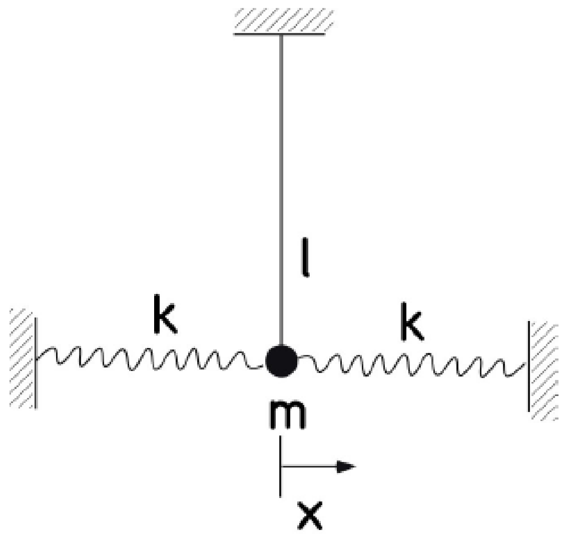
# QP 1

The drawing shows a pendulum attached to 2 springs. The pendulum moves side to side (in the plane of the paper).

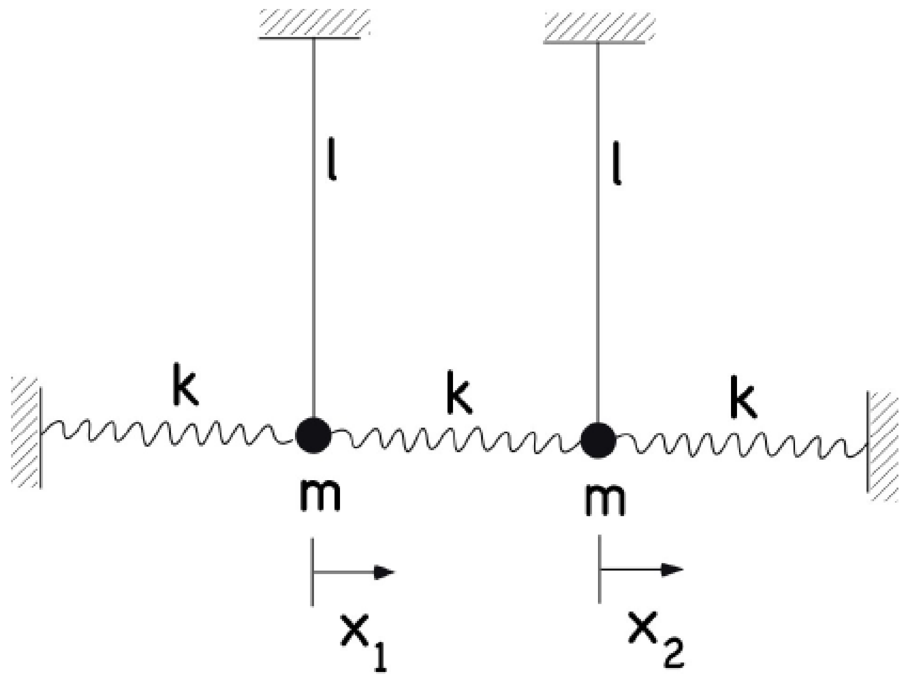



(a) [1 pt] Write down a differential equation for small displacements of the mass *m* [1]from equilibrium in the x-direction.

(b) [1 pt] What is the characteristic frequency in terms of *m, g, k, l?*

# QP 1 (continued)

Now 2 pendula are attached as shown:




(c) [2 pt] Write down the coupled linear differential equations for small displacements, $x_1$ and $x_2$, from equilibrium

(d) [2 pt] What are the two eigenfrequencies (normal mode frequencies)?

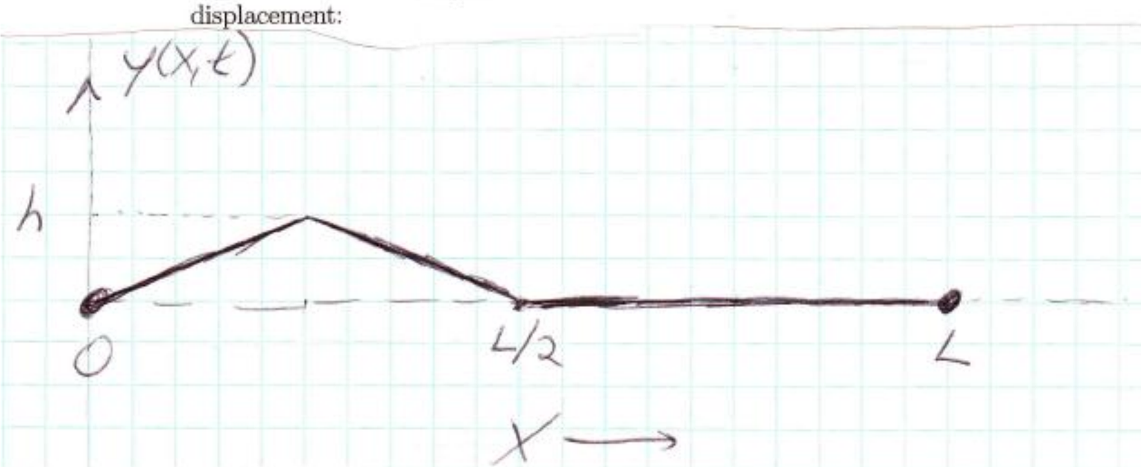
(e) [2 pt] Derive and sketch the 2 eigenfunctions (normal modes).

(f) [2 pt] If $kl = mg/10$ sketch (approximately) the motion of both masses if at t=0, $x_1=x_0$, $dx_1/dt=0$, and $x_2=dx_2/dt=0$.

# QP 2

A string with length $L$, tension $T$, and mass per unit length $\mu$ is fixed at both ends.

1. What is the angular frequency $\omega_1$ of the fundamental mode ($n = 1$)? [1 point]
2. Write down the most general series equation for the time $t$ and space $x$ dependence of the displacement $y(x,t)$ for the string. [1 point]
3. Suppose that the string is released at time $t = 0$ with the following displacement:




I.e.,

$$y(x,t=0) = \begin{cases} \frac{4xh}{L} & 0 < x < \frac{L}{4} \\ 2h - \frac{4xh}{L} & \frac{L}{4} < x < \frac{L}{2} \\ 0 & \frac{L}{2} < x < L. \end{cases} \quad (1)$$

Write down an expression for the first four coefficients $A_n$ (for $n = 1, 2, 3, 4$) in the series expansion for $y(x,t)$. (Assume the string is released from rest.) [2 points]

4. Are any of these first four coefficients zero? If so, can you think of a reason why? (Hint: Think about the symmetry of the characteristic modes.) [2 points]
5. Sketch the approximation for $y(x, t = 0)$ obtained by using only the first four terms in the series expansion. [2 points]
6. Sketch the approximation for $y(x,t)$, at $t = \pi/\omega_1$, obtained by using only the first four terms in the series expansion. [2 points]

# QP 3

Several Nobel prizes in physics have been given out for discoveries made with pulsars. A *neutron star* is an extremely dense remnant of a star, after that star has burned up all the nuclear fuel that makes it shine. It has a mass comparable to the mass of the Sun, but occupies a space about the size of Pasadena. A *pulsar* is a rapidly spinning neutron star; they may spin around up to 1000 times per second.

Pulsars emit extremely short bursts of radiation at regular intervals, every time they spin around. We see these as periodic radio signals, with a period equal to the spin period of the pulsar.

Consider now a single burst of radio emission from a pulsar at a distance $D = 3 \times 10^{20}$ m away from us. The radio signal from the pulsar has to propagate through interstellar space before reaching us. That Interstellar space is filled with a plasma, a gas of ionized atoms, with an electron density $n_e = 2 \times 10^{-8}\ \mathrm{m}^{-3}$.

As you know, electromagnetic waves propagate through a vacuum at the speed of light, but if they propagate through a medium, they may travel a bit slower. Electromagnetic waves in a plasma have a dispersion relation,

$$\omega^2 = \omega_p^2 + k^2 c^2, \tag{1}$$

between the (angular) frequency $\omega$ of the wave and its wavenumber $k$ ($= 2\pi/\lambda$, where $\lambda$ is the wavelength). Here, $\omega_p$ is the (angular) "plasma frequency" given by

$$\omega_p^2 = \frac{n_e e^2}{m_e \epsilon_0}, \tag{2}$$

where $e = 1.6 \times 10^{-19}$ C is the electron charge, $m_e = 9.11 \times 10^{-31}$ kg is the electron mass, and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m is the permittivity of the vacuum.

1. Evaluate the (angular) plasma frequency $\omega_p$ for interstellar space [2 points].

2. Write down an expression for the phase velocity $v_{\mathrm{ph}}$ in terms of the wavenumber $k$ [2 points].

# QP 3 (continued)

3. Write down an expression for the group velocity $v_g(k)$, in terms of the wavenumber $k$ [2 points].

4. Suppose the pulsar emits simultaneous pulses centered at two different (angular) frequencies: $\omega_1 = 1$ GHz and $\omega_2 = 2$ GHz (note: GHz= $10^9$ Hz). These will now propagate through interstellar space at slightly different velocities. Calculate the difference in the arrival times of these two pulses [2 points].

5. These radio signals are observed at the Arecibo Telescope, in Puerto Rico. This telescope has a diameter of 305 m. What is the angular resolution of the telescope for the $\omega = 1$ GHz pulse [2 points].

# QP4

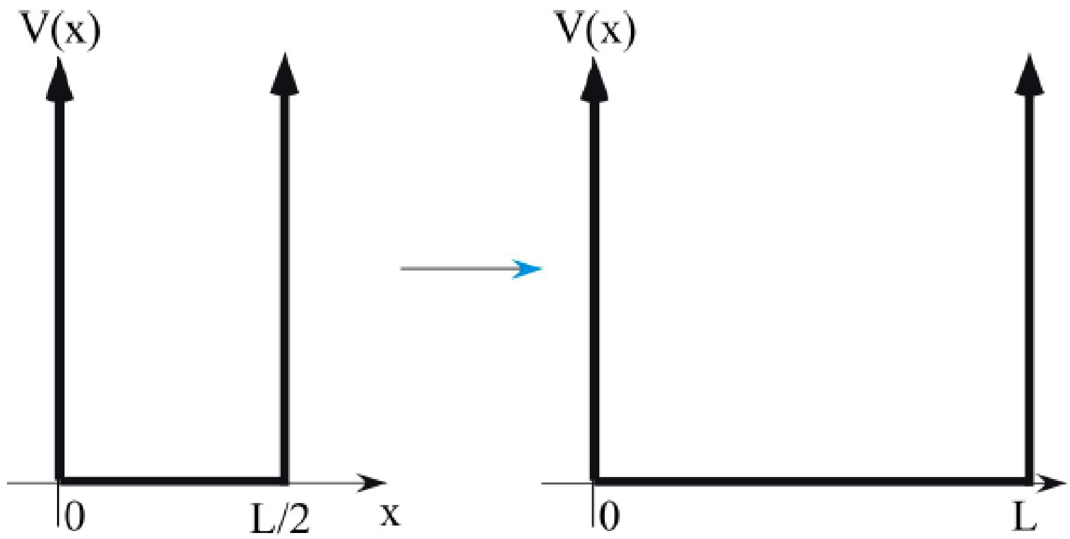



Figure 1 Figure 2

(a) (1 point) Consider a particle of mass m in the ground state of an infinite square well potential of width L/2 (Figure 1). What is the probability of detecting the particle at x=L/4 in a range of Δx=0.01L? (You do not need to integrate).

(b) (1 point) Assume that the particle is in the (properly normalized) state $\Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x)$ at time t=0 (a superposition of the ground state and first excited state) . What is the time dependent wave function $\Psi(x,t)$?

(c) (1 point) What is the expectation value of the Energy $\langle E \rangle$?

(d) (2 points) Now the wave function is returned to the ground state of part (a). At t=0 the well suddenly changes to an infinite square well of width L without affecting the wave function (Figure 1 → Figure 2). Find the probability that a measurement of energy just after the expansion will yield the ground state energy of the expanded square well:

$$E_1 = \frac{\hbar^2\pi^2}{2mL^2}$$

# QP5

Consider the following time-independent, one-dimensional wavefunction $\Psi(x)$ with $x_o$ a constant and $b$ *is* an integer with $b > 1$. Only consider positive values of x (x>0).

$$\Psi(x) = A\left(\frac{x}{x_o}\right)^b \exp(-x/x_o)$$

**(a) [3 pts]** Find $A$ through normalization, you may assume it is real and positive. The following integral may be useful in this problem.

$$\int_0^\infty x^n \exp(-ax)dx = \frac{n!}{a^{n+1}}$$

**(b) [3 pts]** Find $\langle x^m \rangle$, where $m$ is an integer $m \geq 1$.

**(c) [3 pts]** Use this result to calculate and simplify $\Delta x$.

**(d) [6 pts]** Use the Schrodinger Equation to find the potential associated with the above eigenfunction for x>0. Find the value of energy $E$ such that $V(x) \to 0$ as $x \to \infty$.