**Problem 5.20** Suppose we use delta function wells, instead of spikes (i.e., switch the sign of $\alpha$ in Equation 5.57). Analyze this case, constructing the analog to Figure 5.6. This requires no new calculation, for the positive energy solutions (except that $\beta$ is now negative; use $\beta = -1.5$ for the graph), but you *do* need to work out the negative energy solutions (let $\kappa \equiv \sqrt{-2mE}/\hbar$ and $z \equiv -\kappa a$, for $E < 0$). How many states are there in the first allowed band?

For the infinite square well (Equation 2.28)

(a) Construct the completely antisymmetric wave function $\psi(x_A, x_B, x_C)$ for three identical fermions, one in the state $\psi_5$, one in the state $\psi_7$, and one in the state $\psi_{17}$.

(b) Construct the completely symmetric wave function $\psi(x_A, x_B, x_C)$ for three identical bosons, (i) if all three are in state $\psi_{11}$, (ii) if two are in state $\psi_1$ and one is in state $\psi_{19}$, and (iii) if one is in the state $\psi_5$, one in the state $\psi_7$, and one in the state $\psi_{17}$.

**Problem 5.23** Suppose you had three (noninteracting) particles, in thermal equilibrium, in a one-dimensional harmonic oscillator potential, with a total energy $E = (9/2)\hbar\omega$.

(a) If they are distinguishable particles (but all with the same mass), what are the possible occupation-number configurations, and how many distinct (three-particle) states are there for each one? What is the most probable configuration? If you picked a particle at random and measured its energy, what values might you get, and what is the probability of each one? What is the most probable energy?

(b) Do the same for the case of identical fermions (ignoring spin, as we did in Section 5.4.1).

(c) Do the same for the case of identical bosons (ignoring spin).
Problem 5.29

(a) Show that for bosons the chemical potential must always be less than the minimum allowed energy. *Hint:* \( n(\varepsilon) \) cannot be negative.

(b) In particular, for the ideal bose gas, \( \mu(T) < 0 \) for all \( T \). Show that in this case \( \mu(T) \) monotonically increases as \( T \) decreases, assuming \( N \) and \( V \) are held constant. *Hint:* Study Equation 5.108, with the minus sign.

(c) A crisis (called **Bose condensation**) occurs when (as we lower \( T \)) \( \mu(T) \) hits zero. Evaluate the integral, for \( \mu = 0 \), and obtain the formula for the critical temperature \( T_c \) at which this happens. Below the critical temperature, the particles crowd into the ground state, and the calculational device of replacing the discrete sum (Equation 5.78) by a continuous integral (Equation 5.108) loses its validity. *Hint:*

\[
\int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx = \Gamma(s)\zeta(s),
\]

where \( \Gamma \) is Euler's **gamma function** and \( \zeta \) is the **Riemann zeta function**. Look up the appropriate numerical values.

(d) Find the critical temperature for \(^4\text{He}\). Its density, at this temperature, is 0.15 gm/cm\(^3\). *Comment:* The experimental value of the critical temperature in \(^4\text{He}\) is 2.17 K. The remarkable properties of \(^4\text{He}\) in the neighborhood of \( T_c \) are discussed in the reference cited in footnote 29.

Problem 5.30

(a) Use Equation 5.113 to determine the energy density in the *wavelength* range \( d\lambda \). *Hint:* Set \( \rho(\omega) \, d\omega = \rho(\lambda) \, d\lambda \), and solve for \( \rho(\lambda) \).

(b) Derive the **Wien displacement law** for the wavelength at which the blackbody energy density is a maximum:

\[
\lambda_{\text{max}} = \frac{2.90 \times 10^{-3}}{T} \text{ mK}.
\]

*Hint:* You'll need to solve the transcendental equation \((5 - x) = 5e^{-x}\), using a calculator or a computer; get the numerical answer accurate to three significant digits.

Problem 5.31 Derive the **Stefan-Boltzmann formula** for the *total* energy density in blackbody radiation:

\[
\frac{E}{V} = \left( \frac{\pi^2 k_B^4}{15\hbar^3 c^3} \right) T^4 = \left( 7.57 \times 10^{-16} \text{ Jm}^{-3}\text{K}^{-4} \right) T^4.
\]

*Hint:* Use Equation 5.110 to evaluate the integral. Note that \( \zeta(4) = \pi^4/90 \).