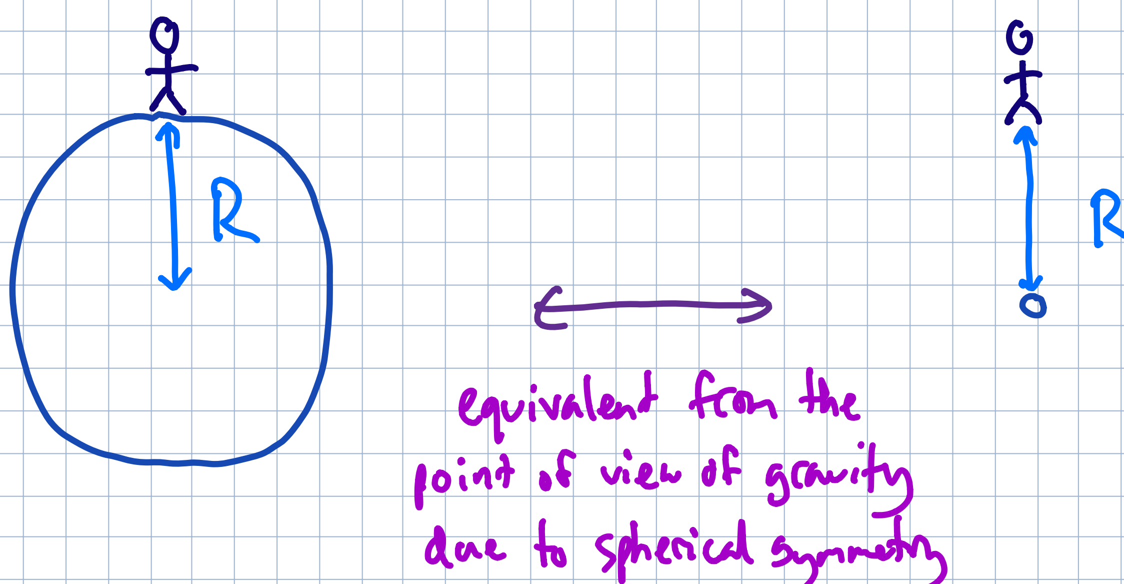
Lecture 5: Gravity and Circular Motion

Newton's huge insight: the laws that govern falling apples also govern the moon

Let's derive $g = 9.8\ m/s^2$.



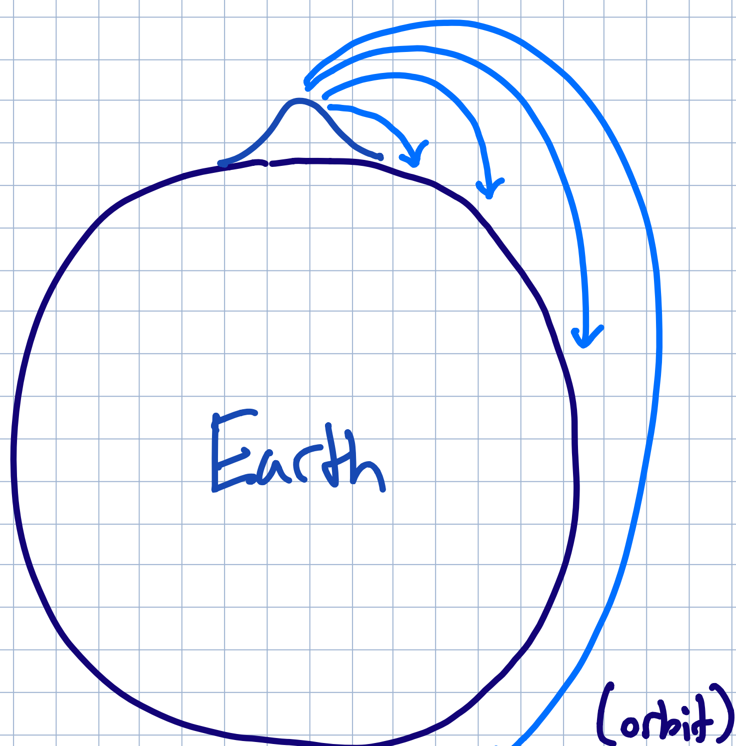
$$F = -\frac{G\,m\,M}{R^2} = -mg$$

effectively constant because the Earth's mass and our distance from Earth's center do not change very much

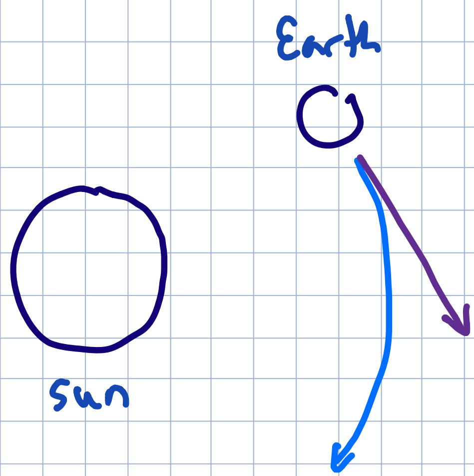
$$\Rightarrow \quad g = \frac{GM}{R^2} = \frac{\left(6.7\times 10^{-11}\ m^3/kg\,s^2\right)\left(6.0\times 10^{24}\ kg\right)}{\left(6.4\times 10^{6}\ m\right)^2}$$

$$\sim 9.8\ m/s^2 \quad \checkmark\checkmark\checkmark$$

Conversely, the moon is falling but missing the Earth.



Circular Motion



something is pushing the Earth off a straight-line course

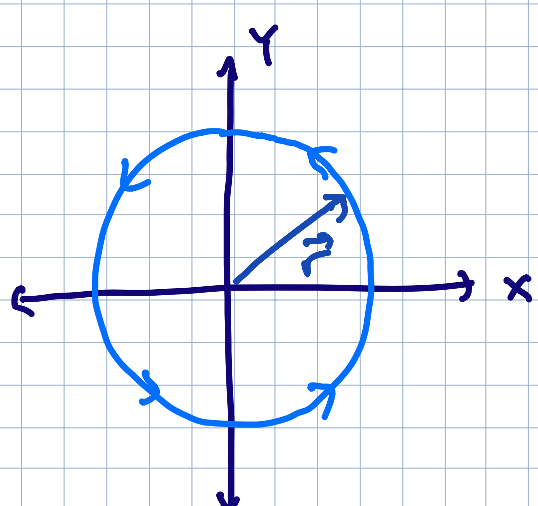
Let's deduce the acceleration.

Consider an object in uniform circular motion:

$$\vec{r}(t) = \big(x(t),\ y(t)\big)$$

$$= \big(r\cos \omega t,\ r\sin \omega t\big)$$

angular frequency



Also, define the period,

$$T = \frac{2\pi}{\omega}$$

the time it takes for one cycle.

$$\vec{v} = \frac{d\vec{r}}{dt} = (-r\omega \sin \omega t, \ r\omega \cos \omega t)$$

Note that $\vec{v} \cdot \vec{r} = 0$.

Also, $v = \sqrt{\vec{v}^2} = \sqrt{r^2\omega^2(\cos^2\omega t + \sin^2\omega t)}$

$$\Rightarrow \boxed{v = \omega r}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = (-r\omega^2 \cos \omega t, \ -r\omega^2 \sin \omega t)$$

$$\Rightarrow \boxed{\vec{a} = -\omega^2 \vec{r}}$$

the faster the rotation,
the more the acceleration

unit vector

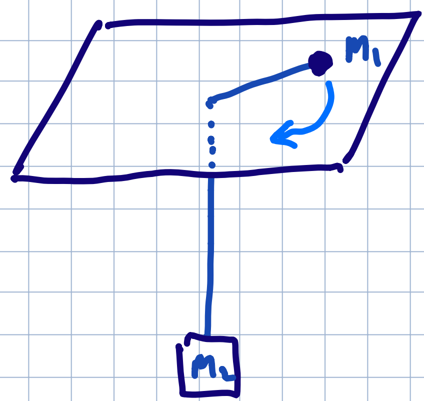
$\hat{r} \equiv \vec{r}/r$

Lifting from acceleration to force:

$$\vec{F}_{centripetal} = m\vec{a} = -m\omega^2 r \hat{r} = -\frac{mv^2}{r} \cdot \hat{r}$$

(eg 1) For this apparatus, $\vec{F}_{centripetal} = \vec{F}_{box}$

$$m_1 \omega^2 r = m_2 g$$



$$\Rightarrow \quad r = \frac{m_2 g}{m_1} \frac{1}{\omega^2} = \frac{m_2 g}{m_1} \left(\frac{T}{2\pi}\right)^2$$

so $\boxed{r \propto T^2}$

《《《《《 demo: "swinging flail" 》》》》》

(eg 2) For the Earth, $\vec{F}_{centripetal} = \vec{F}_{grav}$

$\downarrow$

mass of Earth

mass of sun

$$-m\omega^2 r \hat{r} = -\frac{GmM}{r^2} \hat{r}$$

bottom line: rotation speed is fixed by radius

$$\Rightarrow \quad r^3 = \frac{GM}{\omega^2} = GM\left(\frac{T}{2\pi}\right)^2$$

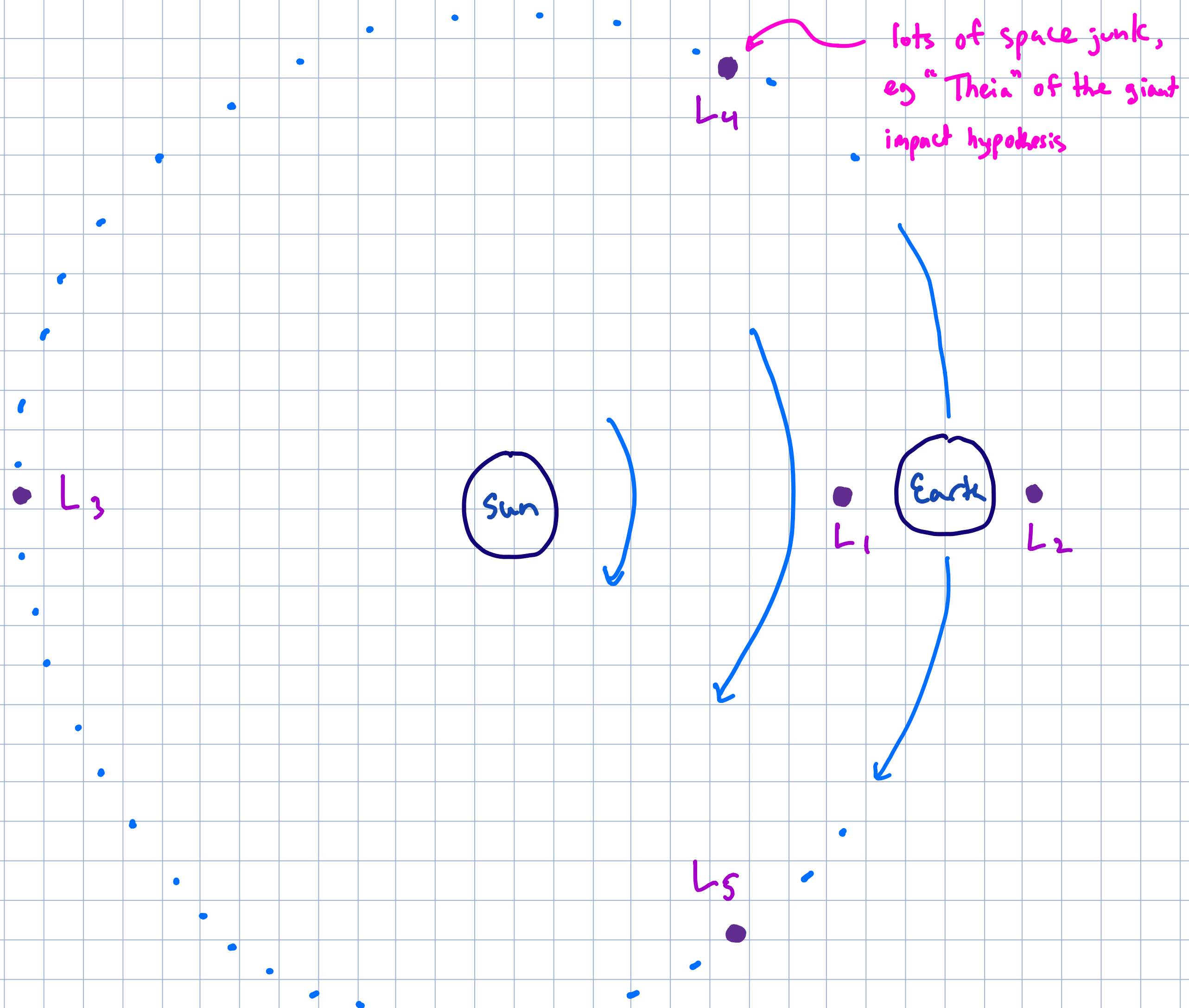
$\Rightarrow$ $\boxed{\text{Kepler's 3rd Law : } r^3 \propto T^2}$

$$T^2 \propto r^3/G$$

farther out, longer period

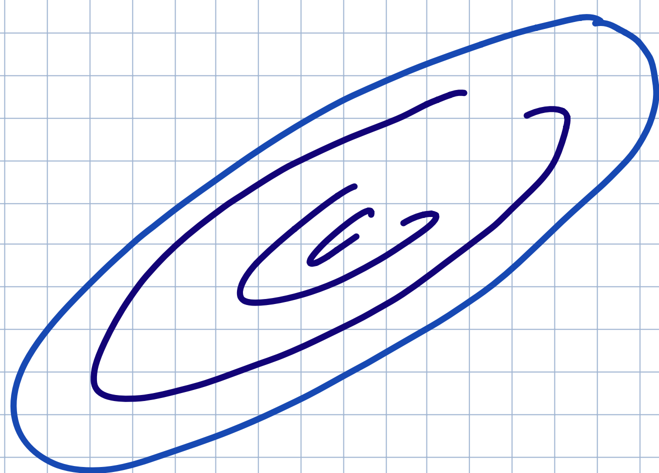
weaken the sun's gravity, longer period

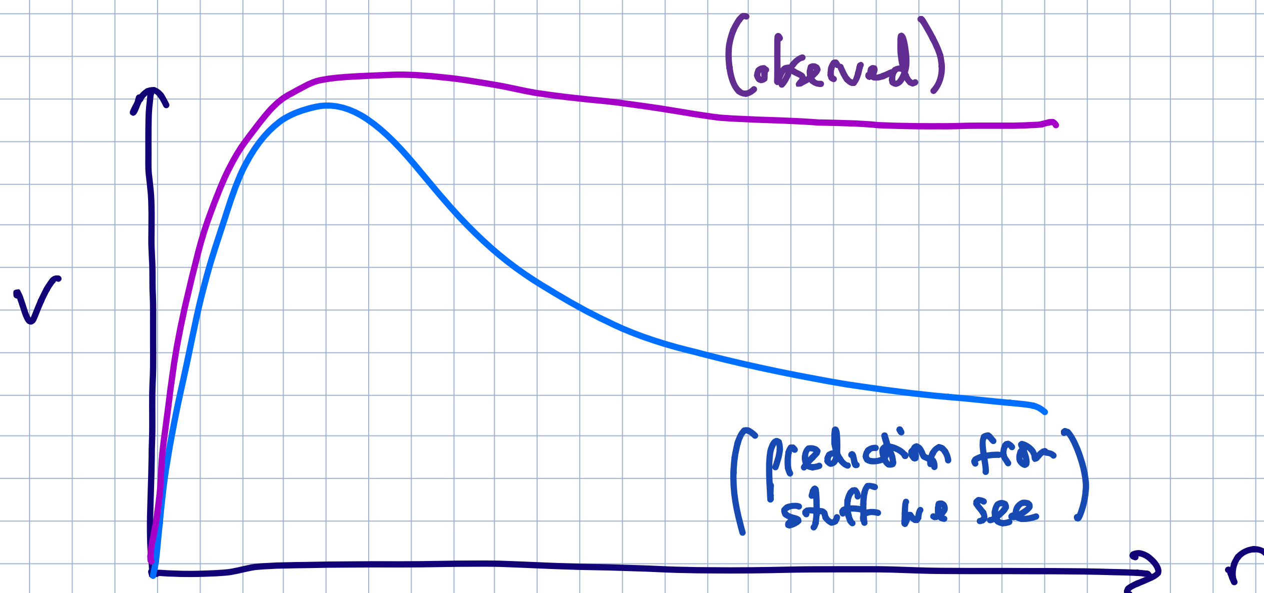
"Lagrange" points orbit the sun at the same frequency as the Earth, ie are rotationally synchronized.



This kind of logic can be applied to galaxies...

.... and it fails!!!





$\Longrightarrow$ gravitational proof of dark matter

Mystery of Universe #2: "What is dark matter?"