Lecture 8 : Momentum

Momentum Conservation

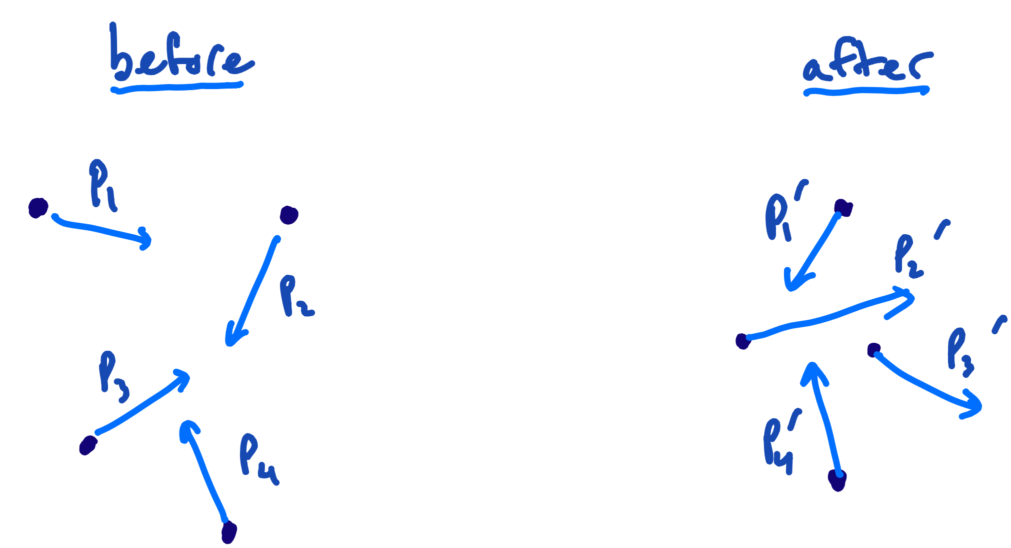
Momentum unifies and quantifies Newton's 1st (principle of inertia) and 3rd (action/reaction) laws :

momentum velocity

$$\vec{p} = m\vec{v}$$

all objects

$$\sum_i \vec{p}_i = \text{constant}$$



$$\rightarrow \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4 = \vec{p}_1{}' + \vec{p}_2{}' + \vec{p}_3{}' + \vec{p}_4{}'$$

demo: "Newton's cradle"

1st law: special case of one object,

$$\vec{p} = m\vec{v} = \text{const}$$

3rd law: special case of two objects,

$$\vec{p}_1 + \vec{p}_2 = \text{const} = \vec{p}_1{}' + \vec{p}_2{}'$$

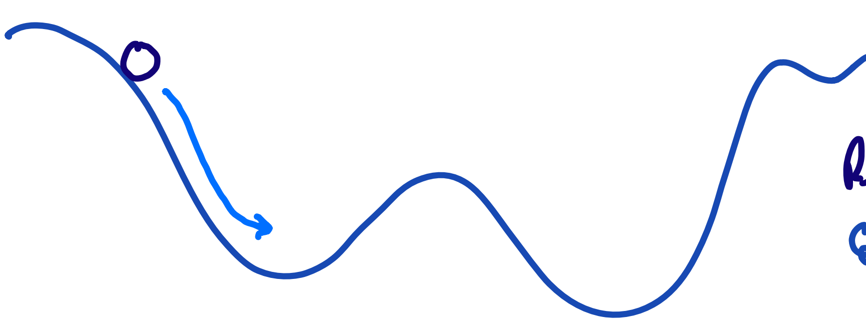
$$\rightarrow \left(\vec{p}_1 - \vec{p}_1{}'\right) = -\left(\vec{p}_2 - \vec{p}_2{}'\right)$$

$$\rightarrow -\frac{\Delta \vec{p}_1}{\Delta t} = \frac{\Delta \vec{p}_2}{\Delta t}$$

$$\rightarrow -m_1 \vec{a}_1 = m_2 \vec{a}_2$$

$$\rightarrow \boxed{-\vec{F}_{12} = \vec{F}_{21}}$$

Naively confusing: What happened to momentum = const ??



Resolution: the Earth has momentum

⟪⟪⟪ demo: "big/small bouncy ball" ⟫⟫⟫

Center of Mass

$$\bar{X} = \text{"C.o.M position"}$$

$$= \text{"where the mass is, on average"}$$

$$= \text{"mass-averaged position"}$$

$$= \frac{m_1}{M} X_1 + \frac{m_2}{M} X_2$$

where $M = m_1 + m_2$

(ex 1) $m_1 = m_2$

$$\bar{x} = \frac{x_1 + x_2}{2}$$

(ex 2) $m_1 \gg m_2$

$$\bar{x} \sim x_1$$

We also define a center of mass velocity,

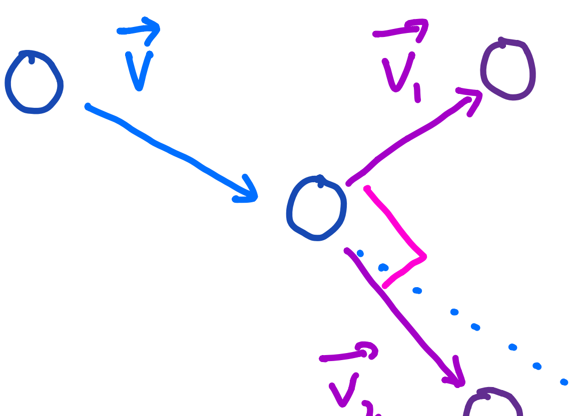
$$\overline{V} = \frac{d\overline{X}}{dt} = \frac{m_1}{M} v_1 + \frac{m_2}{M} v_2$$

$$= \frac{p_1 + p_2}{M}$$

$$\Longrightarrow \underbrace{M\overline{v}}_{\overline{P} = \text{"C.o.M. momentum"}} = p_1 + p_2 = \text{const} \quad \left(\begin{array}{c}\text{by momentum}\\ \text{conservation}\end{array}\right)$$

[Bottom line: C.o.M position of a closed system moves in a straight line]

《demo: "airtrack billiards"》



i) $\frac{1}{2} m \vec{v}^2 = \frac{1}{2} m (\vec{v}_1^2 + \vec{v}_2^2)$

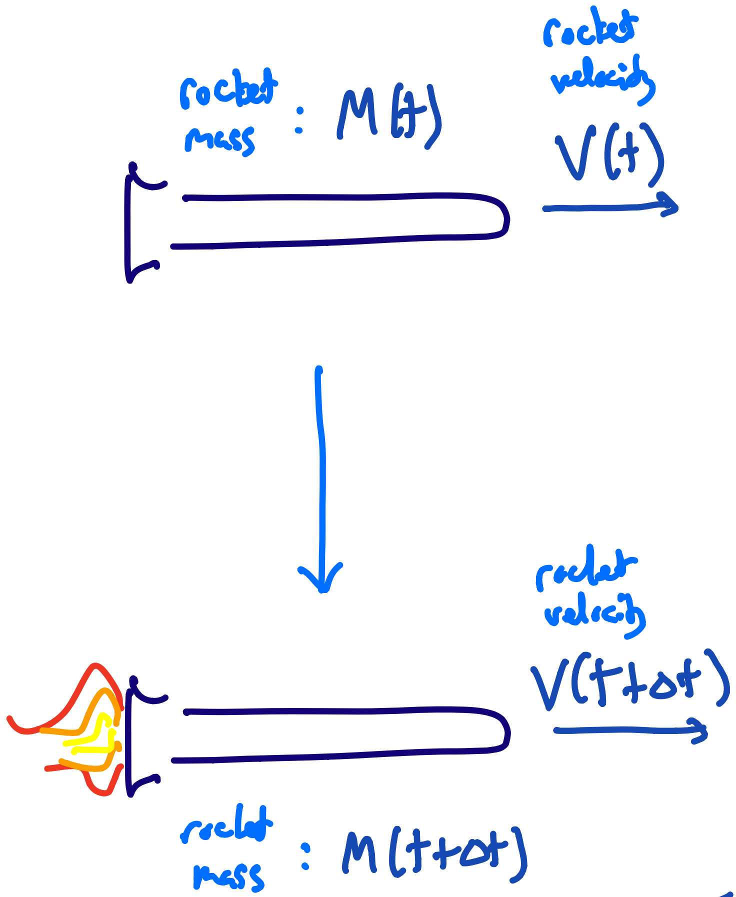
ii) $m\vec{v} = m(\vec{v}_1 + \vec{v}_2)$

$$\Downarrow$$

$$\vec{v}^2 = \vec{v}_1^2 + 2\vec{v}_1 \vec{v}_2 + \vec{v}_2^2$$

$$\Longrightarrow \vec{v}_1 \cdot \vec{v}_2 = 0$$

Rocket Science



$$P_{initial} = P_{final} = \underbrace{M(t+\Delta t)\,V(t+\Delta t)}_{\text{find momentum of rocket}}$$

$$+ \underbrace{\left[M(t) - M(t+\Delta t)\right]}_{\text{mass of exhaust}}\left[V(t) - \underbrace{V_{ex}}_{\substack{\text{velocity of exhaust in}\\ \text{rocket frame}}}\right]$$

$$\parallel$$

$$\underbrace{M(t)\,V(t)}_{\text{initial momentum of rocket}}$$

$$0 = MV\Big|_{t+\Delta t} - MV\Big|_{t} + \left(M\Big|_{t} - M\Big|_{t+\Delta t}\right)\left(V - V_{ex}\right)$$

$$\Rightarrow \frac{d}{dt}(Mv) - \frac{dM}{dt}(v - v_{ex}) = 0$$

$$\| $$

$$\cancel{\frac{dM}{dt} v} + M \frac{dv}{dt} - \cancel{\frac{dM}{dt} v} + \frac{dM}{dt} v_{ex}$$

$$\Rightarrow M \frac{dv}{dt} = -\frac{dM}{dt} v_{ex}$$

Solving the diff eq, we obtain

$$\int \frac{dv}{dt} dt = -v_{ex} \int \frac{dM/dt}{M} dt$$

$$\| \qquad\qquad\qquad \|$$

$$v(t) \qquad\qquad -v_{ex} \log M(t) + \text{const}$$

$$\boxed{v(t) = v(0) + v_{ex} \log \frac{M(0)}{M(t)}}$$

bottom line: you need to lose an exponentially high fraction of your rocket mass to increase your speed $\sim v_{ex}$.

$\Rightarrow$ rocket science is hard.