Lecture 16: Keplers Laws

Previously, we derived a formula for orbital motion:

$$r(\theta) = \frac{L^2}{Gm^2M} \frac{1}{1 + e\cos\theta}$$

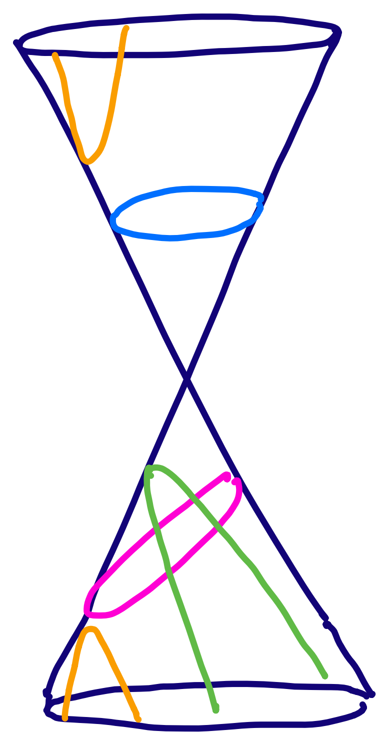
You should recognize this as a general conic section.

$e = 0$: circle

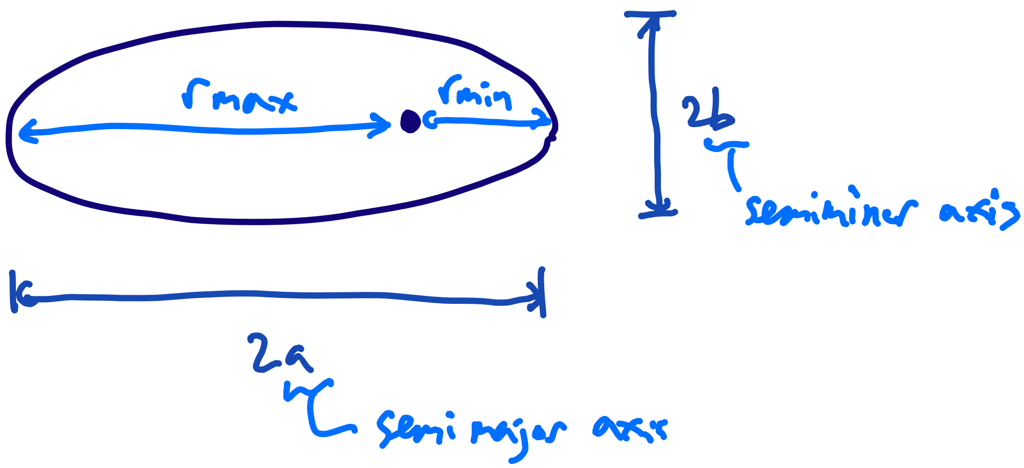
$0 < e < 1$: ellipse

$e = 1$: parabola

$e > 1$: hyperbola



Elliptical Orbits



$$r_{min} = r(\theta = 0) = \frac{\text{const}}{1+e}$$

$$r_{max} = r(\theta = \pi) = \frac{\text{const}}{1-e}$$

$$2a = r_{min} + r_{max} = \text{const}\left(\frac{1}{1+e} + \frac{1}{1-e}\right)$$

$$= \text{const} \times \frac{2}{1-e^2}$$

$$\Rightarrow \text{const} = a(1-e^2)$$

so $$\boxed{r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}}$$

Next, we compute the geometric properties of the orbit from energy and angular momentum.

by inspection: $a(1-e^2) = \dfrac{L^2}{Gm^2M}$

Now use the energy, so

$$E = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

$$= \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + \underbrace{\left(r\frac{d\theta}{dt}\right)^2}_{=\left(\frac{L}{mr}\right)^2}\right] - \frac{GmM}{r}$$

E and L are constants of motion. So for convenience, let us compute them at $\theta = 0$, where $\frac{dr}{dt} = 0$.

$$E = \frac{1}{2}m\left(\frac{L}{mr_{min}}\right)^2 - \frac{GmM}{r_{min}} \quad \left(\text{recall that } r_{min} = \frac{L^2}{GmM^2}\frac{1}{1+e}\right)$$

$$= \left(\frac{GmM}{L}\right)^2 m\left[\frac{1}{2}(1+e)^2 - (1+e)\right]$$

$$= \left(\frac{GmM}{L}\right)^2 \frac{m}{2}\left(e^2 - 1\right)$$

$$\Rightarrow \boxed{e = \sqrt{1 + \frac{2EL^2}{G^2 m^3 M^2}}}$$

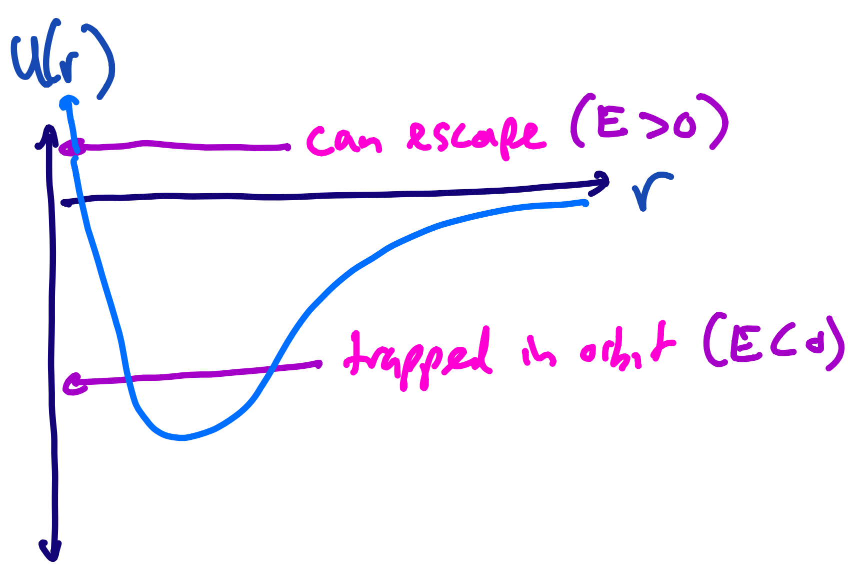
and from before,

$$a = \frac{L^2}{Gm^2 M} \frac{1}{1-e^2}$$

$$= \frac{L^2}{Gm^2 M}\left(-\frac{m}{2E}\right)\left(\frac{GmM}{L}\right)^2$$

$$\Rightarrow \boxed{a = -\frac{GmM}{2E}}$$

Note the relation between eccentricity and escape velocity



bounded orbits $(E<0)$: $e<1$ (ellipse)

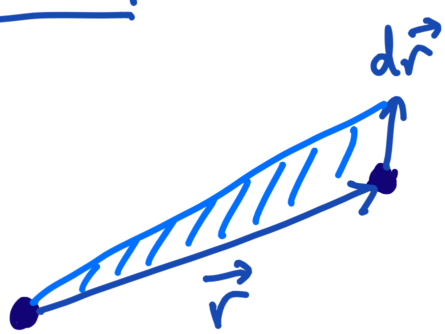
unbounded orbits $(E>0)$: $e>1$ (hyperbola)

Kepler's Laws

i) planetary orbits are elliptical ✓✓✓

ii) orbit sweeps out equal areas during equal time intervals

$\boxed{\text{Proof:}}$



$$dA = \overbrace{\frac{1}{2}\left|\vec{r} \times d\vec{r}\right|}^{\text{definition of area of triangle}}$$

$$=$$

$$\frac{1}{2}dt\left|\vec{r} \times \vec{v}\right|$$

$$=$$

$$\frac{1}{2}dt\left|\vec{r} \times \left(\cancelto{0}{\frac{dr}{dt}\hat{r}} + r\frac{d\theta}{dt}\hat{\theta}\right)\right|$$

$$dA = \frac{1}{2}\, dt\, r^2 \dot{\theta}$$

$$\Downarrow$$

$$\boxed{\dot{A} = \frac{1}{2}\frac{L}{m} = \text{const}} \quad \checkmark\checkmark\checkmark$$

iii) $(\text{orbital period})^2 \propto (\text{semimajor axis})^3$

$$\Longrightarrow \quad T^2 \propto a^3$$

$\boxed{\text{Proof:}}$

$$\text{total area swept} = \int_0^T \dot{A}\, dt = \frac{1}{2}\frac{L}{m}\cdot T$$

$$\parallel$$

$$\underbrace{\pi ab = \pi a^2 \sqrt{1-e^2}}_{\text{area of ellipse}}$$

use that $a(1-e^2) = \dfrac{L^2}{G m^2 M}$

$$\Longrightarrow \quad T = \frac{2\pi m a^2}{L}\sqrt{1-e^2}$$

$$\Rightarrow \quad T^2 = \left(\frac{2\pi m a^2}{L}\right)^2 \cdot \frac{L^2}{aGm^2M}$$

$$= \frac{4\pi^2 \cancel{m^2} a^4 \cancel{L^2}}{\cancel{L^2} a G \cancel{m^2} M} = \frac{4\pi^2 a^3}{GM}$$

so $\boxed{T^2 \propto a^3}$ ✓✓✓