Lecture 15: Orbits II

Now that we have mastered polar coordinates, let us consider orbital motion from gravity.

$$m\vec{a} = \vec{F} = -\frac{GmM}{r^2}\hat{r}$$

Lets be devious and use $\frac{d\hat{\theta}}{dt} = -\hat{r}\frac{d\theta}{dt}$, so

$$m\vec{a} = -\frac{GmM}{r^2}\left(-\frac{d\hat{\theta}/dt}{d\theta/dt}\right)$$

use that $L = mr^2\frac{d\theta}{dt}$

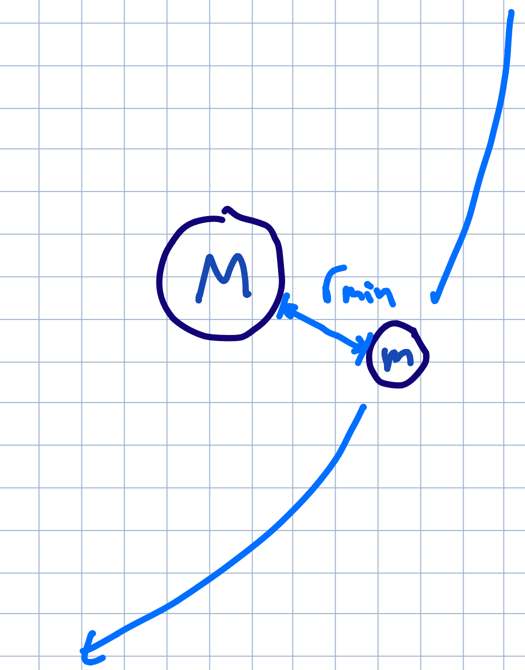
$$= +\frac{GmM}{r^2}\frac{d\hat{\theta}/dt}{L}mr^2$$

$$\Longrightarrow \vec{a} = \underbrace{\frac{GmM}{L}}_{\text{constant}}\frac{d\hat{\theta}}{dt}$$

integrate $\Longrightarrow$

$$\boxed{\vec{v} = \frac{GmM}{L}\hat{\theta} + \vec{c}}$$

constant of integration induces ellipticity



define t_0 = "time of closest approach"

such that $r(t_0) = r_{min}$

Since r_{min} is a minimum,

$$\left.\frac{dr}{dt}\right|_{t_0} = 0$$

Expanding the velocity vector,

$$\left.\left(\vec{v} = \cancelto{0}{\frac{dr}{dt}}\,\hat{r} + r\frac{d\theta}{dt}\hat{\theta} = \frac{GmM}{L}\hat{\theta} + \vec{c}\right)\right|_{t_0}$$

Taking the $\hat{\theta}$ component,

$$\Rightarrow r\frac{d\theta}{dt} = \frac{GmM}{L} + \overbrace{\vec{c}\cdot\hat{\theta}}^{= C\cos\theta}$$

$$\parallel$$

$$\frac{rL}{mr^2} = \frac{L}{mr}$$

$$\Rightarrow \boxed{\frac{1}{r} = \frac{Gm^2M}{L^2}\left(1 + e\cos\theta\right)}$$

rescaled and renamed c to e = "eccentricity"

this defines an ellipse / parabola / hyperbola in polar coordinates