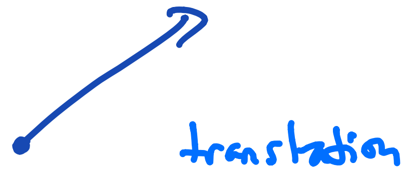
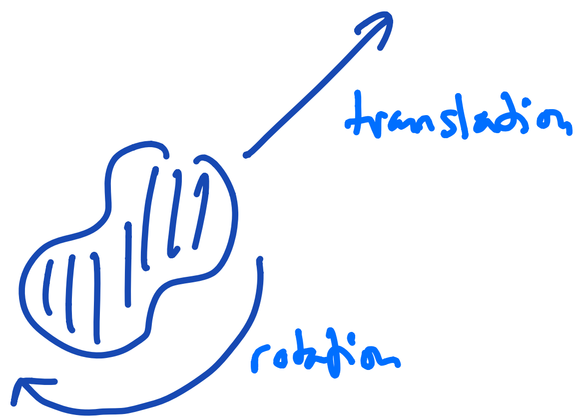
Lecture 9: Angular Momentum

Force and Torque

point particle moves one way :



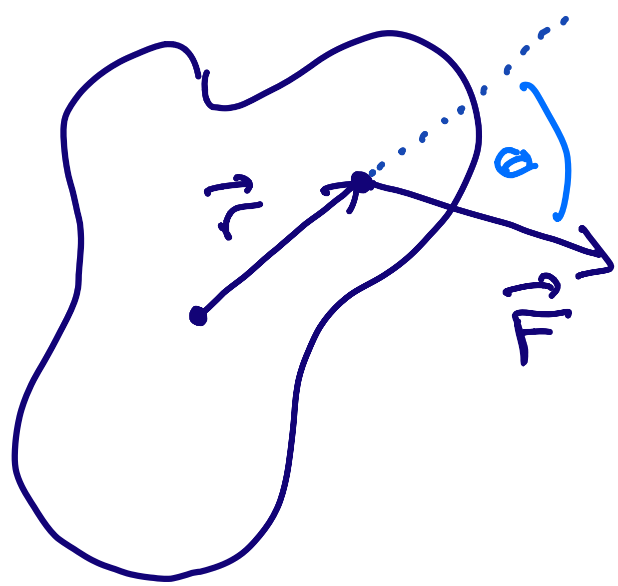
rigid extended object moves in two ways :



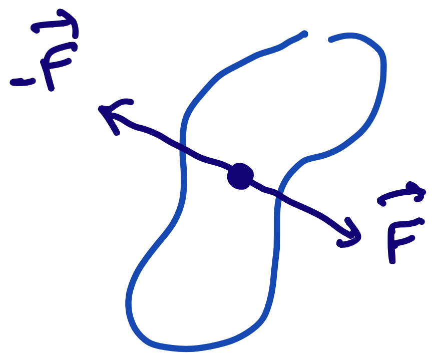
$$\vec{\tau} = \vec{r} \times \vec{F}$$

so $\tau = r F \sin\theta$

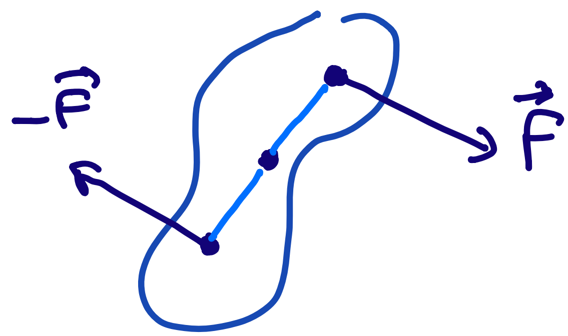
(into the page, by the right hand rule)



Bottom line: With extended objects it matters *where* the forces are exerted.

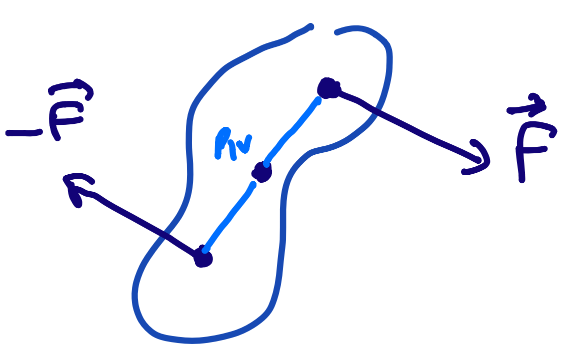


VS.

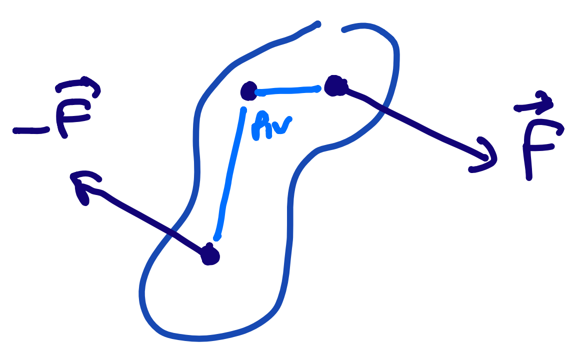


(motionless) (will rotate)

Note that for statics ($\vec{F} = \sum_i \vec{F}_i = 0$) we can measure the torques from *any* pivot point.



OR



Proof: $\vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i$

$$\Big\downarrow \text{ (shift pivot by } \vec{d}\text{)}$$

$$\sum_i (\vec{r}_i + \vec{d}) \times \vec{F}_i$$

$$=$$

$$\vec{\tau} + \sum_i \vec{d} \times \vec{F}_i$$

$$=$$

$$\vec{\tau} + \vec{d} \times \overbrace{\sum_i \vec{F}_i}^{=0}$$

Next, let us draw an analogy:

- For an object in force equilibrium,

$$0 = \sum_i \vec{F}_i = \sum_i m_i \vec{a}_i = \frac{d}{dt}\left(\sum_i m_i \vec{v}_i\right)$$

$$\sum_i m_i \vec{v}_i = \underbrace{\boxed{\sum_i \vec{p}_i = \text{const}}}_{\text{momentum conserved}}$$

- For an object in torque equilibrium,

$$0 = \sum_i \vec{\tau}_i = \sum_i \vec{r}_i \times \vec{F}_i = \sum_i m_i \vec{r}_i \times \vec{a}_i$$

$$= \frac{d}{dt}\left(\sum_i m_i \vec{r}_i \times \vec{v}_i \right)$$

$$\text{constant} = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

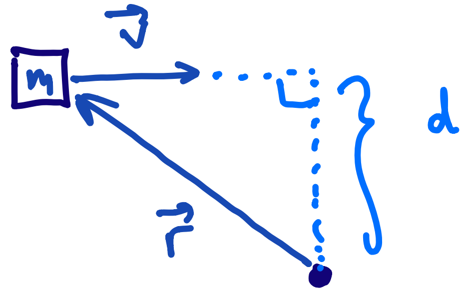
$$\left(\text{since } \frac{d}{dt}\left(\vec{r}_i \times \vec{v}_i\right) = \underbrace{\cancel{\vec{v}_i \times \vec{v}_i}}_{0} + \vec{r}_i \times \vec{a}_i \right)$$

$$\text{angular momentum} = \vec{L}_i = \vec{r}_i \times \vec{p}_i$$

$$\text{where } \underbrace{\boxed{\sum_i \vec{L}_i = \text{const}}}_{\text{angular momentum conserved}}$$

⟪⟪ demo: "spinning screw" ⟫⟫

(ex. 1) linear motion

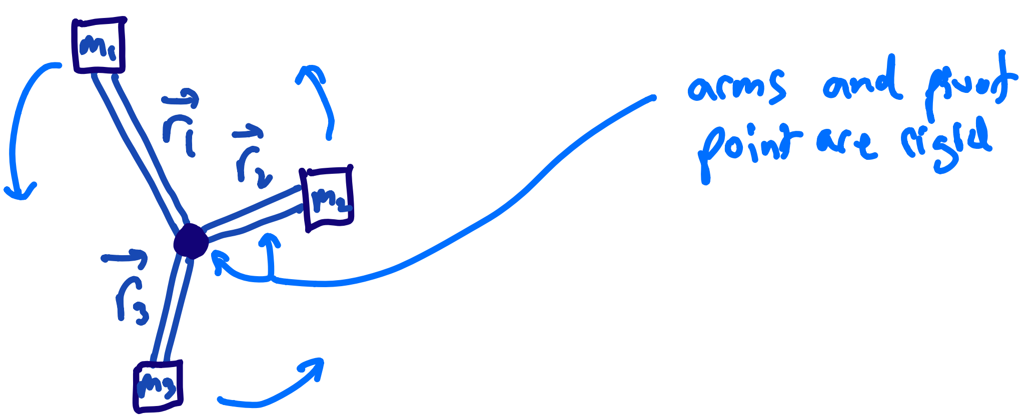


choose an arbitrary point in space as the axis of rotation

$$L = \left| m\,\vec{r} \times \vec{v} \right| = m\,d\,v = \text{constant} \;\checkmark\checkmark\checkmark$$

all constant!

(ex. 2) circular motion



$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

distance from rotation axis

$$\Rightarrow L = \sum_i m_i r_i v_i = \underbrace{\sum_i m_i r_i^2}_{\sum_i m_i r_i^2 \equiv I} \omega$$

"moment of inertia"

Moment of inertia encodes all the geometric properties of the object relevant to rotation.

《《《《《 demo: "bicycle wheel" 》》》》》