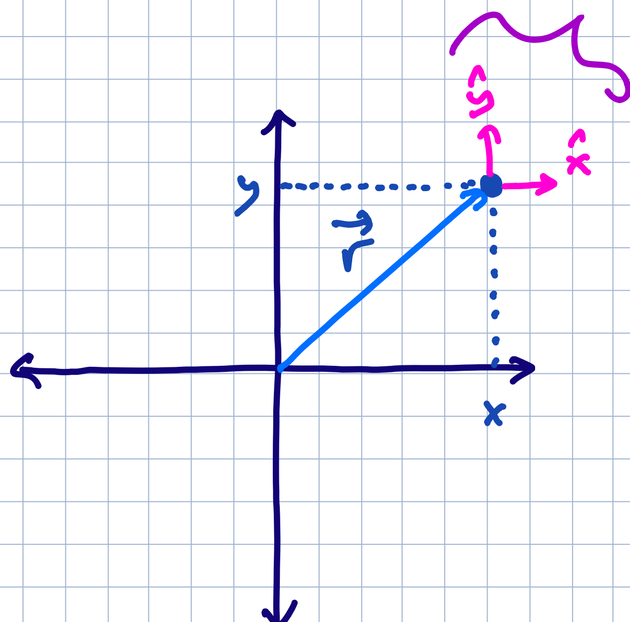
Lecture 14: Orbits I

In the next series of lectures we will discuss orbital dynamics governing the motions of planets.

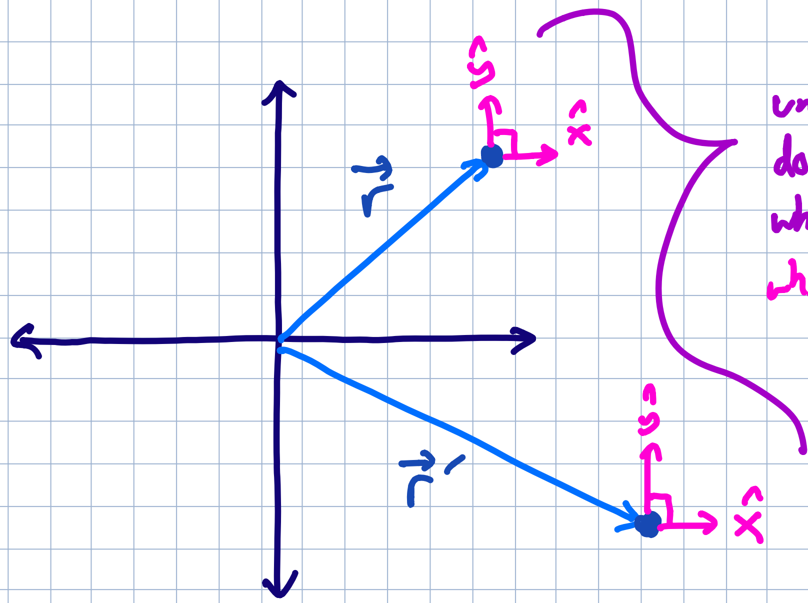
There is some mathematical heavy lifting needed. So let us review some basic facts.

- Cartesian coordinates

"unit" vectors where $\hat{x}^2 = \hat{y}^2 = 1$



$$\vec{r} = \underbrace{x}_{\text{component in } \hat{x} \text{ direction}}\hat{x} + \underbrace{y}_{\text{component in } \hat{y} \text{ direction}}\hat{y}$$



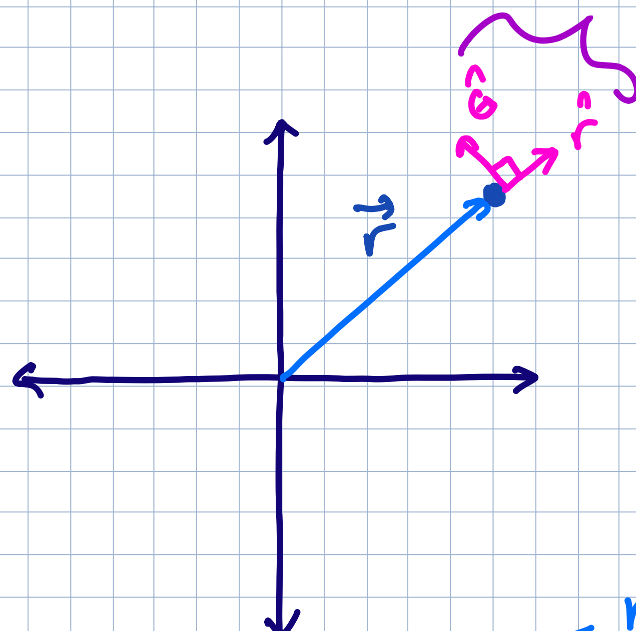
unit vectors do not depend on where you are when in Cartesian

constant

circular motion: $\vec{r}(t) = r(t)\cos\theta(t)\,\hat{x} + r(t)\sin\theta(t)\,\hat{y}$

- Polar Coordinates

"unit" vectors where $\hat{r}^2 = \hat{\theta}^2 = 1$

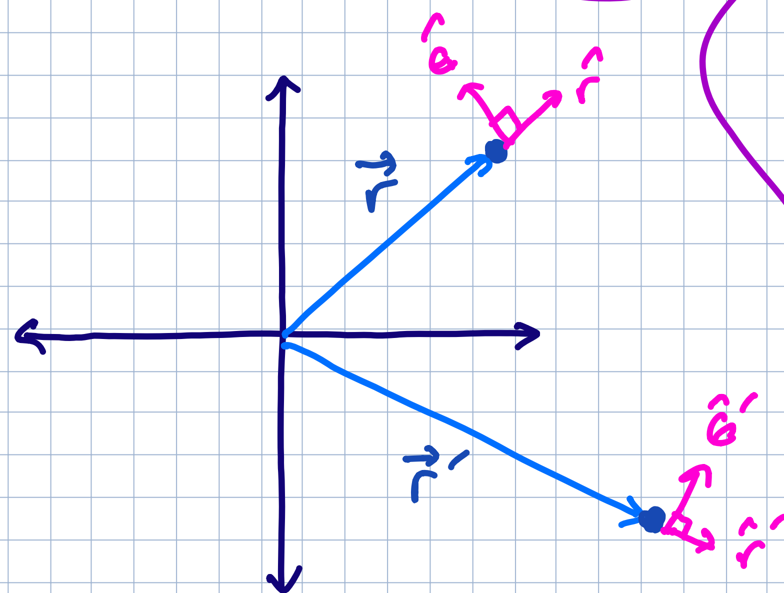


no component in $\hat{\theta}$ direction!

$$\vec{r} = r\,\hat{r}$$

The radial unit vector is conversely defined as $\hat{r} = \vec{r}/r$.

unit vectors DO depend on where you are when in polar



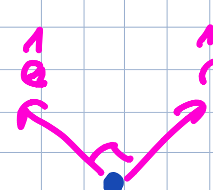
unit vector varies in t

circular motion: $\vec r(t) = r(t)\,\hat r(t)$

How does $\hat r(t)$ change with time???

equate Cartesian and polar eqs: $\cancel{r}\cos\theta\,\hat x + \cancel{r}\sin\theta\,\hat y = \cancel{r}\,\hat r$

$$\hat r = \cos\theta\,\hat x + \sin\theta\,\hat y$$



$$\hat\theta = -\sin\theta\,\hat x + \cos\theta\,\hat y$$

$$\frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} = \hat{\theta}$$

$$\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{x} - \sin\theta\,\hat{y} = -\hat{r}$$

while $\dfrac{d\hat{r}}{dr} = \dfrac{d\hat{\theta}}{dr} = 0$ (no r dependence!)

Applying the chain rule,

$$\boxed{\begin{aligned} \frac{d\hat{r}}{dt} &= \frac{d\theta}{dt}\frac{d\hat{r}}{d\theta} = \hat{\theta}\,\frac{d\theta}{dt} \\ \frac{d\hat{\theta}}{dt} &= \frac{d\theta}{dt}\frac{d\hat{\theta}}{d\theta} = -\hat{r}\,\frac{d\theta}{dt} \end{aligned}}$$

Let's return to some physical quantities.

(ex 1) velocity

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$\boxed{\vec{v} = \underbrace{\dot{r}}_{v_r}\,\hat{r} + \underbrace{r\dot{\theta}}_{v_\theta = r\omega}\,\hat{\theta}}$$

(ex 2) angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = m\,\vec{r} \times \vec{v}$$

$$= m\,\vec{r} \times \left(\cancel{\dot{r}\,\hat{r}} + r\dot{\theta}\,\hat{\theta} \right)$$

by cross prod identity

$$= m r^2 \dot{\theta} \underbrace{\left(\hat{r} \times \hat{\theta} \right)}_{= \hat{z}}$$

so $\boxed{\vec{L} = m r^2 \dot{\theta}\,\hat{z}}$