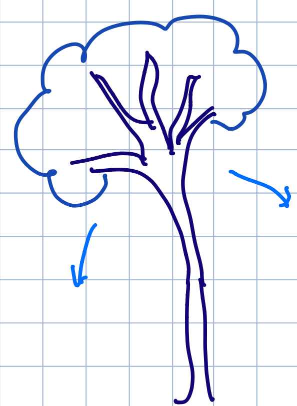
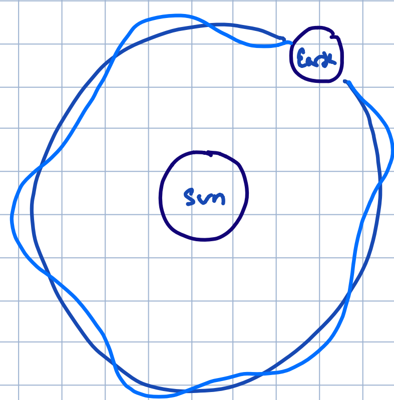
Lecture 12: Oscillatory Motion I

Oscillatory motion is ubiquitous in the natural world.

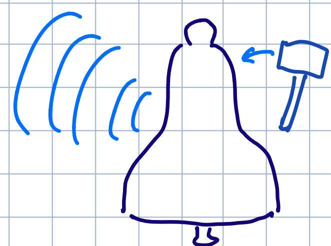
trees :



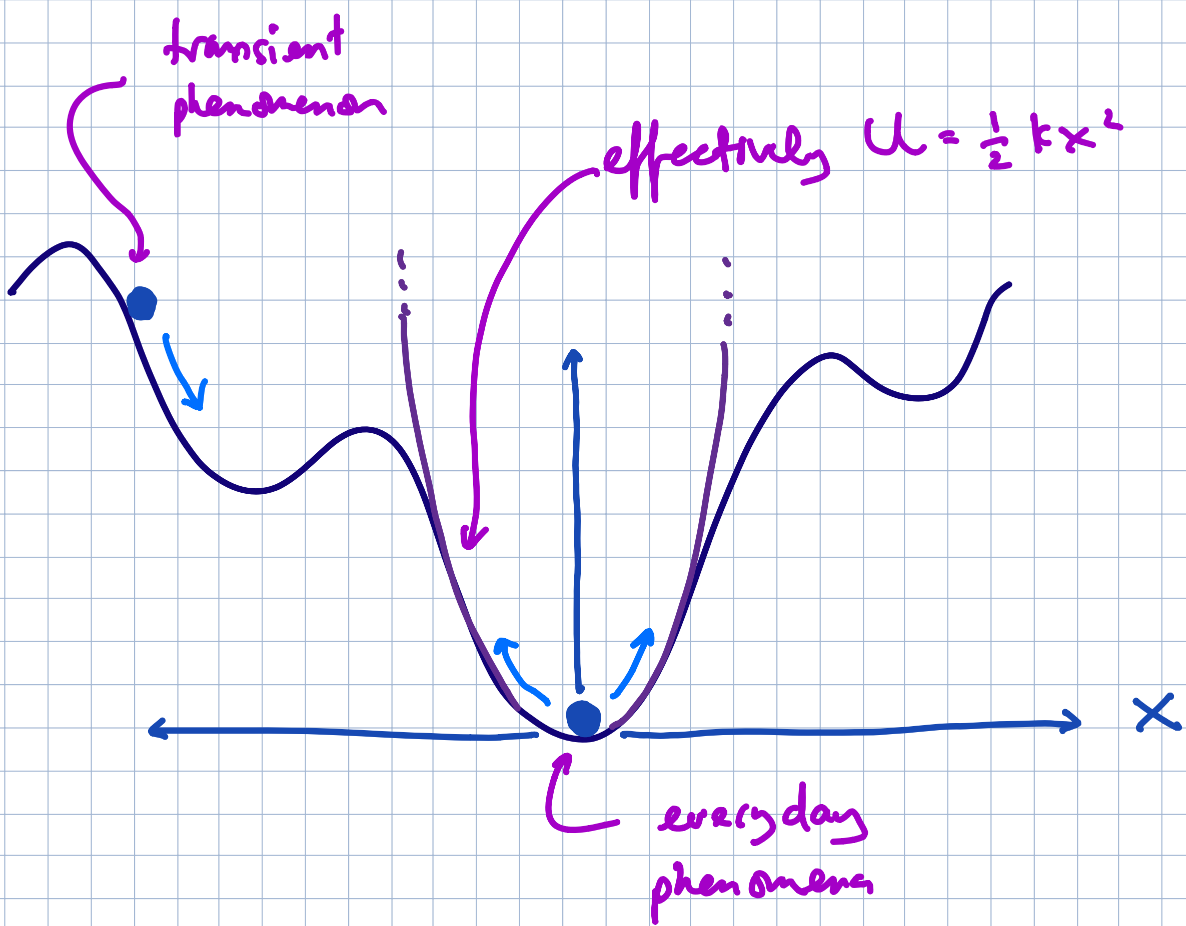
planets :



sound :



$\Longrightarrow$ why does everything oscillate / vibrate ???



Super Deep Fact #4: oscillators are everywhere because we encounter objects in equilibrium!

⟪ demo: "simple & spring pendulum" ⟫

Harmonic Oscillator

$$U = \frac{1}{2} k x^2$$

$$m\ddot{x} = F = -\frac{dU}{dx} = -kx$$

$$\longrightarrow \boxed{\ddot{x} = -\frac{k}{m}x}$$

harmonic oscillator
equation of motion

Let's solve this equation by guessing.

- What function is proportional to itself after applying $\frac{d^2}{dt^2}$???

$$\left. t^n \right\} \longrightarrow \text{NO}$$

$$\left.\begin{matrix} \cos(t) \\ \sin(t) \\ \exp(t) \end{matrix}\right\} \text{YES,}$$

but let's check the equation

- If x_1 and x_2 are both solutions of $\ddot{x} = -\frac{k}{m}x$, then $A_1 x_1 + A_2 x_2$ is also a solution!

arbitrary constants

A natural guess is then

overall constant

$$x(t) = A \cos \omega_0 t$$

oscillation freq

$$\rightarrow \quad \dot{x} = -\omega_0 A \sin \omega_0 t$$

$$\rightarrow \quad \ddot{x} = -\omega_0^2 A \cos \omega_0 t = -\omega_0^2 x$$

$$\xRightarrow{\text{this requires}} \quad \boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

The same works for

$$x(t) = B \sin \omega_0 t$$

So the general solution is

$$\boxed{x(t) = A \cos \omega_0 t + B \sin \omega_0 t}$$

two not yet determined parameters

Given the two initial conditions,

$$\begin{aligned} x(0) &= x_0 & & & x(0) &= A \\ \dot{x}(0) &= v_0 & \xleftrightarrow[\text{compare}]{} & & \dot{x}(0) &= \omega_0 B \end{aligned}$$

So we deduce that

$$\boxed{A = x_0 \quad \text{and} \quad B = v_0/\omega_0}$$

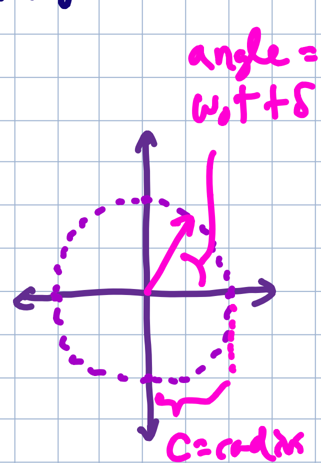
Finally, note that $x(t)$ can be recast as

$$x(t) = \overbrace{C}^{\text{normalization}} \cos(\omega_0 t + \underbrace{\delta}_{\text{phase shift}})$$

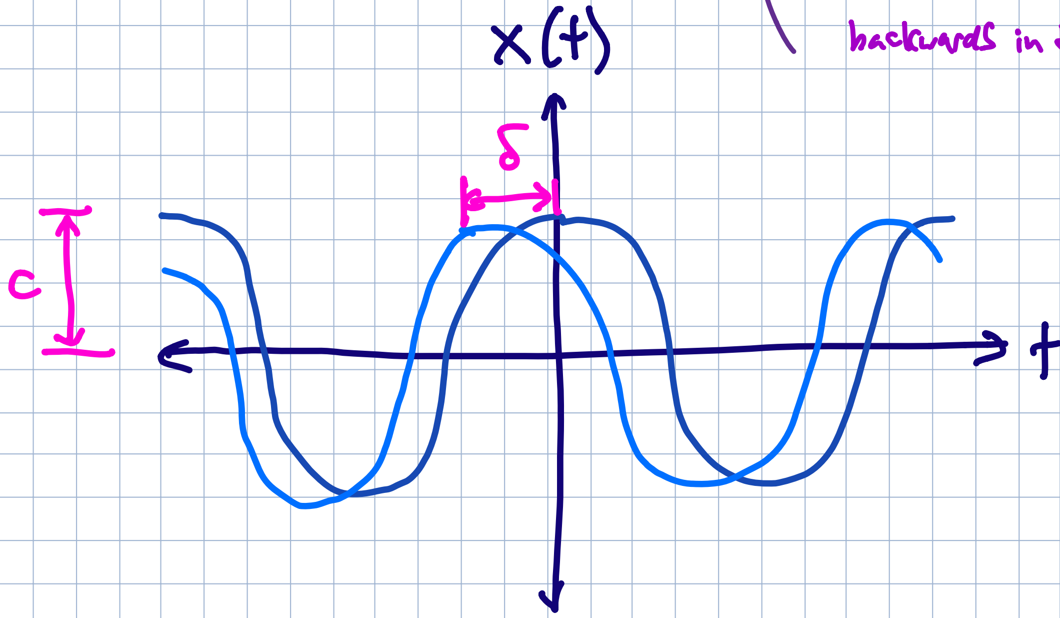
$$= C\left(\cos\omega_0 t \cos\delta - \sin\omega_0 t \sin\delta\right)$$

$$= \underbrace{(C\cos\delta)}_{=A} \cos\omega_0 t + \underbrace{(-C\sin\delta)}_{=B} \sin\omega_0 t$$

⟪⟪⟪ demo: "rotator viewed from side" ⟫⟫⟫



(Note: oscillations look similar forward vs backwards in time)



Damped Harmonic Oscillator

$$m\ddot{x} = F = -kx \underbrace{- \gamma \dot{x}}_{\text{friction}}$$

fights the direction of motion

$\beta \equiv \frac{\gamma}{m}$

$$\rightarrow \quad \ddot{x} + \beta \dot{x} + \omega_0^2 x = 0$$

$\omega_0 = \sqrt{k/m}$

Again, let's guess our way to the answer. We will deduce from various limits.

$\beta = 0$: $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$

$\omega_0 = 0$: $x(t) = C e^{-\beta t}$ (pure exponential decay)

So a natural guess that interpolates between limits is

not yet known

$$x(t) = C e^{-bt} \cos(\omega_1 t + \delta)$$

$$\dot{x} = C\left(-b e^{-bt}\cos(\cdots) - e^{-bt}\omega_1 \sin(\cdots)\right)$$

$$\ddot{x} = C\Big(b^2 e^{-bt}\cos(\cdots) + b\omega_1 e^{-bt}\sin(\cdots) + b\omega_1 e^{-bt}\sin(\cdots) - \omega_1^2 e^{-bt}\cos(\cdots)\Big)$$

$$\ddot{x} + \beta\dot{x} + \omega_0^2 x = 0$$

$$\|$$

$$Ce^{-bt} \sin(\cdots) \times \left\{ \underbrace{2b\omega_1 + \beta(-\omega_1)}_{=0 \to b = \beta/2} \right\}$$

$$+$$

$$Ce^{-bt} \cos(\cdots) \times \left\{ (-\omega_1^2 + b^2) + \beta(-b) + \omega_0^2 \right\}$$

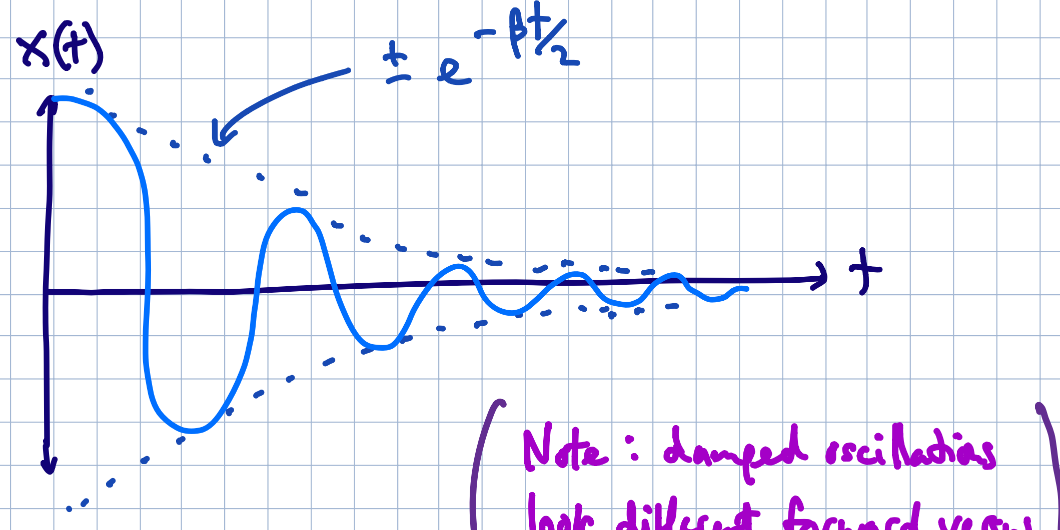
$$0 = \omega_0^2 - \omega_1^2 - \frac{\beta^2}{4} \to \omega_1 = \sqrt{\omega_0^2 - \beta^2/4}$$

Thus we obtain the solution to the damped oscillator:

$$\boxed{x(t) = Ce^{-\beta t/2} \cos\left(\sqrt{\omega_0^2 - \beta^2/4}\, t + \delta\right)}$$

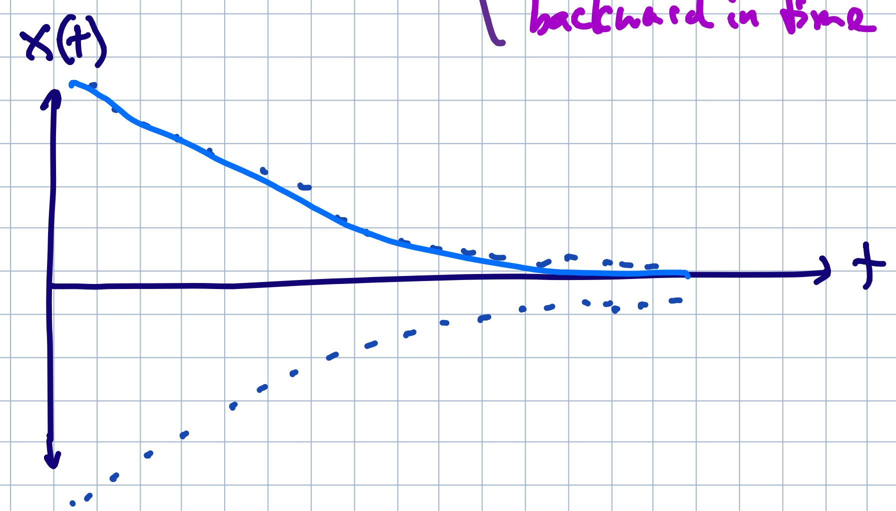
applies when $\omega_0 \geq \beta/2$

underdamped
$\beta < 2\omega_0$



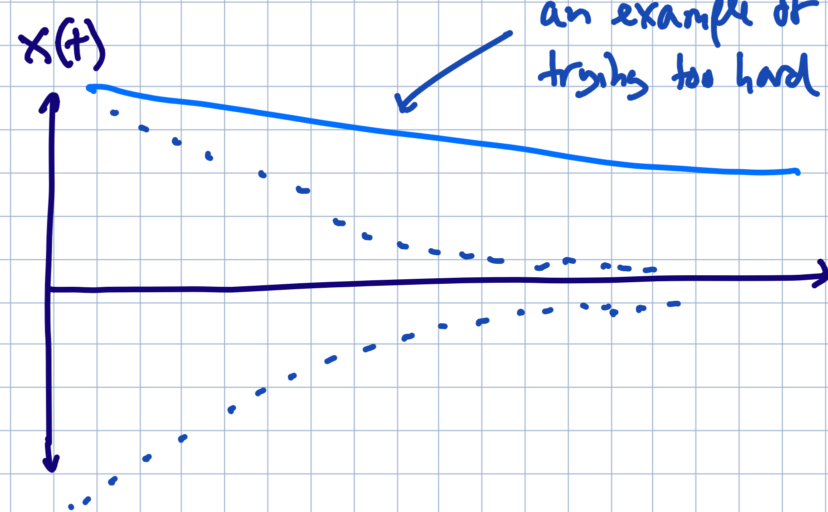
(Note: damped oscillations look different forward versus backward in time)

critically damped
$\beta = 2\omega_0$



over damped
$\beta = 2\omega_0$

an example of trying too hard



<<<< demo: "tuning fork" >>>>