Problem 5.1 Typically, the interaction potential depends only on the vector \( \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \) between the two particles. In that case the Schrödinger equation separates, if we change variables from \( \mathbf{r}_1, \mathbf{r}_2 \) to \( \mathbf{r} \) and \( \mathbf{R} = (m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2)/(m_1 + m_2) \) (the center of mass).

(a) Show that \( \mathbf{r}_1 = \mathbf{R} + (\mu/m_1)\mathbf{r}, \mathbf{r}_2 = \mathbf{R} - (\mu/m_2)\mathbf{r} \), and \( \nabla_1 = (\mu/m_2)\nabla_R + \nabla_r, \nabla_2 = (\mu/m_1)\nabla_R - \nabla_r \), where

\[
\mu = \frac{m_1 m_2}{m_1 + m_2}
\]  

is the reduced mass of the system.

(b) Show that the (time-independent) Schrödinger equation becomes

\[
-\frac{\hbar^2}{2(m_1 + m_2)} \nabla_R^2 \psi - \frac{\hbar^2}{2\mu} \nabla_r^2 \psi + V(\mathbf{r})\psi = E\psi.
\]

(c) Separate the variables, letting \( \psi(\mathbf{R}, \mathbf{r}) = \psi_R(\mathbf{R})\psi_r(\mathbf{r}) \). Note that \( \psi_R \) satisfies the one-particle Schrödinger equation, with the total mass \( m_1 + m_2 \) in place of \( m \), potential zero, and energy \( E_R \), while \( \psi_r \) satisfies the one-particle Schrödinger equation with the reduced mass in place of \( m \), potential \( V(\mathbf{r}) \), and energy \( E_r \). The total energy is the sum: \( E = E_R + E_r \). What this tells us is that the center of mass moves like a free particle, and the relative motion (that is, the motion of particle 2 with respect to particle 1) is the same as if we had a single particle with the reduced mass, subject to the potential \( V \). Exactly the same decomposition occurs in classical mechanics; it reduces the two-body problem to an equivalent one-body problem.

Problem 5.2 In view of Problem 5.1, we can correct for the motion of the nucleus in hydrogen by simply replacing the electron mass with the reduced mass.

(a) Find (to two significant digits) the percent error in the binding energy of hydrogen (Equation 4.77) introduced by our use of \( m \) instead of \( \mu \).

(b) Find the separation in wavelength between the red Balmer lines \( n = 3 \rightarrow n = 2 \) for hydrogen and deuterium.

(c) Find the binding energy of positronium (in which the proton is replaced by a positron—positrons have the same mass as electrons, but opposite charge).

(d) Suppose you wanted to confirm the existence of muonic hydrogen, in which the electron is replaced by a muon (same charge, but 206.77 times heavier). Where (i.e., at what wavelength) would you look for the “Lyman-\(\alpha\)” line \( n = 2 \rightarrow n = 1 \)?
Problem 5.4

(a) If \( \psi_a \) and \( \psi_b \) are orthogonal, and both normalized, what is the constant \( A \) in Equation 5.10?

(b) If \( \psi_a = \psi_b \) (and it is normalized), what is \( A \)? (This case, of course, occurs only for bosons.)

Problem 5.6 Imagine two noninteracting particles, each of mass \( m \), in the infinite square well. If one is in the state \( \psi_n \) (Equation 2.28), and the other in state \( \psi_l \) (\( l \neq n \)), calculate \( \langle (x_1 - x_2)^2 \rangle \), assuming (a) they are distinguishable particles, (b) they are identical bosons, and (c) they are identical fermions.

Problem 5.9

(a) Suppose you put both electrons in a helium atom into the \( n = 2 \) state; what would the energy of the emitted electron be?

(b) Describe (quantitatively) the spectrum of the helium ion, He\(^+\).

Problem 5.14 The ground state of dysprosium (element 66, in the 6th row of the Periodic Table) is listed as \( ^3I_8 \). What are the total spin, total orbital, and grand total angular momentum quantum numbers? Suggest a likely electron configuration for dysprosium.