1a) \[ U = \frac{3}{2} N k a o \left( \frac{N}{V} \right)^{2/3} \exp \left( \frac{S}{(\frac{3}{2}) N k} - \frac{5}{3} \right) \]

So, we have \( U = U(N,V) \) and we need to find \( U = U(N,V,T) \).

Since \( T = \left( \frac{2U}{3S^2} \right)_N V \),
we have \( T = a o \left( \frac{N}{V} \right)^{2/3} \exp \left( \frac{S}{(\frac{3}{2}) N k} - \frac{5}{3} \right) \).

Thus, \[ U = \frac{3}{2} N k T \]

6) Before equilibrium:  
\[
\begin{array}{c|c}
U_0 & 2U_0 \\
T_0, S_0 & T_0, S_0 \end{array}
\]

After equilibrium:  
\[
\begin{array}{c|c|c}
U_1 & U_2 \\
T_1, S_1 & T_2, S_2 & N, V
\end{array}
\]

Since \( U = \frac{3}{2} N k T \), we conclude that initial temperature of the 1st box is \( T_0 = \frac{2U_0}{3Nk} \).

Initial temperature of the 2nd box is \( T_2 = \frac{2U_0}{3Nk} \) given by

At the equilibrium, \( T_1 = T_2 \). At implies \( U_1 = U_2 \)

Since, the total energy is conserved, we have
\[ U_0 + 2U_0 = U_1 + U_2 = 2U_1 \Rightarrow U_1 = U_2 = \frac{3}{2} U_0 \]

and \( U_1 = \frac{3}{2} N k T_1 = \frac{3}{2} U_0 \Rightarrow T_1 = \frac{U_0}{Nk} \).

So, \( T_1 = T_2 = \frac{U_0}{Nk} \).
\[ U = \frac{3}{2} N k T = \frac{3}{2} N k \alpha_0 \left( \frac{\mu}{V} \right)^{2/3} \exp \left( \frac{S}{3Nk} - \frac{S}{3Nk} \right) \]

gives
\[ T = \alpha_0 \left( \frac{\mu}{V} \right)^{2/3} \exp \left( \frac{S}{3Nk} - \frac{S}{3Nk} \right) \]

\[ \ln \left( \frac{1}{\alpha_0} \left( \frac{\mu}{V} \right)^{2/3} \right) = \frac{S}{3Nk} - \frac{S}{3Nk} \]

\[ \implies S = \frac{5}{2} N k + \frac{3}{2} N k \ln \left( \frac{1}{\alpha_0} \left( \frac{\mu}{V} \right)^{2/3} T \right) = \]

\[ = \frac{5}{2} N k + \frac{3}{2} N k \ln \left( \frac{1}{\alpha_0} \left( \frac{\mu}{V} \right)^{2/3} \right) + \frac{3}{2} N k \ln T, \]

Thus, if we fix \( N \) and \( V \),

\[ \Delta S = \frac{3}{2} N k \Delta \ln T \rightarrow S^1 - S^0 = \frac{3}{2} N k \ln \frac{T^1}{T_0^1} \]

\[ \implies \Delta S^1 = \frac{3}{2} N k \ln \frac{3}{2} \text{ is an entropy change for the first box.} \]

\[ \Delta S^2 = \frac{3}{2} N k \ln \frac{T^2}{T_0^2} = \frac{3}{2} N k \ln \frac{3}{2} \text{ is an entropy change for the second box.} \]

The total change of the entropy is

\[ \Delta S = \Delta S^1 + \Delta S^2 = \frac{3}{2} N k \ln \frac{3}{2} > 0. \]
Quiz #1
Solutions

Problem #2

a) This problem is identical to two masses connected by spring, with one mass connected by a second spring to a wall. Just substitute \( I \) for \( m, c_1 \) for \( k_1, c_2 \) for \( k_2, \theta_1 \) for \( \psi_1 \), etc.

The equations of motion can be written down by inspection
\[
\begin{align*}
I \ddot{\theta}_1 &= -c_1 \theta_1 - c_2 (\theta_1 - \theta_2) \\
I \ddot{\theta}_2 &= -c_2 (\theta_2 - \theta_1)
\end{align*}
\]  
(1)
(2)

or in matrix form
\[
\frac{d^2}{dt^2} \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix} = \begin{pmatrix}
-(c_1 + c_2)/I & c_2/I \\
c_2/I & -c_2/I
\end{pmatrix} \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix}
\]  
(3)

Note that the diagonal terms are negative as required for the stability of the system. Furthermore, the constant \( c_2 \) appears in four terms as required by Newton’s third law. The constant \( c_1 \) appears only once since the relevant “spring” is attached to the wall.

b) If we guess a solution of the form \( \theta_1 = A \cos(\omega t), \theta_2 = B \cos(\omega t) \), the derivatives can be replaced by \(-\omega^2\)
\[
-\omega^2 \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix} = \begin{pmatrix}
-(c_1 + c_2)/I & c_2/I \\
c_2/I & -c_2/I
\end{pmatrix} \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix}
\]  
(4)

The term on the left can be moved to the right and added if we insert the \( 2 \times 2 \) identity matrix before \( \omega^2 \). The resulting equations
\[
0 = \begin{pmatrix}
-(c_1 + c_2)/I + \omega^2 & c_2/I \\
c_2/I & -c_2/I + \omega^2
\end{pmatrix} \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix}
\]  
(5)

have a nontrivial solution only if the determinant of the matrix vanishes
\[
0 = \det \begin{pmatrix}
-(c_1 + c_2)/I + \omega^2 & c_2/I \\
c_2/I & -c_2/I + \omega^2
\end{pmatrix}
\]  
(6)

Hence we obtain a second order equation in \( \omega^2 \) for the two normal mode frequencies of the system
\[
0 = \left( -\frac{c_1 + c_2}{I} + \omega^2 \right) \left( -\frac{c_2}{I} + \omega^2 \right) - \left( \frac{c_2}{I} \right) \left( \frac{c_2}{I} \right)
\]  
(7)

\[
0 = (\omega^2)^2 - c_1 + 2c_2 \left( \omega^2 \right) + \frac{c_1 c_2}{I^2}
\]  
(8)
The equation cannot be factored so we use the general equation for second order polynomials:

$$\omega^2 = \frac{c_1 + 2c_2}{2I} \pm \frac{\sqrt{c_1^2 + 4c_2^2}}{2I}$$ \hspace{1cm} (9)

The final solutions are

$$\omega_1 = \sqrt{\frac{c_1 + 2c_2}{2I} + \frac{\sqrt{c_1^2 + 4c_2^2}}{2I}}$$ \hspace{1cm} (10)

$$\omega_2 = \sqrt{\frac{c_1 + 2c_2}{2I} - \frac{\sqrt{c_1^2 + 4c_2^2}}{2I}}$$ \hspace{1cm} (11)