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Quiz 4 Solutions

Consider an infinite well potential (i.e., a closed box) stretching between $0 < x < L$. A particle of mass m is trapped in the box.

/2 1. Show that the expectation value of the momentum in all the infinite-well solutions of the TISE (time-independent Schrödinger equation) vanishes. I.e., $\langle \hat{p} \rangle = 0$ for $\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$.

/1 2. What is the uncertainty in momentum of the state $\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$. ?

At time $t = 0$ the particle's wave function is:

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} + e^{i\phi} \sqrt{\frac{2}{L}} \sin \frac{2\pi x}{L} \right) \tag{1}$$

with ϕ a real number.

/1 3. What is the probability that the particle is found in the left half of the box, $x < L/2$ at $t = 0$? Find it as a function of the phase ϕ.

4. When we consider the time evolution of the wave function $\Psi(x, t)$ we find:

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$$\Psi(x, t) = \frac{1}{\sqrt{L}} \left(f_1(t) \sin \frac{\pi x}{L} + e^{i\phi} f_2(t) \sin \frac{2\pi x}{L} \right). \tag{2}$$

What are $f_1(t)$ and $f_2(t)$?

/1 5. What is the probability density at $x = L/4$ as a function of time? If you could not find $f_{1,2}(t)$ above, express the answer in terms of these functions.

/1 6. What is the momentum expectation value for the initial wave function, $\Psi(x, t = 0)$? (This part may be a bit long, so you might go and try 7 and 8 first if you get stuck here.)

/1 7. A measuring device manages to determine that the particle is in the region $\frac{L}{4} - \frac{L}{20} < x < \frac{L}{4} + \frac{L}{20}$. Estimate the minimum uncertainty in the momentum immediately after the measurement. You can assume that the probability density is roughly constant in this region.

/1 8. After the measurement (which happens at t_0), the wave-function of the particle is actually a superposition of many normal modes (or eigenfunctions) of the Schrödinger equation:

$$\Psi(x, t) = \sum_{n>0} \alpha_n e^{-i\omega_n t} \psi_n(x) \tag{3}$$

with $\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$. This makes the particle quickly expand over the entire box again. But at some time after the measurement, t_1, the particle's wave function returns exactly to its shape at t_0. At what time $t_1 > t_0$ does this happen for the first time time? Hint: ω_n have the form $n^2 \cdot \omega_1$.

Possibly useful integrals: (In 5 and 6 $n \neq m$).

$$\int dx \sin^2 x = \frac{2x - \sin 2x}{4}, \tag{4}$$

$$\int dx \sin(nx) \cos(mx) = -\frac{1}{2} \left(\frac{1}{n+m} \cos[(n+m)x] + \frac{1}{n-m} \cos[(n-m)x] \right) \tag{5}$$

$$\int dx \sin(nx) \sin(mx) = \frac{1}{2} \left(\frac{1}{n-m} \sin[(n-m)x] - \frac{1}{n+m} \sin[(n+m)x] \right) \tag{6}$$

Quiz 4 solution

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1) $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$ $\langle \hat{p} \rangle = \int_0^L \psi_n^*(x) \frac{\hbar}{i}\frac{d}{dx}\psi_n(x)dx = \frac{2}{L}\frac{\hbar}{i}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\frac{d}{dx}\sin\left(\frac{n\pi x}{L}\right)dx$

$$= \frac{2}{L}\frac{\hbar}{i}\frac{n\pi}{L}\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx}_{=0} = 0 = \langle \hat{p} \rangle$$

Proof: $\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx =$ $u = \cos\left(\frac{n\pi x}{L}\right)$ $du = \frac{-n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)$
$dv = \sin\left(\frac{n\pi x}{L}\right)$ $v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$$= \left[-\frac{L}{n\pi}\cos^2\left(\frac{n\pi x}{L}\right)\right]_0^L - \int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow 2\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \left[-\frac{L}{n\pi}\cos^2\left(\frac{n\pi x}{L}\right)\right]_0^L \to 0$$

$$\Rightarrow \boxed{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0}$$

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2) To quantify the uncertainty in the momentum, let's find the variance: $= (\Delta\hat{p})^2 = \langle (\hat{p} - \langle\hat{p}\rangle)^2 \rangle = \langle \hat{p}^2 \rangle - \cancel{\langle \hat{p} \rangle^2} = \langle \hat{p}^2 \rangle$

$$\langle \hat{p}^2 \rangle = \int_0^L \psi_n^*(x)(-\hbar^2)\frac{d^2}{dx^2}\psi_n(x)dx = -\frac{2}{L}\hbar^2\int_0^L \sin\left(\frac{n\pi x}{L}\right)\frac{d^2}{dx^2}\sin\left(\frac{n\pi x}{L}\right)dx =$$

$$= \frac{2\hbar^2}{L}\left(\frac{n\pi}{L}\right)^2\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{2\hbar^2}{L}\left(\frac{n\pi}{L}\right)^2\cdot\frac{L}{2} = \boxed{\frac{\hbar^2 n^2\pi^2}{L^2} = \langle \hat{p}^2 \rangle = (\Delta\hat{p})^2}$$

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3) We know the probability of finding a particle between $x=a$ and $x=b$ at time t is $\int_a^b |\psi(x,t)|^2dx$ so the probability of finding the particle between 0 and $\frac{L}{2}$ at time $t=0$ $= \int_0^{L/2} |\psi(x,t=0)|^2dx =$

$$= \frac{1}{2}\frac{2}{L}\int_0^{L/2}\left(\sin\left(\frac{\pi x}{L}\right) + e^{-i\phi}\sin\left(\frac{2\pi x}{L}\right)\right)\left(\sin\left(\frac{\pi x}{L}\right) + e^{i\phi}\sin\left(\frac{2\pi x}{L}\right)\right)dx =$$

$$= \frac{1}{L}\left[\int_0^{L/2}\sin^2\left(\frac{\pi x}{L}\right)dx + e^{i\phi}\int_0^{L/2}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)dx + e^{-i\phi}\int_0^{L/2}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)dx\right.$$

$$\left. + \int_0^{L/2}\sin^2\left(\frac{2\pi x}{L}\right)dx\right.$$

- now you can use the integral tables provided. The answer is as follows.

$$\Rightarrow \frac{1}{L}\left[\frac{L}{4} + e^{i\phi}\frac{2}{3}\frac{L}{\pi} + e^{-i\phi}\frac{2}{3}\frac{L}{\pi} + \frac{L}{4}\right] \Rightarrow$$

$$\Rightarrow \frac{1}{4} + \frac{1}{4} + \frac{2}{3\pi}\left(e^{i\phi} + e^{-i\phi}\right) = \boxed{\frac{1}{2} + \frac{4}{3\pi}\cos(\phi) = \int_0^{L/2} |\Psi(x,t=0)|^2 \, dx}$$

4) The answer to this question comes directly from the form of seperable solutions to the schrödinger equation.

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$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi \qquad \Psi(x,t) = \psi(x)\phi(t)$$

$$i\hbar \frac{d\phi}{dt}\psi(x) = \frac{-\hbar^2}{2m}\phi(t)\frac{d^2\psi}{dx^2} + V\psi\phi \quad \text{divide by } \psi\phi \Rightarrow \quad i\hbar\frac{d\phi}{dt}\cdot\frac{1}{\phi} = \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2}\cdot\frac{1}{\psi} + V$$

both sides must be equal to a constant. That constant is E.

$$\Rightarrow i\hbar\frac{d\phi}{dt}\cdot\frac{1}{\phi} = E \quad \Rightarrow \quad \frac{d\phi}{dt} = -\frac{iE\phi}{\hbar} \quad \Rightarrow \quad \boxed{\phi(t) = Ae^{\frac{-iEt}{\hbar}}}$$ This is the time dependence

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2}\cdot\frac{1}{\psi} + V = E \quad \Rightarrow \quad \boxed{\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi}$$ time-independent schrödinger equation.

So to get the time-dependence, we simply "tack" on a factor of $e^{-iEt/\hbar}$ to our time-independent schrödinger equation solutions.

$$\Rightarrow \boxed{\psi(x,t) = \frac{1}{\sqrt{L}}\left(\sin\left(\frac{\pi x}{L}\right)e^{-iE_1t/\hbar} + e^{i\phi}e^{-iE_2t/\hbar}\sin\left(\frac{2\pi x}{L}\right)\right)}$$

where E_1, E_2 are the first, second energies of the infinite square well

$$\boxed{E_1 = \frac{\hbar^2\pi^2}{2mL^2}} \qquad \boxed{E_2 = \frac{4\hbar^2\pi^2}{2mL^2}}$$

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5) The probability density at $x = \frac{L}{4}$ as a function of time $= |\psi(\frac{L}{4},t)|^2 =$

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$$= \frac{1}{L}\left(\sin\left(\frac{\pi}{4}\right)e^{iE_1t/\hbar} + e^{-i\phi}e^{iE_2t/\hbar}\sin\left(\frac{\pi}{2}\right)\right)\left(\sin\left(\frac{\pi}{4}\right)e^{-iE_1t/\hbar} + e^{i\phi}e^{-iE_2t/\hbar}\sin\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{L}\left[\frac{1}{2} + 1 + \frac{\sqrt{2}}{2}e^{i\phi}e^{i(E_1-E_2)t/\hbar} + \frac{\sqrt{2}}{2}e^{-i\phi}e^{-i(E_1-E_2)t/\hbar}\right] =$$

$$= \boxed{\frac{1}{L}\left[\frac{3}{2} + \frac{\sqrt{2}}{2}\cos(\phi + (E_1-E_2)t/\hbar)\right] = |\psi(\tfrac{L}{4},t)|^2}$$

6) $\langle \hat{p} \rangle_{t=0} = \frac{1}{2}\frac{2}{L}\int_0^L \left(\sin\left(\frac{\pi x}{L}\right) + e^{-i\phi}\sin\left(\frac{2\pi x}{L}\right)\right)\frac{\hbar}{i}\frac{d}{dx}\left(\sin\left(\frac{\pi x}{L}\right) + e^{i\phi}\sin\left(\frac{2\pi x}{L}\right)\right)dx$

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$$= \frac{\hbar}{iL}\int_0^L \left(\sin\left(\frac{\pi x}{L}\right) + e^{-i\phi}\sin\left(\frac{2\pi x}{L}\right)\right)\left(\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right) + e^{i\phi}\left(\frac{2\pi}{L}\right)\cos\left(\frac{2\pi x}{L}\right)\right)dx =$$

$$= \frac{\hbar}{iL}\left[\underbrace{\int_0^L \frac{\pi}{L}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)}_{0 \text{ by integral in part 1}} + \underbrace{\frac{2\pi}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)dx}_{0 \text{ by integral in part 1}}\right.$$

$$\left. + \frac{2\pi}{L}e^{i\phi}\int_0^L \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)dx + e^{-i\phi}\frac{\pi}{L}\int_0^L \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)dx\right]$$

The last two integrals we can get from those provided.

the answer is: (remember when using the tables to use a variable substitution $x' = \frac{n\pi x}{L}$ $dx' = \frac{n\pi}{L}dx$)

$$= \frac{\hbar}{iL}\left[\frac{2\pi}{L}e^{i\phi}\left(-\frac{2}{3}\frac{L}{\pi}\right) + \frac{\pi}{L}e^{-i\phi}\left(\frac{4}{3}\frac{L}{\pi}\right)\right] = -\frac{4}{3}\left(e^{i\phi} - e^{-i\phi}\right) =$$

$$= \left[-\frac{4}{3}(2i\sin\phi)\right]\frac{\hbar}{iL} = \boxed{\frac{-8\hbar}{3L}\sin\phi = \langle \hat{p} \rangle_{t=0}}$$

(7) Okay, this problem is just an order of magnitude estimate. Factors of 2 or π don't really matter.

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Heisenberg uncertainty principle : $\Delta x \Delta p \geq \frac{\hbar}{2}$

if the particle is confined to $\frac{L}{4} - \frac{L}{20} < x < \frac{L}{4} + \frac{L}{20}$

we can say $\Delta x \approx \frac{L}{20}$

$\Rightarrow \frac{L}{20} \Delta p \geq \frac{\hbar}{2} \quad \Rightarrow \quad \Delta p \geq \frac{\hbar}{2} \cdot \frac{20}{L} = \frac{10\hbar}{L}$

$\Rightarrow$ $\boxed{\text{minimum uncertainty in momentum} = \frac{10\hbar}{L}}$

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8) in the expression $\Psi(x,t) = \sum_{n>0} a_n e^{-i\omega_n t} \psi_n(x)$

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$\omega_n = \frac{E_n}{\hbar}$ we know this because the way to expand a wavefunction in terms of seperable solutions of the Schrödinger equation is $\Psi(x,t) = \sum_{n>0} a_n e^{-iE_n t/\hbar} \psi_n(x)$

now we know $E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}$ so $\omega_n = \frac{\hbar n^2 \pi^2}{2mL^2}$

So all ω_n are integer multiples of ω_1.

So when $\omega_1 t^* = 2\pi$ then $\omega_n t^* = 2\pi n^2 = (\text{integer multiple}) \cdot 2\pi$

So when $t^* = \frac{2\pi}{\omega_1}$ all the time functions will have repeated.

$\Rightarrow t^* = \frac{2\pi}{E_1/\hbar} = \frac{2\pi\hbar}{E_1} = \frac{2\pi\hbar \cdot 2mL^2}{\hbar^2 n^2 \pi^2} = \boxed{\frac{4mL^2}{\hbar n^2 \pi} = t^*}$