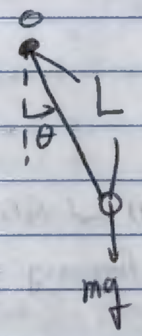
Final Review

Ph2a/Caltech.
Fall 2011

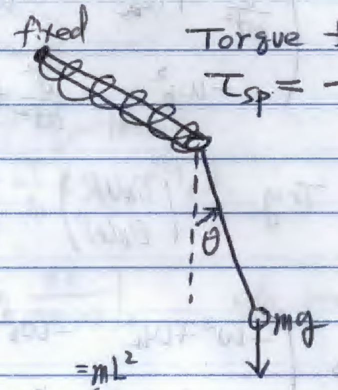
[1] Simple Harmonic Oscillator



EOM:

$$\tau = I\ddot{\theta} = -mgL\sin\theta$$

$$\rightarrow \ddot{\theta} + \frac{g}{L}\theta = 0$$



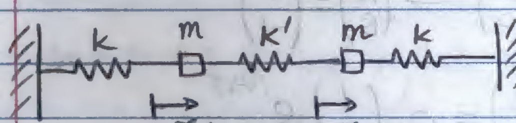
Torque from spring
$\tau_{sp} = -K\theta$

EOM:

$$\tau = \overbrace{I}^{=mL^2}\ddot{\theta} = -mgL\sin\theta - k\theta$$

$$\rightarrow \ddot{\theta} + \left(\frac{g}{L} + k\right)\theta = 0$$

[2] Coupled Oscillator

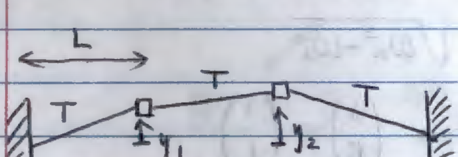
① 

EOM:

$$m\ddot{x}_1 = -kx_1 + k'(x_2 - x_1)$$

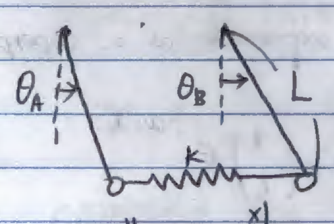
$$m\ddot{x}_2 = -kx_2 - k'(x_2 - x_1)$$

$$\Rightarrow \begin{pmatrix} \frac{d^2}{dt^2} + \frac{k}{m} + \frac{k'}{m} & -\frac{k'}{m} \\ -\frac{k'}{m} & \frac{d^2}{dt^2} + \frac{k}{m} + \frac{k'}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

② 

EOM:

$$\left.\begin{aligned} m\ddot{y}_1 &= -T\frac{y_1}{L} + \frac{T}{L}(y_2 - y_1) \\ m\ddot{y}_2 &= -T\frac{y_2}{L} - \frac{T}{L}(y_2 - y_1) \end{aligned}\right\} \Rightarrow \begin{pmatrix} \frac{d^2}{dt^2} + \frac{2T}{mL} & -\frac{T}{mL} \\ -\frac{T}{mL} & \frac{d^2}{dt^2} + \frac{2T}{mL} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

③ 

$$\begin{pmatrix} \frac{d^2}{dt^2} + \frac{g}{L} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{d^2}{dt^2} + \frac{g}{L} + \frac{k}{m} \end{pmatrix} \begin{pmatrix} \theta_A \\ \theta_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \tau_A &= I\ddot{\theta}_A = -m\underset{\times L}{g}\sin\theta_A + L\cos\theta_A\, k(L\sin\theta_B - L\sin\theta_A) \\ \tau_B &= I\ddot{\theta}_B = -m\underset{\times L}{g}\sin\theta_B - L\cos\theta_B\, k(L\sin\theta_B - L\sin\theta_A) \end{aligned}\right\}$$

General EOM of coupled oscillator

$$\begin{pmatrix} \frac{d^2}{dt^2}+\omega_a^2 & -\omega_b^2 \\ -\omega_b^2 & \frac{d^2}{dt^2}+\omega_a^2 \end{pmatrix}\begin{pmatrix} \theta_A(t) \\ \theta_B(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Try $\begin{pmatrix} \theta_A(t) \\ \theta_B(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\omega t}$ (Note: $\theta_A(t)$ and $\theta_B(t)$ are oscillating w/ the same frequency $\Rightarrow$ Normal mode)

$$\Rightarrow \begin{pmatrix} -\omega^2+\omega_a^2 & -\omega_b^2 \\ -\omega_b^2 & -\omega^2+\omega_a^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} e^{i\omega t} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det = 0 = (-\omega^2+\omega_a^2)^2 - (\omega_b^2)^2 \rightarrow \omega = \sqrt{\omega_a^2 \pm \omega_b^2}$$

i) Normal mode 1: $\omega_1 = \sqrt{\omega_a^2+\omega_b^2}$

$$\begin{pmatrix} -\omega_b^2 & -\omega_b^2 \\ -\omega_b^2 & -\omega_b^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigenfunction: $\begin{pmatrix} \theta_A(t) \\ \theta_B(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_1 t}$

ii) Normal mode 2: $\omega_2 = \sqrt{\omega_a^2-\omega_b^2}$

$$\begin{pmatrix} \omega_b^2 & -\omega_b^2 \\ -\omega_b^2 & \omega_b^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Eigenfunction: $\begin{pmatrix} \theta_A(t) \\ \theta_B(t) \end{pmatrix} = C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_2 t}$

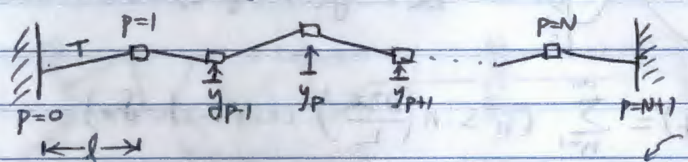
* Any oscillating motion can be expressed as a superposition of normal modes:

$$\begin{pmatrix} \theta_A(t) \\ \theta_B(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_1 t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_2 t}$$

Exercise) Determine C_1 and C_2 for given initial $\theta_A(0), \theta_B(0), \dot{\theta}_A(0), \dot{\theta}_B(0)$.

3 N-mass system.

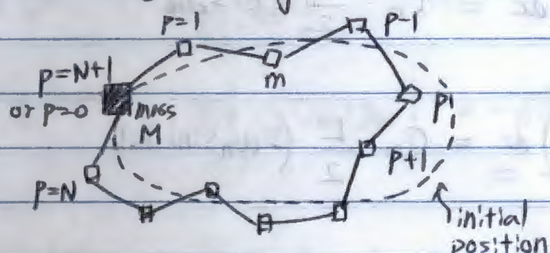
1) Fixed Boundary condition : $y(p=0) = y(p=N+1) = 0$



EOM : $m\ddot{y}_p = -\frac{T}{\ell}(y_p - y_{p-1}) + \frac{T}{\ell}(y_{p+1} - y_p)$

Normal (mode) frequency : $\omega_n = 2\omega_0 \left| \sin \frac{n\pi}{2(N+1)} \right|$ w/ $\omega_0 = \sqrt{\frac{T}{m\ell}}$

Different geometry of the same problem :



with $M \to \infty$, $y(p=0) = y(p=N+1) = 0$

Question : If we replace M by m , what's Normal mode frequency ? Which mode is still good normal mode for "periodic Boundary condition" ?

$y(p=0) = y(p=N+1)$

Answer : $\omega_{n'} = 2\omega_0 \left| \sin \frac{n'\pi}{(N+1)} \right|$

(Only even-n modes are good. $n' = \frac{n}{2}$)

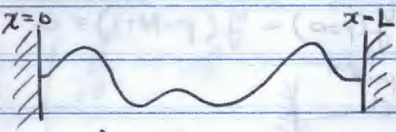
Note : In the limit of $N \to \infty$ & $\ell \to 0$ w/ fixed $(N+1)\ell = L$,

$$\omega_n = 2\sqrt{\frac{T}{m\ell}} \left| \sin \frac{n\pi\ell}{2(N+1)\ell} \right| = 2\sqrt{\frac{T}{m\ell}} \left| \sin\left(\frac{n\pi}{2L} \cdot \ell\right) \right|$$

$$\simeq 2\sqrt{\frac{T}{m\ell}} \cdot \frac{n\pi}{2L} \cdot \ell = \sqrt{\frac{T}{m/\ell}} \frac{n\pi}{L} \Longleftrightarrow \omega_n = \sqrt{\frac{T}{\mu}} \frac{n\pi}{L}$$

compare this to normal mode frequency of string

4 Normal modes of continuous system.



BC: $y(0) = y(L) = 0$

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t - \delta_n)$$

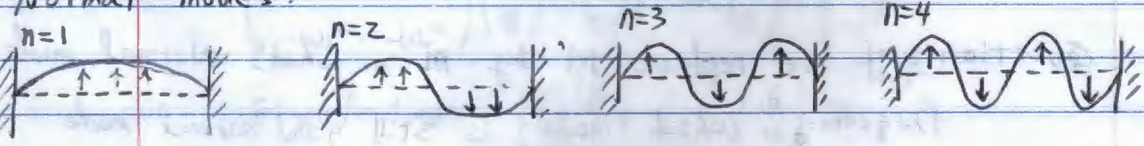
satisfying $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{T}{\mu} \dfrac{\partial^2 y}{\partial x^2} \Rightarrow \omega_n = v \cdot k_n = \sqrt{\dfrac{T}{\mu}} \cdot \dfrac{n\pi}{L}$

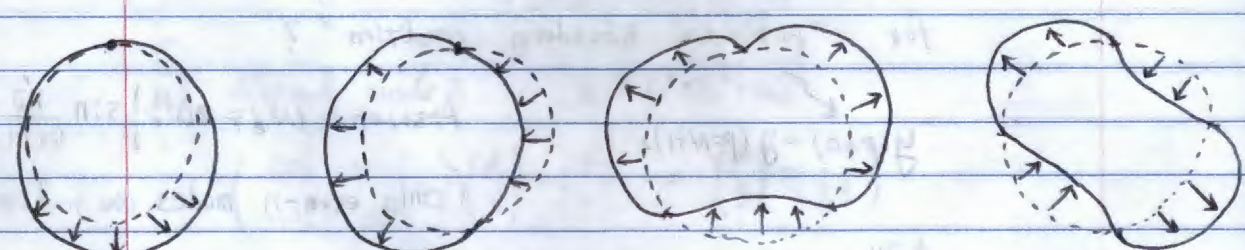
Fourier Analysis

$$\int_0^L y(x,0) \sin\left(\frac{m\pi x}{L}\right) dx = C_m \cdot \frac{L}{2} \cdot \cos\delta_m$$

$$\int_0^L \frac{\partial y(x,0)}{\partial t} \cdot \sin\left(\frac{m\pi x}{L}\right) dx = C_m \cdot \frac{L}{2} \left(-\omega_n \sin(-\delta_m)\right)$$

Normal modes.





Question: Which mode is satisfying Periodic Boundary condition: $y(0) = y(L)$, $\left.\dfrac{\partial y}{\partial x}\right|_0 = \left.\dfrac{\partial y}{\partial x}\right|_L$

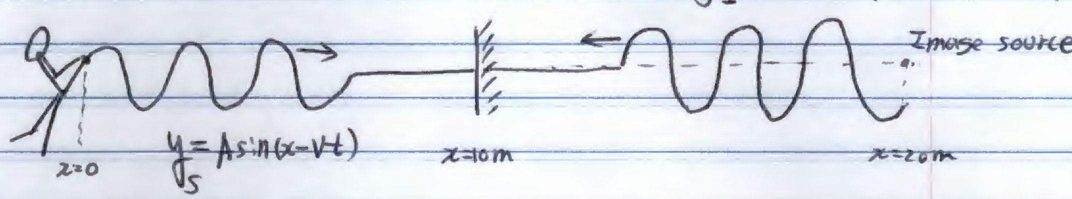
Answer: Even-n modes: $n = 2, 4, 6, \cdots$

Question: Is $n=0$ mode allowed in PBC?
what's the frequency of $n=0$ mode?

[5] Traveling wave and image source

Quiz 3 problem 1.

$y_I = A\sin((x-20m)+vt)$



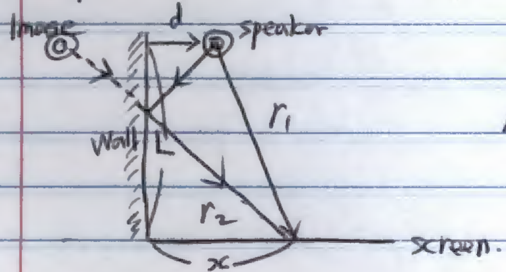
At $x=10m$, $y_S + y_I = A\sin(10-vt) + A\sin(-10+vt)$

$= 0$ ← satisfying BC at $x=10m$.

$$F_y(x=0) = -T\frac{\partial y_s}{\partial x}\Big|_{x=0} = -TA\cos(-vt)$$

$$F_y(x=10m) = -T\frac{\partial}{\partial x}(y_s+y_I)\Big|_{x=10m} = -TA\cos(10-vt) - TA\cos(-10+vt)$$
$$= -2TA\cos(10-vt).$$

Speaker in a room problem.



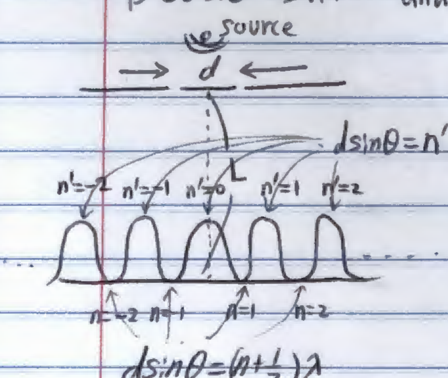
$r_1 = \sqrt{(x-d)^2+L^2}$

$r_2 = \sqrt{(x+d)^2+L^2}$

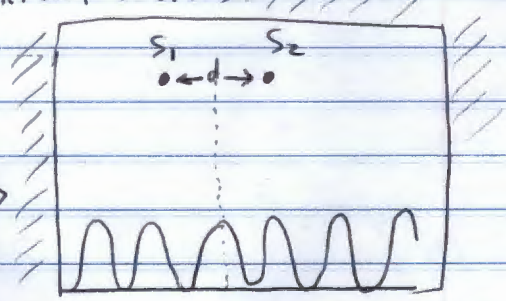
If $r_2 - r_1 = n\lambda$, destructive Interference

If $r_2 - r_1 = (n+\frac{1}{2})\lambda$, constructive 〃.

Double slit and two coherent sources



$\Longleftrightarrow$



spacing between two maxima

$$\Delta X = L\tan\theta_1 - L\tan\theta_0 \simeq \frac{L\lambda}{d}$$

If source S_1 and S_2 are coherent, this system is equivalent ~~two~~ ~~to~~ to Double slit system (left).