QP 1

Consider 3 particles, all of equal mass, in a 1D Harmonic Oscillator potential with combined total energy $E_{TOT} = 3\hbar\omega$.[1] A specific *configuration* of the system is specified by $(N_0, N_1, N_2, \ldots)$ where N_n is the number of particles with energy $E_n = n\hbar\omega$. In the case of distinguishable particles there are multiple *microstates* specified by (n_1, n_2, n_3), the set of 3 quantum numbers *n* for each particle.

In all of the following, you need not write down any wavefunctions. You simply need to enumerate the allowed *configurations* $(N_0, N_1, N_2, \ldots)$ and each of the distinct combinations of quantum numbers (n_1, n_2, n_3) that correspond to each *configuration.* Assume for all parts that the particles are *in equilibrium.*

Assume for parts (a)-(d) that the particles are *distinguishable.* (a) (4 points) List the possible configurations $(N_0, N_1, N_2, \ldots)$ for the system (remember there is a fixed total Energy), and the multiplicity (number of microstates (n_1, n_2, n_3)) for each configuration. Hint: You will find a total of 3 configurations.

(b) (1 point) What is the most probable configuration?

(c) (3 points) If you picked a particle at random and measured *E*, what values might you get? What is the probability you would measure $E = \hbar\omega$? What is the probability you would measure $E = 0$? What is the most probable single particle energy?

(d) (1 points) Now assume that the particles are identical fermions, ignoring spin (i.e., assume that there is only one particle allowed in each spatial eigenmode of the harmonic oscillator potential). What is the number of distinct microstates (multiplicity) for each configuration?

(e) (1 points) Now assume that the particles are identical bosons. What is the number of distinct microstates (multiplicity) for each configuration.

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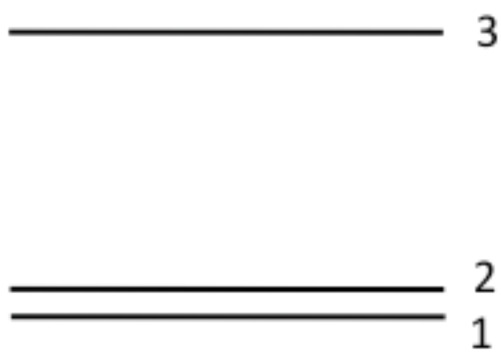
[1] Note that this is not correct quantum mechanically, since it ignores the zero point energy, but is consistent with the Kittel & Kromer notation.

QP 2

The Big Bang filled the universe with "cosmic background" radiation that acts as a thermal bath (reservoir). The bath temperature is 3K.

Imagine that an atom exists today (T_0=3K) with 3 energy levels like this:



The energy spacing between the lowest two levels is δ. The energy spacing between the top two levels is $\Delta=1000\delta$. Assume that the atom is always in thermal equilibrium with the universe and by chance the temperature of the universe today is almost exactly the energy difference d. ($\tau_0=\delta$ or $T_0=\delta/k$).

(a) [2 pts] Write an expression for the probability of finding the atom in the highest level.

(b) [3 pts] What is the (approximate) average energy of the atom?

(c) [2 pts] If there is a cloud of these atoms, we can measure the relative number in level 1 (n_1) and level 2 (n_2) by observing absorption rates of the atoms using a background source of light (a distant quasar perhaps). **What value do we predict for the ratio of level populations** n_2/n_1 **?**

(d) [3 pts] Because the universe is expanding, at earlier times the temperature of the Cosmic Background was much higher. **At what temperature T' was the probability for finding an atom in level 3 1/6?** Make reasonable approximations.

QP3

Consider a system of N in thermal equilibrium with a reservoir at temperature τ. Treat the atoms as localized and distinguishable objects with internal energy states but no translational motion and no interactions with the other atoms. Each atom has the following non-degenerate energy spectrum: (i) A ground state with $\varepsilon_0 = 0$; (ii) A first excited state with $\varepsilon_1 = \Delta$; (iii) A second excited state with $\varepsilon_2 = 2\Delta$.

(a) (2 pts) Evaluate the one-particle partition function, Z_1.

(b) (1 pts) Evaluate the N-particle partition function_Z_N

(c) (1 pts) What is the N-particle Helmholtz free energy, $F(\tau)$?

(d) (2 pts) What is the entropy of the N-particle system?

(e) (2 pts) What is the energy of the N-particle system $U(\tau)$?

(f) (2 pts) What is the energy in the limits $\tau \to 0$, $\tau \to \infty$ and $\tau \to -\infty$?

QP4 **Surface Temperature of Venus**

In problem 4.5 in K&K, you calculated that the temperature of the Earth was 280K by assuming that the Earth was a spherical, isothermal blackbody in radiative equilibrium with the Sun. In this problem, we will try to understand the surface temperature of Venus. Venus is different from the Earth in 3 respects that are important for this calculation: (i) it is only 70% as far from the Sun. (ii) its thick clouds reflect 77% of all incident sunlight and (iii) its atmosphere is much more opaque to the infrared thermal emission from the planet/s surface (the so-called "green-house" effect).

a) (2 points) Calculate the temperature that Venus would have in the absence of any atmosphere, approximating it as an isothermal, spherical blackbody.

b) (1 point) Calculate the temperature again, now taking into account the reflectivity of the clouds (but not the green-house effect of the atmosphere).

c) (2 points) Now model (crudely) the green-house effect as a spherical shell below the layer of the clouds and above the surface of the planet that is perfectly transparent to sunlight but that is perfectly black to thermal emission from the surface of the planet. Calculate the equilibrium temperature of the shell and of the planet below. (*Hint: The shell is heated by the emission from the planet below, and re-radiates in two directions: back towards the planet and out to space. Assume that the area of the shell and of the planet are equal.)*

d) (2 points) The model in part (c) corresponds to an optical depth to thermal radiation of unity. The real optical depth of the atmosphere of Venus is much higher. Derive an expression for the temperature of the planet if it is surrounded by N identical shells that are thermally isolated from one another except for the exchange of thermal radiation. How many shells are necessary to give a surface temperature close to the true value of 740K? The number of shells is a good estimate of the optical depth of the atmosphere to thermal radiation. *(Hint: work out what happens for 2 and 3 shells – the pattern will be clear).*

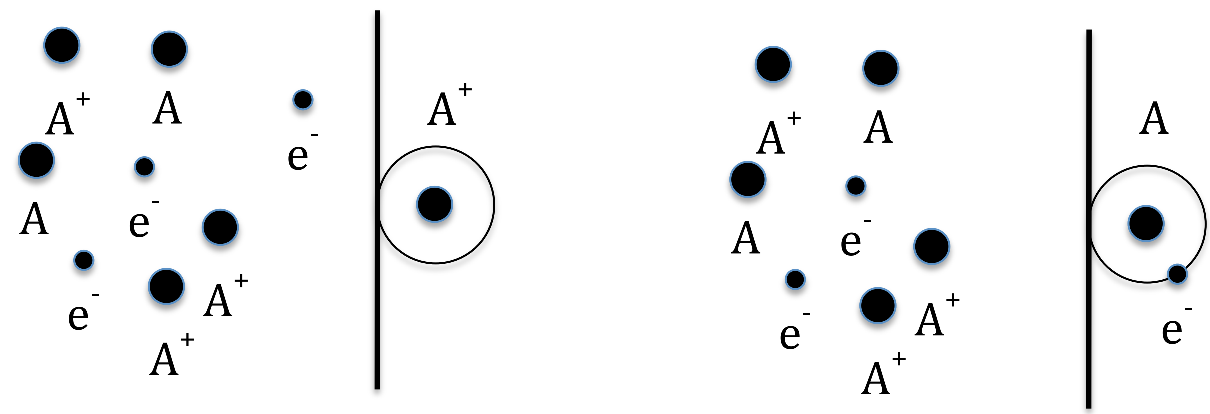
e) (1 point) What temperature would Venus be if it orbited the sun at the same distance as the Earth?

f) After you finish this quiz, and for the remainder of your life, think hard about ways to reduce carbon emissions, which raise the optical depth of the Earth's atmosphere to thermal radiation.

QP5 -- Deriving the Saha equation

Let's use the Gibbs Sum to determine the ionization balance in a hot plasma in the Sun. Let's consider ion A^+ and its neutral counterpart A. The ionization energy is I=10 eV. We consider that electrons are in an ideal gas in equilibrium with A and A^+, and that the plasma is neutral (on average).

(a) A single ion in the system can be considered in 2 states: unoccupied (no electron) and occupied.



Unoccupied Occupied

Write down the Gibbs sum (in both cases for electrons) in terms of the chemical potential for electrons μ and I. Derive the ionization ratio $\frac{N_{A^+}}{N_A}$ in terms of μ, τ and I.

(b) Now use the fact that the electrons are in an ideal gas to show that

$$\frac{n_{A^+} n_e}{n_A} = \left(\frac{M_e \tau}{2\pi\hbar^2}\right)^{3/2} e^{-I/\tau}$$

(c) The center of the sun has a pressure and temperature of 3 x 10^{17} dynes cm^{-2} and 1.5 x 10^7 K while the surface has pressure and temperature of 10^6 dynes cm^{-2} and 6000K. Estimate the ionization ratio for the two cases.

QP 6

Part 1 -- Photon Gas

For a reversable, isothermal photon gas in a cavity (a black body) from volume V to volume 2V, do the following quantities increase, decrease, or stay the same? Answer separately for each one. No calculations need to be shown, but explain your reasoning in one sentence.

* Entropy, σ
* Pressure, p
* Total number of photons, N
* Free energy, F

Part 2 -- Two black-body sheets

Two large sheets of material face each other, and are held at T_a and T_b, respectively, with $T_a < T_b$. They are placed a distance apart small compared to their dimensions. Both are well approximated by black bodies. The only source of heat transfer is radiation.

(a) Calculate the net rate of energy transfer per unit area (the flux) between the two sheets.

(b) You are given a sheet of foil that has one black side, and one reflective side (reflectance, *r*). If the foil is oriented with the shiny side toward *a*, calculate the energy flux between the two sheets.

(c) Calculate the temperature of the foil.

QP7 -- Conduction electrons from ionized impurity atom

Consider a system consisting of a single impurity atom in a semiconductor. The impurity has an "extra" electron compared to the neighboring atoms, which can easily be removed, leaving behind a positively charged ion (a "hole"). The ionization energy (energy it takes to remove the electron) is I. The ionized electron is called a conduction electron, because it is free to move through the material. The impurity is called a donor, because it can donate an electron. In this problem you must include the spin degeneracy.

(a) Assuming the conduction electrons behave like an ideal gas in the classical limit (with two spin states per particle), write their chemical potential μ in terms of the number of conduction electrons per unit volume, $N_c/V = n$, the electron mass m, and the temperature τ. Is the chemical potential positive, negative, or zero?

(b) Write a formula for the probability of a single donor atom to be ionized. Do not neglect the fact that the electron, if present, can have two independent spin states. Express your answer in terms of the temperature, ionization energy, and the chemical potential of the electron "gas".

(c) Assume that every conduction electron comes from an ionized donor atom. In this case the number of electrons is equal to the number of ionized donor atoms. Use this condition to derive a quadratic equation for N_c in terms of the number of donor atoms (N_d), thereby eliminating μ.

(d) Solve for N_c using the quadratic formula (introduce $x = N_c / N_d$, $t = \tau / I$ for convenience). What is the limit of N_c as $t \to 0$? What is the limit as $t \to \infty$? (Note that $\left(\frac{N_c}{V}\right) / n_q \ll 1$, and for small x, $\sqrt{1+x} \approx 1 + \frac{x}{2}$).

QP8 -- Relativistic Fermi Gas

The energy for relativistic electrons is $\varepsilon = pc$, where p is the electron momentum. For electrons in a cube of volume $V = L^3$ the momentum is $p = (\pi\hbar / m)n$, where n is the quantum number designating the state (just as in the non-relativistic case).

(a) Show that in the relativistic limit the Fermi energy of a gas of N electrons is:

$$\varepsilon_F = \hbar\pi c\left(\frac{3N}{\pi V}\right)^{1/3}$$

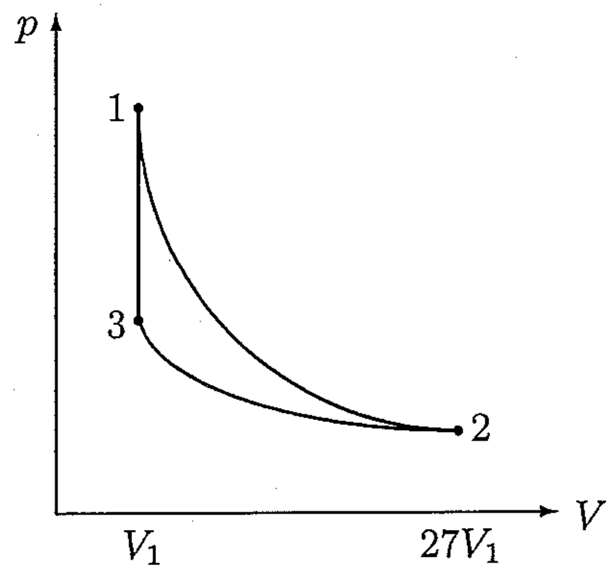
(b) Show that the total energy of the ground state of the gas is

$$U_0 = \frac{3}{4}N\varepsilon_F$$

(c) What is the chemical potential if the electron gas is in the ground state?

QP 9

5 Heat engine. **[20 points]** Consider a heat engine consisting of a monatomic, ideal gas, operating between temperatures τ_1 and τ_2. The engine begins at pressure p_1, volume V_1 and executes a three leg cycle. Along leg $1 \to 2$, the gas expands adiabatically to a volume $V_2 = 27V_1$. Along leg $2 \to 3$, the gas compresses isothermally back to the initial volume $V_3 = V_1$. Finally, along leg $3 \to 1$, the pressure increases (at constant volume) back to the initial pressure p_1.
Express all of your answers in terms of p_1 and V_1, except for the efficiencies which are to be evaluated numerically.



(a) [3 points] Find p_2V_2 (the product of the pressure and volume at point 2).

(b) [3 points] Find p_3V_3.

(c) [4 points] Find the total work W done *by* the gas over one complete cycle.

(d) [4 points] Find the heat Q_h input *to* the gas and the heat Q_l output *by* the gas over one complete cycle.

QP 9

(e) [3 points] Find the efficiency η of the engine.

(f) [3 points] Find the efficiency η_C of a Carnot cycle operating between the same two temperatures τ_1 and τ_2. Is the engine considered in this problem reversible?