Lecture 13: Oscillatory Motion II

Energy of Oscillation

Like all physical systems, oscillators have energy.

harmonic oscillator: $\ddot{x} = -\frac{k}{m} x$

$\downarrow$ multiply by $\dot{x}$ and integrate

$$\int \left(\dot{x}\ddot{x} = -\frac{k}{m} x\dot{x} \right) dt$$

$\downarrow$

$$\frac{\dot{x}^2}{2} = -\frac{k}{m}\frac{x^2}{2} + \text{const}$$

$\downarrow$

energy conservation!

$$\underbrace{\frac{m\dot{x}^2}{2}}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}kx^2}_{\text{potential energy}} = \underbrace{\text{const}}_{\text{total energy}} = E$$

Plugging the harmonic oscillator, $X(t) = C\cos(\omega_0 t + \delta)$:

$$E_{HO} = \frac{m}{2}\left(-C\omega_0 \sin(\ldots)\right)^2 + \frac{1}{2}k\left(C\cos(\ldots)\right)^2$$

$$= C^2\left(\underbrace{\frac{m\omega_0^2}{2}}_{=\frac{k}{2}}\sin^2(\ldots) + \frac{k}{2}\cos^2(\ldots)\right)$$

$$= \frac{kC^2}{2}\left(\underbrace{\sin^2(\ldots) + \cos^2(\ldots)}_{=1}\right)$$

$$\Longrightarrow \boxed{E_{HO} = \frac{kC^2}{2}}$$

increasing amplitude increases energy

For the damped harmonic oscillator, the amplitude of oscillation falls exponentially, $\sim Ce^{-\beta t/2}$

$$\Longrightarrow \boxed{E_{DHO} \sim \frac{kC^2}{2}e^{-\beta t}}$$

friction dissipates away energy and sets arrow of time

The "Q" Factor

$$Q = \text{"quality factor"}$$

where $\frac{1}{Q}$ = fraction of energy lost in one radian of oscillation

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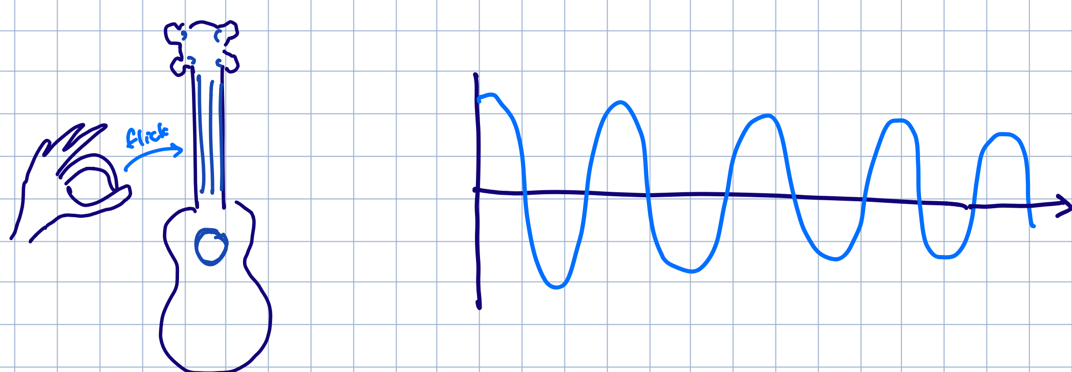
high Q (guitar string)



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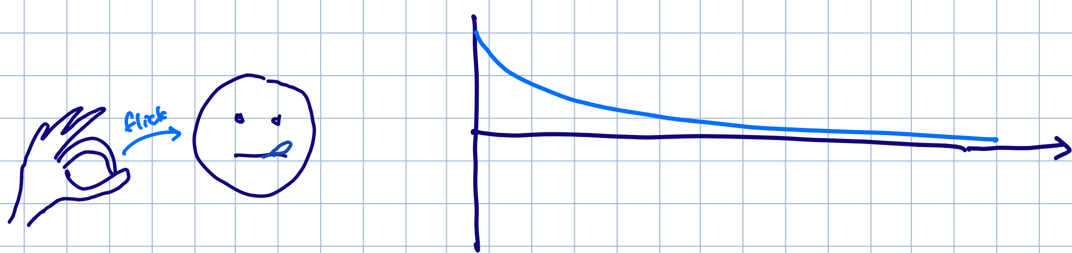
low Q (your face)



From this definition we find

$$Q = \frac{E(t)}{E(t) - E(t + \underbrace{1/\omega_0}_{\text{one radian later}})}$$

$$= \frac{e^{-\beta t}}{e^{-\beta t} - e^{-\beta(t + 1/\omega_0)}}$$

For high quality oscillators, $\beta \ll \omega_0$, so $e^{-\beta/\omega_0} \sim 1 - \beta/\omega_0$

$$\longrightarrow \quad Q \sim \frac{e^{-\beta t}}{\cancel{e^{-\beta t}} - e^{-\beta t}(\cancel{1} - \beta/\omega_0)}$$

$$\boxed{Q \sim \omega_0/\beta}$$

Driven Damped Harmonic Oscillator

$$m\ddot{x} = F = -kx - \underbrace{\gamma\dot{x}}_{\text{friction}} + \overbrace{F_0 \sin \omega t}^{\text{external drive}}$$

$$\Longrightarrow \quad \ddot{x} + \beta\dot{x} + \omega_0^2 x = a_0 \sin \omega t$$

$$a_0 = F_0/m$$

As usual, behold the power of guess and check:

frequency of drive

$$x(t) \overset{??}{=} A \sin(\omega t - \alpha)$$

$$\dot{x} = \omega A \cos(\omega t - \alpha)$$

$$\ddot{x} = -\omega^2 A \sin(\omega t - \alpha)$$

Rewrite $\sin \omega t = \sin(\omega t - \alpha + \alpha)$

$$= \sin(\omega t - \alpha)\cos\alpha + \cos(\omega t - \alpha)\sin\alpha$$

Combining terms, we obtain

$$\ddot{x} + \beta\dot{x} + \omega_0^2 x = a_0 \sin\omega t$$

$$\begin{array}{ccc} \| & & \| \\ A\sin(\cdots)\times\left\{-\omega^2+\omega_0^2\right\} & & a_0\sin(\cdots)\times\left\{\cos\alpha\right\} \\ + & & + \\ A\cos(\cdots)\times\left\{\beta\omega\right\} & & a_0\cos(\cdots)\times\left\{\sin\alpha\right\} \end{array}$$

$(\omega t-\alpha)$

$\Longrightarrow$ since $\sin(\cdots)$ and $\cos(\cdots)$ are independent:

$$A\left(\omega_0^2-\omega^2\right) = a_0\cos\alpha$$

$$A\beta\omega = a_0\sin\alpha$$

amplitude is maximized when $\omega_0=\omega$

$$\Longrightarrow \boxed{\tan\alpha = \frac{\beta\omega}{\omega_0^2-\omega^2} \quad \text{and} \quad A = \frac{a_0}{\sqrt{(\beta\omega)^2+(\omega_0^2-\omega^2)^2}}}$$

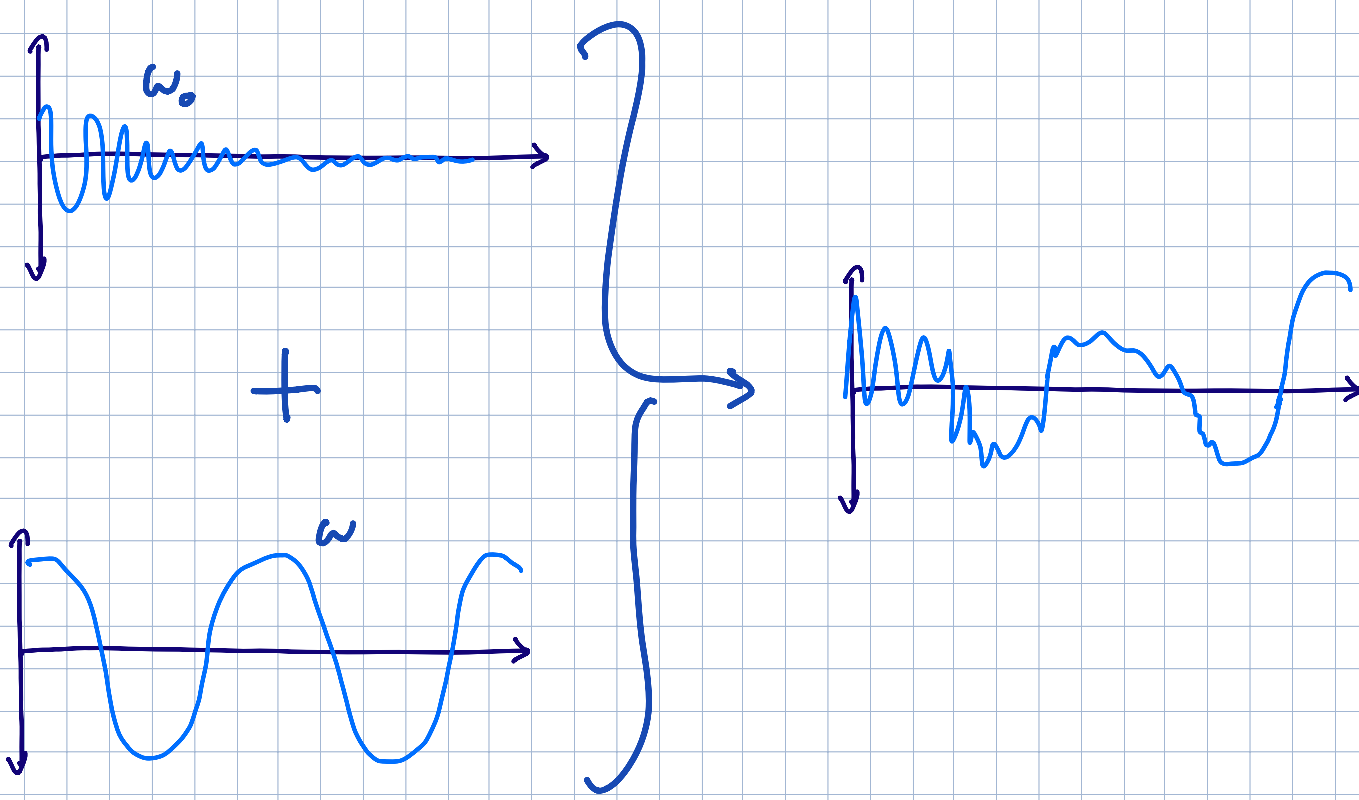
We can add this "particular" solution to the "homogeneous" solution we computed before.

$$X_{full}(t) = X_{hom}(t) + X_{part}(t)$$

$$C e^{-\beta t/2} \cos(\omega_1 t + \delta) \qquad\qquad A \sin(\omega t - \alpha)$$

fixed by initial cond   fixed by system

At late times, the $e^{-\beta t/2}$ squashes one solution.



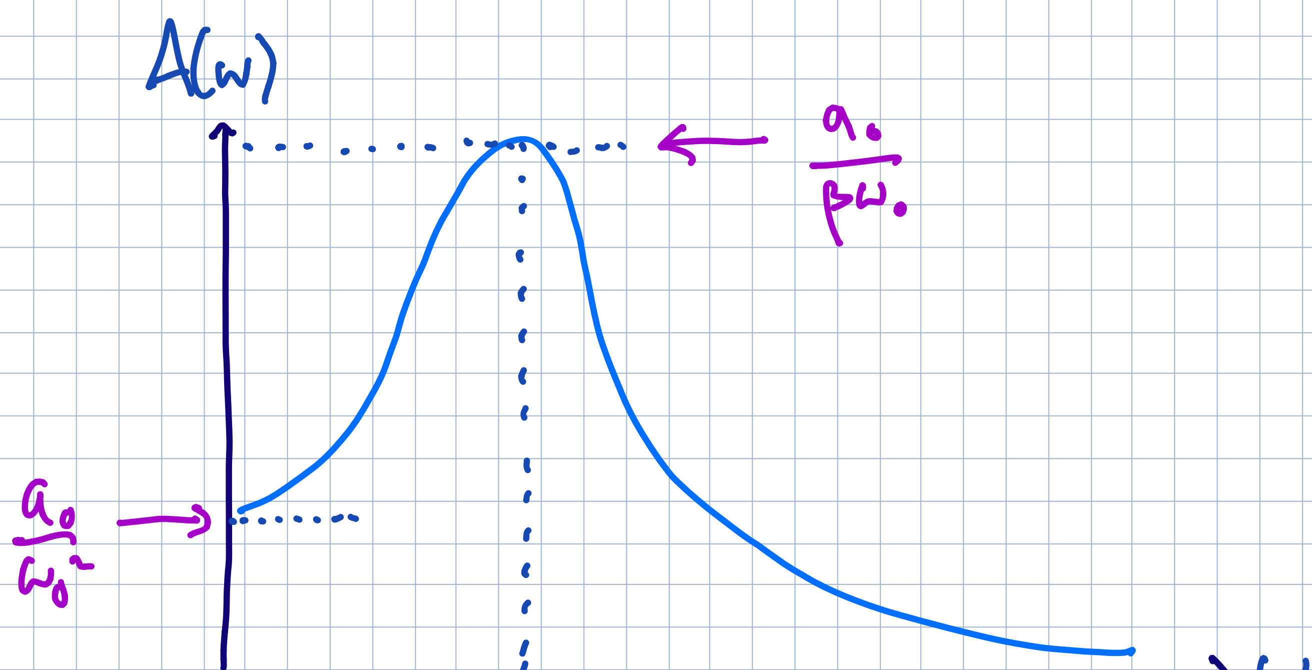
We saw before that the homogeneous solution dies off from damping while the particular solution persists.

$\Longrightarrow$ particular or "steady state" solution

$$x(t) = A \sin(\omega t - \alpha)$$

$$A = \frac{a_0}{\sqrt{(\beta\omega)^2 + (\omega_0^2 - \omega^2)^2}} \qquad \tan\alpha = \frac{\beta\omega}{\omega_0^2 - \omega^2}$$

- Consider amplitude $A(\omega)$ as a function of ω



demo: "driven pendulum"

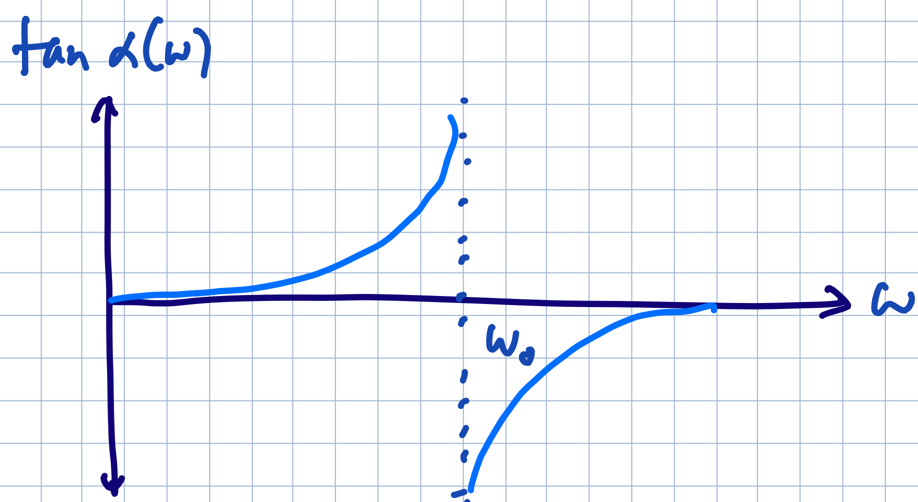
For $\beta \ll \omega_0$, the maximum is at $\omega = \omega_0$.

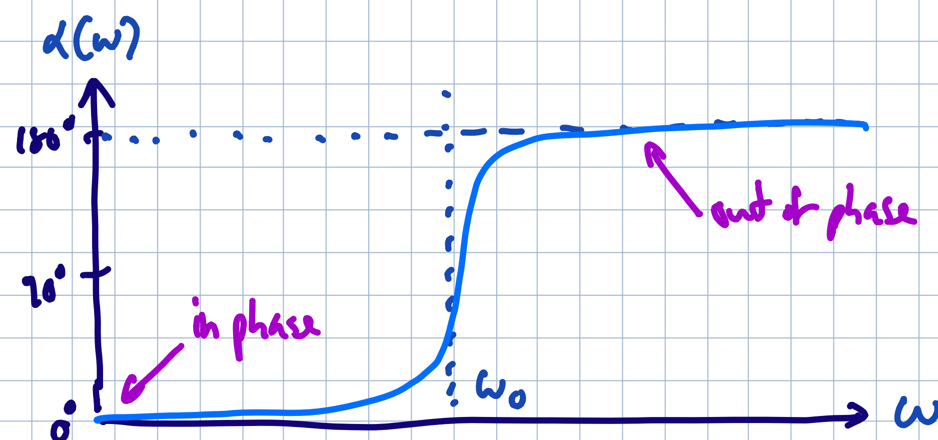
Note also that:

$$\frac{A(\omega_0)}{A(0)} = \frac{a_0/\beta\omega_0}{a_0/\omega_0^2} = \frac{\omega_0}{\beta} = Q$$

$\Longrightarrow$ $Q \sim$ height of resonance peak

- Consider phase lag $\alpha(\omega)$ as a function of ω





demo: "shattering wine glass"