Recitation 9

Problem 1: Pressure and Displacement in Tube

Tube open at one end, length L, velocity of sound v

a) Find an expression for the allowed frequencies of vibration, in Hz.

b) Draw the spatial dependence of the longitudinal atomic displacement and pressure in the tube for the first two normal modes.

c) A diaphragm located half way down the tube is used to cause a slight over-pressure in the closed end of the tube, as shown in the sketch below. The diaphragm is suddenly removed at t=0. In the subsequent motion, what are the normal amplitudes for a normal mode expansion of the atomic displacement in the tube?

\[ p(x,t=0) \]

\[ \Delta p \]

\[ \frac{L}{2} \]

\[ x \]

Solutions:

a) \( \xi(x) = A_n \cos k_n x + B_n \sin k_n x \)

open end at \( t=0 \) \( \Rightarrow \rho = 0 = \frac{\partial \xi}{\partial x} = 0 \Rightarrow B_n = 0 \forall n \)

closed end at \( x=L \) \( \Rightarrow \cos k_n L = 0 \)
a) \[ k_n L = \frac{2n-1}{n} \pi \quad n=1, 2, 3. \]

\[ j_n = \frac{\nu k_n \omega}{2\pi} = \frac{2n-1}{n} \frac{\nu}{L} \quad n=1, 2, 3 \]

b) \[ P = -\rho c^2 \frac{d^2 \phi}{dx^2} = -\rho c^2 \frac{d^2 \phi}{dx^2} \]

c) \[ \xi (x, t) = \sum_n A_n \cos (k_n x) \cos (\omega_n t + \phi_n) \]

\[ \text{rest at } t=0 \Rightarrow \phi_n = 0 \, \forall n \]

\[ P(x, t) = \alpha \sum_n k_n A_n \sin k_n x \cos \omega_n t \]

\[ P(x, 0) = \alpha \sum_n k_n A_n \sin k_n x = \begin{cases} 0 & x < \frac{L}{2} \\ \Delta P & x > \frac{L}{2} \end{cases} \]

Multiply by \[ \sin k_n x \] and integrate from 0 to L on both sides:

\[ \alpha k_n A_n \int_0^L \sin^2 k_n x \, dx = \Delta P \int_{\frac{L}{2}}^L \sin k_n x \, dx \]
Thus we get:

\[
A_n = \frac{2}{L} \Delta \rho \int_0^L \sin k_n x \, dx = \frac{2}{L} \Delta \rho \int_{knL}^{\frac{k_n L}{2}} \sin \Theta \, d\Theta =
\]

\[
= \frac{2}{L} \frac{\Delta \rho}{\alpha k_n^2} \left[ -\cos \Theta \right]_{knL}^{\frac{k_n L}{2}} = 2 \frac{\Delta \rho}{L} \frac{1}{\alpha k_n^2} \left[ -\cos (knL) + \cos \left( \frac{2n-1}{4} \pi \right) \right]
\]

\[
= \frac{8}{\pi^2} \frac{\Delta \rho L}{\alpha} \left( \frac{1}{2n-1} \right)^2 \cos \left( \frac{2n-1}{4} \pi \right)
\]

Plot to find values

\[
A_n = \frac{8}{\pi^2} \frac{\Delta \rho L}{\alpha} \left( \frac{1}{2n-1} \right)^2 \cos \left( \frac{2n-1}{4} \pi \right)
\]

\[
\begin{align*}
\text{n=1} & \quad \frac{1}{\sqrt{2}} \\
\text{n=2} & \quad -\frac{3}{\sqrt{2}} \\
\text{n=3} & \quad -\frac{5}{\sqrt{2}} \\
\text{n=4} & \quad \frac{7}{\sqrt{2}} \\
\text{n=5} & \quad \frac{9}{\sqrt{2}}
\end{align*}
\]