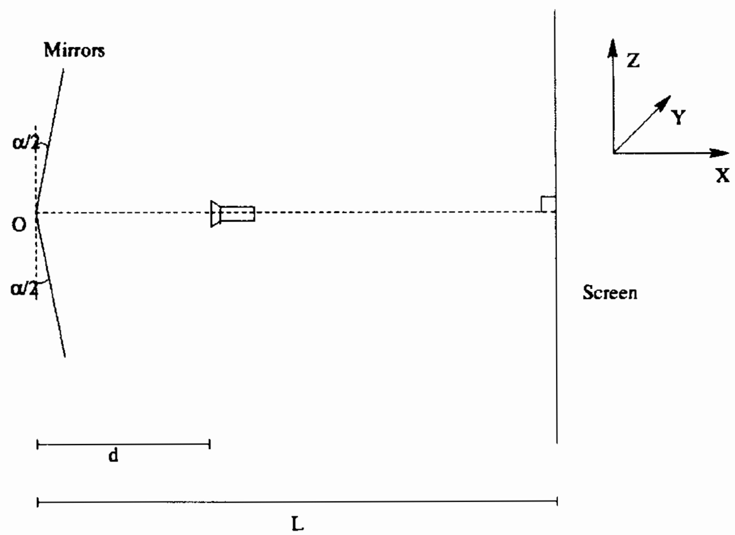
Interference: You are given a slightly bent mirror. It consists of two large plane mirrors at an angle α to each other. A coherent light source of wavelength λ is shone at a distance d from the mirrors. A large plane screen is placed at a distance L from the point O (see figure) and it is ensured that all the light hitting it has reflected off the mirrors. (i.e., No light directly from the source reaches the screen).

(a) (4 pts) Show that the interference pattern is the same as what you would expect from a Young's double slit experiment. (It is sufficient to show that the geometry of the setup is identical in both experiments)

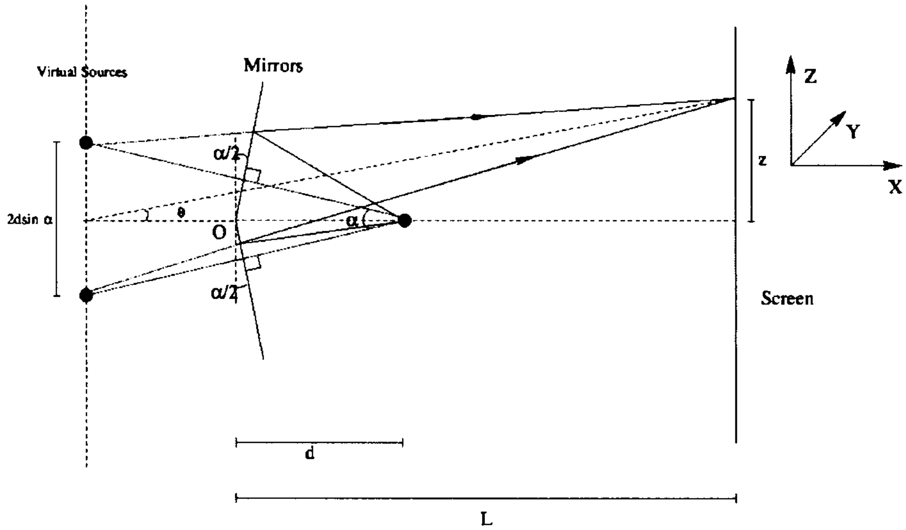
(b) (4 pts) Obtain the separation between two consecutive maxima on the screen in terms of d, α and L . To simplify calculations make an assumption that the $L \gg d$ and α and is very small.

(c) (2pts) Now consider that the source used is red light (wavelength $\lambda = 650$nm), the mirrors make an angle $\alpha = (\pi/100)$rad and the screen is at a distance $L = 1$m. How far must you place the source in the given setup so that you can distinguish two consecutive maxima on the screen? Assume that it is possible to distinguish two maxima if they are 0.5cm apart. Note that you still have to use the approximations that $L \gg d$ and α is small.

Note : In this problem you **do not** have to consider the effect of the shadow of the source on the screen and **ignore** the fact that the interference pattern is observed in a finite region in the $\hat{z}$ direction.



1. Show that the interference pattern is the same as what you would expect from a Young's double slit experiment. (It is sufficient to show that the geometry of the setup is identical in both experiments)
   Ans: Look at the geometry in the diagram shown. It looks like the double slit experiment but just with two virtual sources instead of slits.



2. Obtain the separation between two consecutive maxima on the screen in terms of d, α and L . To simplify calculations make an assumption that the $L \gg d$ and α and is very small.
   Ans: The two virtual sources are in phase with each other. The separation between the two 'sources' is $2dsin(\alpha)$. Therefore the separation between two consecutive maxima is given by

$$2dsin(\alpha)sin(\theta) = \lambda \tag{1}$$

For $L \gg d$ we can approximate $sin(\theta) \approx tan(\theta) = \frac{z}{L+2dcos^2(\frac{\alpha}{2})-d} = \frac{z}{L+dcos(\alpha)} \approx \frac{z}{L}$. So we have

$$\frac{2dzsin(\alpha)}{L} = \lambda \tag{2}$$

$$\Rightarrow \quad z = \frac{\lambda L}{2dsin(\alpha)} \approx \frac{\lambda L}{2d\alpha} \tag{3}$$

3. Now consider that the source used is red light (wavelength λ = 650nm), the mirrors make an angle $\alpha = (\pi/100)$rad and the screen is at a distance L = 1m. How far must you place the source in the given setup so that you can distinguish two consecutive maxima on the screen? Assume that it is possible to distinguish two maxima if they are 0.5cm apart. Note that you still have to use the approximations that $L \gg d$ and α is small.
   Ans: To distinguish two maxima we require $z > 0.5$ cm. We have, $\lambda =$

650nm, $\alpha = (\pi/100)$rad and $L = 1$m. Using these values in the formula derived in the previous part we have,

$$z = \frac{650 \times 10^{-9} \times 1}{2 \times d \times \frac{\pi}{100}} > 5 \times 10^{-3} \quad (4)$$

$$\Rightarrow \quad d < \frac{650 \times 10^{-9} \times 1}{2 \times 5 \times 10^{-3} \times \frac{\pi}{100}} \quad (5)$$

Therefore we must have $d < \frac{6.5}{\pi} \times 10^{-3}$m.

If we consider the fact that the interference is limited to a finite region in z direction, that would give us a lower bound on d after which one cannot observe two maxima.

Problem 3

A Solar System "Atom" (borrowed from Griffiths Introduction to Quantum Mechancs) Consider the earth-sun system as a gravitational analog to the hydrogen atom. Let M denote the mass of the sun, m be the mass of the earth, and G be Newton's gravitational-constant. Recall that the gravitational potential energy is $\frac{-GMm}{r}$.

(a) [5 pts] What is the "Bohr radius," a_g, for the earth-sun system in terms of M, m, G, and $\hbar$? Use the fact that $M = 2.0\times 10^{30} kg$, $G = 6.7\times 10^{-11}\frac{m^3}{kg\cdot s}$, and $\hbar = 1.1\times 10^{-34} J\cdot s$ to compute a numerical value.

(b) [5 pts] Write down the gravitational "Bohr formula" for the energy E_n. Then, set this equal to the classical energy of a planet in a circular orbit of radius r_0 (hint: you may recall from Ph1a that this is given by $\frac{-GMm}{2r_0}$) to show that the orbital quantum number $n = \sqrt{\frac{r_0}{a_g}}$. Using the fact that $r_0 \approx 1AU = 1.5\times 10^{11} m$ compute the numerical value of n.

(c) [5pts] Finally, suppose that the earth makes a transition from the value of n calculated in part (b) to the next lower level n-1. How much energy, in Joules, would be released? (hint: in this calculation you may need to use a taylor expansion in order to get a numerical answer. hint hint: $(1-x)^{-2} \cong (1+2x)$ for very small x). What is the wavelength of the photon (or graviton if you like) produced by this release of energy? (For reference, 1 light year=9.5×10^{15}m).

A Solar System "Atom" solution

(a) The thing to realize here is that, mathematically, we are solving the same problem as the Schrödinger equation for the hydrogen atom. Hydrogen atom: $V = \frac{-e^2}{4\pi\epsilon_0} \cdot \frac{1}{r}$ Earth-Sun: $V = \frac{-GMm}{r}$

when solving the hydrogen atom we found $a = \frac{4\pi\epsilon_0 \hbar^2}{me^2}$
Everything will carry over to this problem, we just need to change the constants. From the potentials $\frac{4\pi\epsilon_0}{e^2} \to \frac{1}{GMm} \Rightarrow \boxed{a_g = \frac{\hbar^2}{GMm^2}}$

$\Rightarrow \boxed{a_g = 2.508\text{e-}138}$

The typical size of the solar system wavefunction is fantastically small!

b) We found for the hydrogen atom: $E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right] \cdot \frac{1}{n^2}$

using our substitution $\frac{e^2}{4\pi\epsilon_0} \to GMm \Rightarrow E_n^{grav} = -\frac{\left[\frac{m}{2\hbar^2}(GMm)^2\right]}{n^2}$

now setting this equal to the classical energy of a mass in a circular orbit we get: $\frac{+GMm}{2r_0} = \frac{+m}{2\hbar^2 n^2}(GMm)^2$

$\Rightarrow n^2 = \frac{2mr_0}{2\hbar^2}(GMm) = \frac{m^2 r_0 GM}{\hbar^2} \Rightarrow \boxed{n = \sqrt{\frac{m^2 r_0 GM}{\hbar^2}} = \sqrt{\frac{r_0}{a_g}}}$

if $r_0 = 1.5\text{e}11 \Rightarrow \boxed{n \cong 2.5\text{e}74}$

This system is in a very high quantum level which means it is well-described classically.

c) $E_n^{grav} - E_{n-1}^{grav}\Big|_{n=2.5e74} = -\frac{\left[\frac{m}{2\hbar^2}(GMm)^2\right]}{(2.5\text{e}74)^2} + \frac{\left[\frac{m}{2\hbar^2}(GMm)^2\right]}{(2.5\text{e}74-1)^2} =$

$= \frac{-\frac{m}{2\hbar^2}(GMm)^2}{(2.5\text{e}74)^2}\left[1 - \left(1 - \frac{1}{2.5}\text{e-}74\right)^{-2}\right] \overset{\text{taylor expansion}}{=} \frac{-\frac{m}{2\hbar^2}(GMm)^2}{(2.5\text{e}74)^2}\left[1 - \left(1 + \frac{2}{2.5}\text{e-}74\right)\right]$

$= \frac{\frac{m}{2\hbar^2}(GMm)^2}{(2.5\text{e}74)^2}\frac{2}{2.5}\text{e-}74 = \boxed{2.05\text{e-}41 \text{ J} = \Delta E}$

$$\Delta E = hf = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{\Delta E} \cong 9.7e15$$

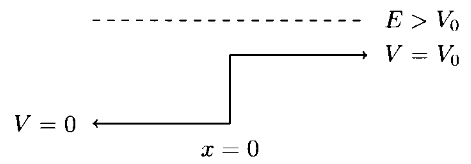
$$\Rightarrow \boxed{\lambda \cong 1 \text{ light year}}$$

wow! coincidence?

Problem 4

Consider the "step" potential:

$$V(x) = \begin{cases} 0, & \text{if } x \le 0, \\ V_0, & \text{if } x > 0. \end{cases}$$



(a) (3 pts) If the incident particle has energy $E > V_0$, what is the amplitude reflection coefficient A_R/A_I and the reflection probability R?

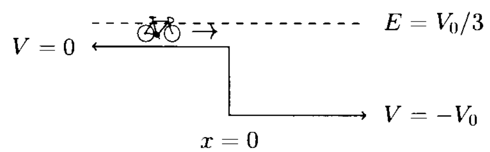
(b) (3 pts) For a potential such as this, which does not go back to zero at the right of the barrier, the transmission probability is *not* simply $|A_T|^2 / |A_I|^2$ (with A_I the incident amplitude and A_T the transmitted amplitude), because the transmitted wave travels at a different *speed*. Show that the transmission probability is

$$T = \sqrt{\frac{E - V_0}{E}} \frac{|A_T|^2}{|A_I|^2},$$

for $E > V_0$. (*Hint:* Use conservation of probability, $T + R = 1$.)

For parts (c) and (d), consider a "cliff" potential,

$$V(x) = \begin{cases} 0, & \text{if } x \le 0, \\ -V_0, & \text{if } x > 0. \end{cases}$$



(c) (3 pts) A particle of mass m and kinetic energy $E > 0$ approaches the abrupt potential drop from the left. What is the probability of reflection if $E = V_0/3$?

(d) (1 pt) When a free neutron enters a nucleus, it experiences a sudden drop in potential energy, from $V = 0$ outside to around -12 MeV inside. Suppose that a neutron, emitted with kinetic energy of 4 MeV by a fission event, strikes such a nucleus. What is the probability that it will be absorbed, thereby initiating another fission? (*Hint:* You calculated the *reflection* in part (a); use $T = 1 - R$ to get the probability of transmission through the surface. Also, neglect the finite radius of the nucleus.)

4. Solution

(a) For $x \leq 0$, the time-independent Schrödinger equation has the general solution $\psi(x) = ae^{ikx} + be^{-ikx}$ with $k = \sqrt{2mE}/\hbar$. For $x \geq 0$, the general solution is $\psi(x) = ce^{ik'x} + de^{-ik'x}$, with $k' = \sqrt{2m(E - V_0)}/\hbar$. Continuity of $\psi(x)$ and $\partial\psi/\partial x$ at $x = 0$ requires that

$$a + b = c + d$$
$$k(a - b) = k'(c - d).$$

This is a system of two equations with four unknowns. By choosing $a = 1$, $b = A_\mathrm{R}$, $c = A_\mathrm{T}$, and $d = 0$, we can interpret the solution as a superposition of an incident and reflected wave on the left and a transmitted wave on the right. This gives us a system of two equations with two unknowns,

$$1 + A_\mathrm{R} = A_\mathrm{T}$$
$$ik(1 - A_\mathrm{R}) = ik'A_\mathrm{T}.$$

The solution is

$$A_\mathrm{R} = \frac{k - k'}{k + k'},$$
$$A_\mathrm{T} = \frac{2k}{k + k'}.$$

The reflection probability when $E > V_0$ is

$$R = |A_\mathrm{R}|^2 = \left(\frac{k - k'}{k + k'}\right)^2.$$

(b) Conservation of probability requires that

$$T = 1 - R = 1 - \left(\frac{k - k'}{k + k'}\right)^2 = 1 - \frac{k^2 - 2kk' + k'^2}{(k + k')^2} = \frac{4kk'}{(k + k')^2}$$
$$= \frac{k'}{k}|A_\mathrm{T}|^2 = \frac{\sqrt{2m(E - V_0)}}{\sqrt{2mE}}|A_\mathrm{T}|^2 = \sqrt{\frac{E - V_0}{E}}|A_\mathrm{T}|^2.$$

(c) The reflection probability is exactly the same as in part (a), except that now $k' = \sqrt{2m(E + V_0)}/\hbar$.

$$R = \left(\frac{k - k'}{k + k'}\right)^2.$$

Substituting $E = V_0/3$, $k = \sqrt{2mV_0/3}/\hbar$, and $k' = \sqrt{2m(V_0/3 + V_0)}/\hbar$,

$$R = \left(\frac{\sqrt{\frac{1}{3}} - \sqrt{\frac{4}{3}}}{\sqrt{\frac{1}{3}} + \sqrt{\frac{4}{3}}}\right)^2 = \frac{1}{9}.$$

(d) In this problem, $V_0 = 12$ MeV and $E = 4$ MeV $= V_0/3$, so we can reuse the answer from part (c). The reflection probability is $1/9$, so the absorption (transmission) probability is $8/9$.

Problem 5

If you liked it you shoulda put a ring on it.
Consider a particle confined to a ring. Its time-independent Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x) \tag{1}$$

where, to describe the ring geometry, x is the position it around the ring, L is the circumference of the ring, with $0 < x \leq L$, $\psi(0) = \psi(L)$, and $\frac{\partial\psi(0)}{\partial x} = \frac{\partial\psi(L)}{\partial x}$ (i.e., the wave function obeys periodic boundary conditions around the ring, and its derivative is continuous everywhere, in particular at $x = 0$, and otherwise behaves like a particle in 1D).

(a) (2.5 pts) It is easy to see that the solutions of this problem are of the form $\psi(x) = \exp(ikx)$. What are the values of k that satisfy the boundary condition?

(b) (2.5 pts) Find *all* the normalized wave functions and their allowed energies. Are there degeneracies, i.e., multiple eigenfunctions for a given energy eigenvalue?

(c) (2.5 pts) A particle on the ring is described by the following wave function at $t = 0$:

$$\Psi(x,0) = \sqrt{\frac{8}{3L}}\sin^2\left(\frac{2\pi x}{L}\right). \tag{2}$$

What is the wave function at time t, given that $\Psi(x,t)$ is a solution of the time-dependent Schrödinger equation? Hint: you can use trigonometric and complex exponent identities to rewrite $\Psi(x,0)$ as a sum of eigenfunctions found in part a.

(d) (2.5 pts) The momentum of the particle described above in c was measured to be 0. What is the particle's wave function *after* the measurement?

problem 5

a) we need to satisfy the periodic boundary conditions.

$\psi(0) = \psi(L) \rightarrow e^{iK\cdot 0} = e^{iKL} \Rightarrow 1 = e^{iKL} \rightarrow KL = n\pi\cdot 2 \quad n = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow \boxed{K = \frac{n\pi\cdot 2}{L} \quad n = 0, \pm 1, \pm 2}$$

The continuity of the derivative is guaranteed from the continuity of the wavefunction in this case.

b) We were already given the form of the wavefunction $\psi(x) = A\exp(iKx)$ I've added a constant if needed for normalization.

$$\int_0^L |\psi|^2 dx = 1 = |A|^2 \int_0^L e^{-iKx} \cdot e^{iKx} dx = |A|^2 \int_0^L 1\, dx = |A|^2 L = 1$$

$$\Rightarrow \boxed{A = \frac{1}{\sqrt{L}} \quad \text{assuming } A \text{ is real}}$$

we already have a quantization constraint on K: $K = \frac{n\cdot 2\pi}{L}$ $n = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow \boxed{\psi_n(x) = \sqrt{\frac{1}{L}}\, e^{i\frac{n2\pi}{L}x}} \rightarrow \text{normalized wavefunctions}$$

from the time-independent schrödinger equation $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi(x)$ we know that $\frac{+\hbar^2 K^2}{2m} = E \Rightarrow \boxed{E_n = \frac{\hbar^2}{2m}\left(\frac{n\cdot 2\pi}{L}\right)^2}$

- now we can answer the degeneracy question. We have a unique wavefunction for every $n = 0, \pm 1, \pm 2, \ldots$ but E_n only depends on n^2. So every energy level is doubly degenerate, except, of course, for $n = 0$, which has no degeneracy.

c) we know that if we have an eigenfunction of the time-independent Schrödinger equation, $\psi_n(x)$, then

$\psi_n(x,t) = \psi_n(x)\cdot e^{-iEt/\hbar}$. So figuring out the time dependence is easy if we can just write our wavefunction as a sum of spatial eigenfunctions.

$$\Psi(x,0) = \sqrt{\frac{8}{3L}}\sin^2\left(\frac{2\pi x}{L}\right)$$

we know: $\sin(x) = \frac{(e^{ix} - e^{-ix})}{2i}$

$$\Rightarrow \Psi(x,0) = \sqrt{\frac{8}{3L}}\cdot\left(\frac{-1}{4}\right)\left(e^{\frac{ix2\pi}{L}} - e^{\frac{-ix2\pi}{L}}\right)^2 = \sqrt{\frac{8}{3L}}\cdot\left(\frac{-1}{4}\right)\left(e^{\frac{i2\cdot2\pi x}{L}} + e^{\frac{-i2\cdot2\pi x}{L}} - 2\right)$$

($n=2$, $n=-2$, $n=0$)

$$\Rightarrow \Psi(x,t) = \sqrt{\frac{8}{3L}}\left(\frac{-1}{4}\right)\left(e^{\frac{i2\cdot2\pi x}{L}}e^{-iE_2t/\hbar} + e^{\frac{-i2\cdot2\pi x}{L}}e^{-iE_2t/\hbar} - 2e^{-iE_0t/\hbar}\right) \Rightarrow$$

$$\Rightarrow \sqrt{\frac{8}{3L}}\left(\frac{-1}{4}\right)\left(e^{\frac{i2\cdot2\pi x}{L}}e^{-iE_2t/\hbar} + e^{\frac{-i2\cdot2\pi x}{L}}e^{-iE_2t/\hbar} - 2\right) = \Psi(x,t)$$

where $E_2 = \frac{\hbar^2}{2m}\left(\frac{2\cdot2\pi}{L}\right)^2$.

d) When you measure a particle's momentum, you collapse it into one of its momentum eigenstates. If $p=0$

then $p^2=0 \Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = 0\cdot\psi$

So by measuring the particle to have momentum zero we have collapsed the wavefunction to the eigenfunction of p^2 with value 0, we found the solution to this earlier.

$$\psi = \frac{1}{\sqrt{L}}e^{ik_0x} = \frac{1}{\sqrt{L}}e^{i\frac{0\cdot2\pi}{L}} = \frac{1}{\sqrt{L}} = \psi$$ right after measurement