Lecture 10: Rotational Dynamics I

translational motion	rotational motion
$\vec{F} = \frac{d\vec{p}}{dt}$	$\vec{\tau} = \frac{d\vec{L}}{dt}$
$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$

Moments of Inertia

depends crucially on where mass is distributed

$$I = \sum_i m_i r_i^2 = \int dm\, r^2$$

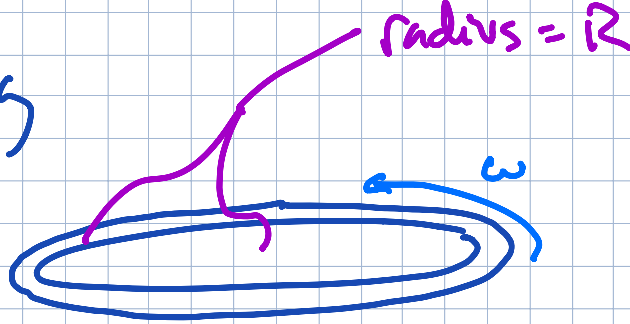
(discrete version) (continuous version)

bottom line: the more outwardly distributed the mass, the greater the rotational inertia

(ex. 1) point-like object

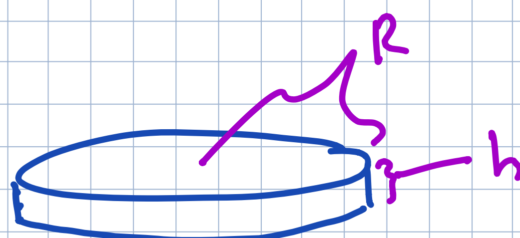
$$\boxed{I_{point} = 0}$$

(ex. 2) ring



$$I_{ring} = \sum_i m_i r_i^2 = \sum_i m_i R^2 = \boxed{MR^2 = I_{ring}}$$

(ex. 3) disc



density

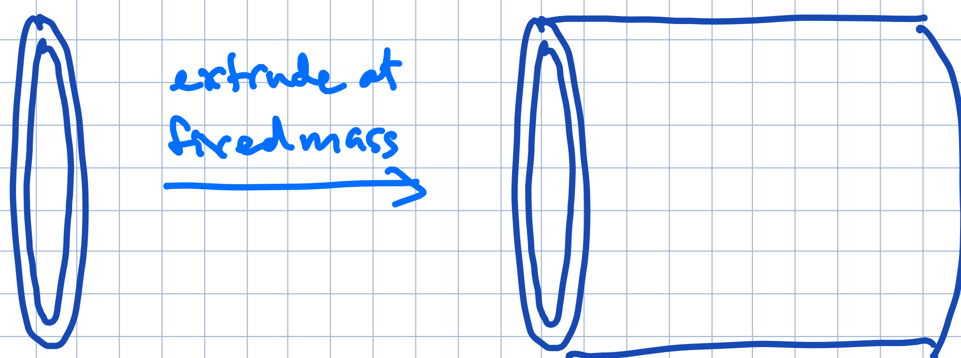
$$I_{disc} = \sum_i m_i r_i^2 = \int_0^R dm\, r^2 = \int_0^R \overbrace{\rho}^{\rho = \frac{M}{\pi R^2 h}} \underbrace{dV}_{2\pi r h\, dr}\, r^2$$

$$= \int_0^R 2\pi h \rho\, r^3\, dr$$

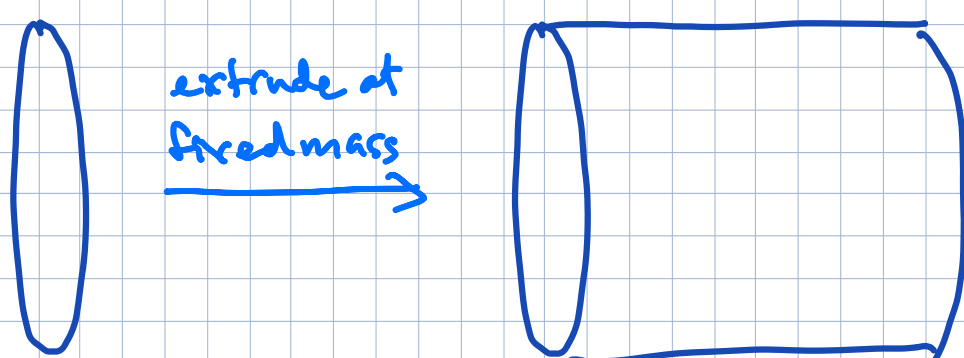
$$= 2\pi h \rho \frac{R^4}{4} = \frac{2\pi h M}{\pi R^2 h} \frac{R^4}{4} = \boxed{\frac{1}{2} MR^2 = I_{disc}}$$

Moments of inertia have some convenient properties.

- only distance from rotational axis matters



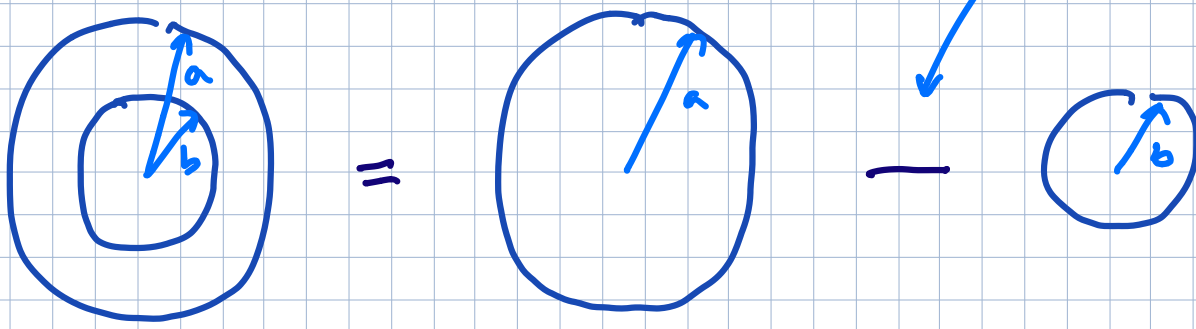
ring	hollow cylinder
MR^2	MR^2



disc	solid cylinder
$\frac{1}{2}MR^2$	$\frac{1}{2}MR^2$

- they add / subtract linearly

"negative mass"



annulus of mass M = disc of mass $M\frac{a^2}{a^2-b^2}$ − disc of mass $M\frac{b^2}{a^2-b^2}$

$$I_{annulus} = \frac{1}{2} M \frac{a^2}{a^2-b^2} a^2 - \frac{1}{2} M \frac{b^2}{a^2-b^2} b^2$$

$$= \frac{1}{2} M \frac{a^4-b^4}{a^2-b^2} = \boxed{\frac{1}{2} M (a^2+b^2) = I_{annulus}}$$

We can change the moment of inertia:

generally is fixed

can be changed

$$\vec{P} = m \vec{v} \qquad\qquad \vec{L} = I \vec{w}$$

const

const

rotation will chan also

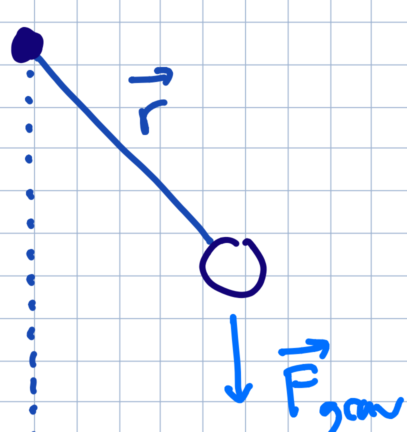
(ex 1) an ice skater

《demo: "physics of ballet"》

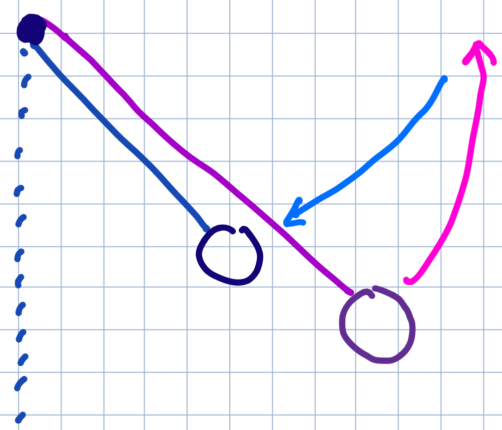
(ex 2) a cat

lands on its feet because it can control rotation by modulating its shape

(ex 3) a swing



The force of gravity always applies torque acting to push the swing to the vertical



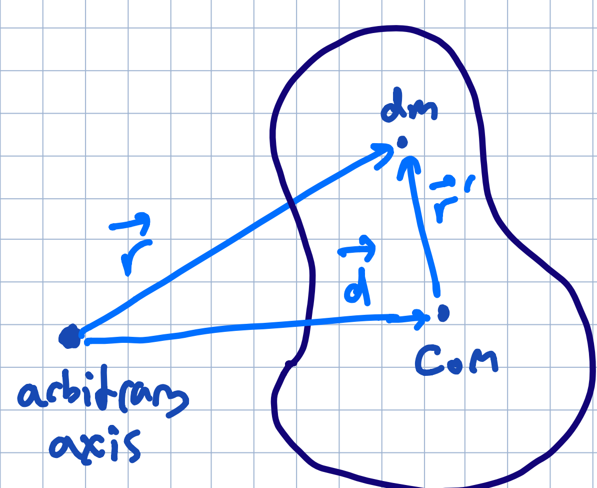
i) during upswing
increase r
increase I
decrease $|\Delta\omega|$

ii) during downswing
decrease r
decrease I
increase $|\Delta\omega|$

Parallel Axis Theorem

The moment of inertia changes with change of rotation axis.

But we can relate I about any arbitrary axis to I about the C.o.M position.



define $\vec{r} = \vec{d} + \vec{r}\,'$

$$I = \int dm\, r^2 = \int dm\, (\vec{d} + \vec{r}\,')^2$$

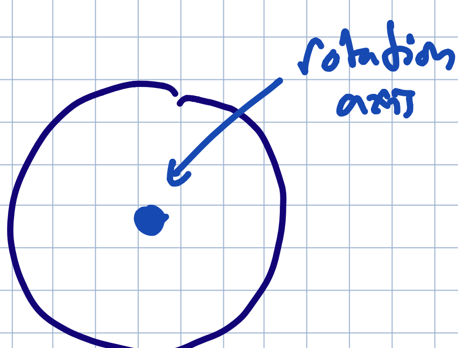
$$= \underbrace{\int dm\, d^2}_{md^2} + \underbrace{\int dm\, r'^2}_{I_{CoM}} + 2\vec{d}\cdot\underbrace{\int dm\, \vec{r}\,'}$$

mass averaged position in C.o.M. frame is zero!

$$\Longrightarrow \boxed{I = I_{CoM} + md^2}$$

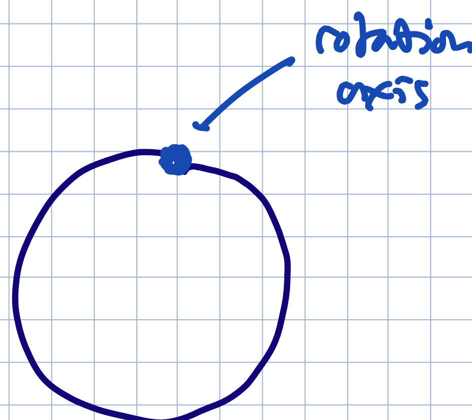
"parallel axis theorem"

(e.g.)



disc

$I = \frac{1}{2} MR^2$

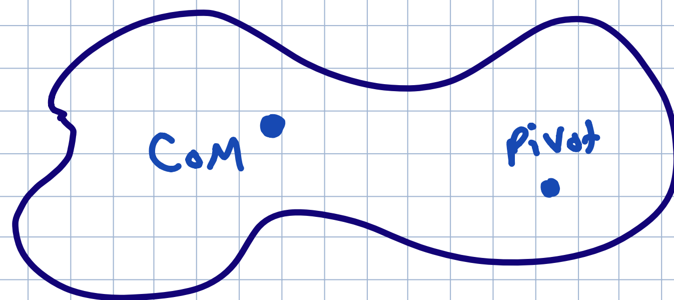


disc

$$I = \frac{1}{2} MR^2 + MR^2$$
$$= \frac{3}{2} MR^2 \ \checkmark\checkmark\checkmark$$

Rotational Kinetic Energy

Let's consider the kinetic energy of an object.



$$K = \sum_i \frac{1}{2} m_i v_i^2$$

$$= \frac{1}{2} \sum_i m_i \overbrace{r_i}^{\text{distance from pivot axis}}{}^2 \omega^2 = \frac{1}{2} I \omega^2$$

$$\Longrightarrow \boxed{K = \frac{1}{2} I \omega^2}$$

If we express I in terms of I_{CoM} via parallel axis theorem, we find:

$$K = \frac{1}{2}\left(I_{CoM} + Md^2\right)\omega^2$$

$$= \underbrace{\frac{1}{2} I_{CoM} \omega^2}_{\text{rotation about CoM}} + \underbrace{\frac{1}{2} M \overbrace{v}^{v_{CoM} = d\omega}{}^2_{CoM}}_{\text{translation of CoM}}$$

demo: "giant yo-yo"