1 Notes for Ph2b Quiz Review 1

Disclaimer: These are the unofficial notes for the first quiz review of Ph2b. It is likely to contain typos and errors and it does not cover the theoretical review.

1.1 Bra-Ket notation

Although it is only introduced by Griffiths in Chapter 3, I will be often using Bra-Ket notation (quantum formulas look much simpler using it). This section is meant as a short guide for those who are unfamiliar with it (Wikipedia contains a much more detailed description). In this notation vector space elements (e.g. functions) are represented in a new notation:

\[ \Psi \rightarrow |\Psi\rangle \]
\[ \Psi^* \rightarrow \langle\Psi| \]

This notation is very useful when we want to write down scalar products and expectation values. The scalar product between \( \Psi_1 \) and \( \Psi_2 \) is defined as

\[ \int_{-\infty}^{\infty} \Psi_1^* (x) \Psi_2(x) dx \]

which we denote as \( \langle\Psi_1|\Psi_2\rangle \). Thus \( \langle\Psi_1|\Psi_2\rangle = \int_{-\infty}^{\infty} |\Psi(x)|^2 dx = 1 \) as the wave functions are normalized.

Expectation values for physical quantity \( \hat{A} \) (\( \hat{A} \) corresponding operator) is defined as

\[ \text{Expval}(A) = \int_{-\infty}^{\infty} \Psi^*(x) \hat{A} \Psi(x) dx = \langle\Psi|\hat{A}|\Psi\rangle \]

in this notation.

1.1.1 Decomposition of wavefunctions

In many cases it is advantageous to decompose the wave function into the eigenfunctions of a certain operator. For eigenfunction \( \langle i \rangle \) it is true that

\[ \hat{A} |i\rangle = A_i |i\rangle , \]

where \( A_i \) is the eigenvalue corresponding to \( |i\rangle \). Since these eigenfunctions form an orthonormal basis\(^2\) we can write any \( \Psi \) as

\[ |\Psi\rangle = \sum_i c_i |i\rangle , \]

where \( |i\rangle \) is the \( i \)th eigenfunction of \( \hat{A} \) and \( c_i \) is just the coefficient. A special case of this is the Fourier decomposition, where the eigenfunctions are \( \sin(ix) \) and \( \cos(ix) \). Since the base is orthonormal we can find the coefficients:

\[ c_i = \langle i|\Psi\rangle = \int_{-\infty}^{\infty} f_i^*(x) \Psi(x) dx. \]

Since \( \langle\Psi|\Psi\rangle = 1 \) thus

\[ \sum_i |c_i|^2 = 1. \]

\(^1\)More precisely: \( |\Psi\rangle \) denotes the \( \Psi \) function, while \( \langle\Psi| \) denotes the distribution that produces \( \int f \Psi^* d\chi \) when acting on \( |f\rangle \).

\(^2\)It can be proven that the eigenfunctions of a physical operator are orthogonal to each other, thus for functions \( f_i \) and \( f_j \):

\[ \langle i|j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \]
To see why this decomposition is very useful let’s calculate the expectation value of A:

\[
\langle \Psi | \hat{A} | \Psi \rangle = \left( \sum_j c_j^* \langle j | \right) \hat{A} \left( \sum_i c_i | i \rangle \right) = \sum_i \sum_j c_i c_j^* A_i \langle j | i \rangle = \sum_i |c_i|^2 A_i.
\] (7)

This means that if we know the \(c_i\) coefficients we can calculate the expectation value easily. It also gives us the probabilities of measuring different results: if we take a quantum system in state \(| \Psi \rangle\) and measure physical quantity \(A\) (e.g. energy) then \(|c_i|^2\) is the probability of getting \(A_i\) as the results (this is one of the reasons QM is very strange, everything is probabilistic).

It is especially useful to decompose into eigenfunctions of the energy operator (\(\hat{H}\) - the Hamiltonian). These satisfy \(\hat{H} | i \rangle = E_i | i \rangle\), which combined with the Schrödinger equation (\(\hat{H} | i \rangle = i\hbar \partial_t | i \rangle\)) gives

\[
|i \rangle = \phi_i(x) \exp\left\{ \frac{i}{\hbar} E_i t \right\}.
\] (8)

This means that if we know \(\Psi\) at some time (e.g. \(t = 0\)) then we can calculate it at later times as:

\[
\Psi(x, t) = \sum_i c_i | i \rangle = \sum_i c_i \phi_i(x) \exp\left\{ \frac{i}{\hbar} E_i t \right\}
\] (9)

\[
c_i = \langle i | \Psi \rangle = \langle i, t = 0 | \Psi, t = 0 \rangle = \int \phi_i^*(x) \Psi(x, 0) dx.
\] (10)

This is exactly the same thing that we did in Ph2a for the "string problems": we had an initial shape which we decomposed into eigenmodes (Fourier) for which we knew the time evolution.

### 1.2 Problem 1: Wave function properties

A particle is in the state

\[
\Psi_1(x) = A \begin{cases} x & -1 < x < 1 \\ 0 \end{cases}
\] (11)

#### a) Find \(A\)!

\[
\langle \Psi_1 | \Psi_1 \rangle = \int_{-\infty}^{\infty} |\Psi_1(x)|^2 dx = |A|^2 \int_{-1}^{1} |x|^2 = \frac{2}{3} |A|^2,
\]

so \(A = \sqrt{\frac{3}{2}}\).

#### b) Find \(< x >\)!

\(< x > = \langle \Psi_1 | \hat{x} | \Psi_1 \rangle = \int ...\), but instead of brute-forcing it, we can realize that we are integrating \(x|x|^2\) on a symmetric region, so \(< x > = 0\).

#### c) Find \(\Delta x\)!

From probability theory: \(\Delta x = \sqrt{< x^2 > - < x >^2}\), so we just need to calculate \(< x^2 > = \int_{-1}^{1} \frac{2}{3} |x|^2 x^2 = \frac{\sqrt{3}}{5}\), thus \(\Delta x = \sqrt{\frac{3}{5}}\).

#### d) Let us introduce

\[
\Psi_2(x) = \begin{cases} \frac{1}{\sqrt{2}} x^2 & -3 < x < 3 \\ 0 \end{cases}
\] (12)

We have another particle in state \(\Psi = \alpha \Psi_1 + \frac{1}{2} \Psi_2\). Find \(\alpha\)!

It is easy to see that \(\Psi_2\) is normalized. It is also orthogonal to \(\Psi_1\) (integral of even×odd), so we can use Eq. 6

\[
\alpha^2 + \frac{1}{4} = 1,
\] (13)

thus \(\alpha = \frac{\sqrt{3}}{2}\).
1.3 Problem 2: Momentum eigenstates

A particle is in the state

\[ |\Psi\rangle = \sqrt{\frac{1}{3}} |p_1\rangle + \sqrt{\frac{2}{3}} |p_2\rangle , \]

(14)

where

\[ |p_n\rangle = \begin{cases} \frac{1}{2} \exp\left\{ i \frac{p_n}{\hbar} x \right\} & -2 < x < 2 \\ 0 & \end{cases} \]

(15)

a) Show that \( |p_i\rangle \) is a momentum eigenstate.
If it is an eigenfunction then \( \hat{P} |p_n\rangle = |p_n\rangle \times \) constant. Since \( \hat{P} = -i\hbar \partial_x \) it is easy to see that it is satisfied with \( p_n \) being the eigenvalue.

b) By measuring the momentum of the particle what possible values can we get and with what probability?
Based on part a) we can get \( p_1 \) with \( 1/3 \) and \( p_2 \) with \( 2/3 \) probability.

c) What is the probability of finding the particle between \( x=0 \) and \( x=1 \)?

\[ \text{Pr} (0 < x < 1) = \int_0^1 \Psi^* \Psi \, dx \]

(16)

\[ \Psi^* \Psi = \frac{1}{12} \left( 3 + \sqrt{2} \exp\left\{ \frac{i}{\hbar} (p_1 - p_2) \right\} + \sqrt{2} \exp\left\{ \frac{i}{\hbar} (p_2 - p_1) \right\} \right) = \frac{1}{12} \left( 3 + 2\sqrt{2} \cos \frac{p_1 - p_2}{\hbar} x \right) \]

(17)

So

\[ \text{Pr} (0 < x < 1) = \frac{1}{12} \left( 3 - 2\sqrt{2} \frac{\sin \frac{p_1 - p_2}{\hbar} x}{\frac{p_1 - p_2}{\hbar}} \right) \]

(18)

1.4 Problem 3: Time evolution of state

A particle of \( m \) mass is confined in an infinite square well (walls at \( x = 0 \) and \( x = a \)) and has the initial state

\[ \Psi(x, 0) = \begin{cases} \frac{1}{\sqrt{a}} & 0 < x < a \\ 0 & \end{cases} \]

(19)

a) Decompose \( |\Psi\rangle \) into energy eigenstates. Hint: the energy eigenfunctions are

\[ |n\rangle = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \exp\left\{ i \frac{n^2 \hbar^2 \pi^2}{2ma^2} t \right\} . \]

We need \( |\Psi\rangle = \sum_n c_n |n\rangle \). Using Eq. [10]

\[ c_n = \langle n | \Psi \rangle = \int_0^a \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \times \frac{1}{\sqrt{a}} \, dx = \begin{cases} \frac{2\sqrt{2}}{n\pi} & \text{odd } n \\ 0 & \text{even } n \end{cases} \]

(20)

b) Find \( \Psi(x, t) \)!

\[ |\Psi\rangle = \sum_n c_n \times |n\rangle = \sum_{l=0}^\infty \frac{2\sqrt{2}}{(2l+1)\pi} \times \sqrt{\frac{2}{a}} \sin \frac{(2l+1)\pi x}{a} \exp\left\{ i \frac{(2l+1)^2 \hbar^2 \pi^2}{2ma^2} t \right\} . \]

(21)